First Order Partial Dynamic Equations on Time Scales

First Order Partial Dynamic Equations on Time Scales

By

Svetlin G. Georgiev

Cambridge Scholars Publishing

First Order Partial Dynamic Equations on Time Scales

By Svetlin G. Georgiev

This book first published 2024

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-0364-0194-4
ISBN (13): 978-1-0364-0194-8

Contents

Preface

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [5] in 1988 (supervised by Bernd Aulbach) in order to unify continuous and discrete analysis and to extend the continuous and discrete theories to cases “in between”. This book presents an introduction to the theory of first order partial dynamic equations (PDEs) on time scales. The book is primarily intended for senior undergraduate students and beginning graduate students of engineering and science courses. Students in mathematical and physical sciences will find many sections of direct relevance. This book contains five chapters, and each chapter consists of results with their proofs, numerous examples, and exercises with solutions. Each chapter concludes with a section featuring advanced practical problems with solutions followed by a section on notes and references, explaining its context within existing literature.

The basic definitions for forward and backward jump operators are due to Hilger. Examples for jump operators on some time scales are given in Chapter 1. The definitions for delta derivative and delta integral are given and some of their properties are explored. Basic definitions and concepts for dynamic and dynamic-algebraic equations on time scales are introduced in Chapter 2. Chapter 3 deals with the first order partial dynamic equations. Chapter 4 is concerned with the Laplace transform method. The Laplace transform is defined and some of its properties are explored. The Laplace transform is applied for solving first order PDEs. The method of separable variables is introduced in Chapter 5.

The aim of this book is to present a clear and well-organized treatment of the concept behind the development of mathematics as well as solution techniques. The text material of this book is presented in a readable and mathematically solid format.

Paris, August 2023 *Svetlin G. Georgiev*

Chapter 1

Elements of the Time Scale Calculus

1.1 Forward and Backward Jump Operators, Graininess Function

Definition 1.1. *A time scale is an arbitrary nonempty closed subset of the real numbers.*

We will denote a time scale by the symbol $\mathbb{T}$.

We suppose that a time scale $\mathbb{T}$ has the topology that inherits from the real numbers with the standard topology.

Example 1.2. $[1,2]$, $\mathbb{R}$, $\mathbb{N}$ *are time scales.*

Example 1.3. $[a,b)$, $(a,b]$, (a,b) *are not time scales.*

Definition 1.4. *For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ as follows*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

We note that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$.

Definition 1.5. *For $t \in \mathbb{T}$, we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

We note that $\rho(t) \le t$ for any $t \in \mathbb{T}$.

Definition 1.6. *We set*

$$\begin{aligned} \inf \emptyset &= \sup \mathbb{T}, \\ \sup \emptyset &= \inf \mathbb{T}. \end{aligned}$$

Definition 1.7. *For $t \in \mathbb{T}$, we have the following cases.*

1. *If $\sigma(t) > t$, then we say that t is right-scattered.*
2. *If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then we say that t is right-dense.*
3. *If $\rho(t) < t$, then we say that t is left-scattered.*
4. *If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say that t is left-dense.*
5. *If t is left-scattered and right-scattered at the same time, then we say that t is isolated.*
6. *If t is left-dense and right-dense at the same time, then we say that t is dense.*

Example 1.8. *Let $\mathbb{T} = \{\sqrt{2n+1} : n \in \mathbb{N}\}$. If $t = \sqrt{2n+1}$ for some $n \in \mathbb{N}$. Then $n = \frac{t^2-1}{2}$ and*

$$\begin{aligned} \sigma(t) &= \inf\{l \in \mathbb{N} : \sqrt{2l+1} > \sqrt{2n+1}\} \\ &= \sqrt{2n+3} \\ &= \sqrt{t^2+2} \quad \textit{for} \quad n \in \mathbb{N}, \\ \rho(t) &= \sup\{l \in \mathbb{N} : \sqrt{2l+1} < \sqrt{2n+1}\} \\ &= \sqrt{2n-1} \\ &= \sqrt{t^2-2} \quad \textit{for} \quad n \in \mathbb{N}, \quad n \ge 2. \end{aligned}$$

For $n = 1$, we have

$$\rho(\sqrt{3}) = \sup \emptyset$$

$$\begin{aligned} &= \inf \mathbb{T} \\ &= \sqrt{3}. \end{aligned}$$

Since

$$\sqrt{t^2-2} < t < \sqrt{t^2+2} \qquad \text{for} \qquad n \geq 2,$$

we conclude that every point $\sqrt{2n+1}$*,* $n \in \mathbb{N}$*,* $n \geq 2$*, is right-scattered and left-scattered, i.e., every point* $\sqrt{2n+1}$*,* $n \in \mathbb{N}$*,* $n \geq 2$*, is isolated.*

Because

$$\begin{aligned} \sqrt{3} &= \rho(\sqrt{3}) \\ &< \sigma(\sqrt{3}) \\ &= \sqrt{5}, \end{aligned}$$

we have that the point $\sqrt{3}$ *is right-scattered.*

Example 1.9. *Let* $\mathbb{T} = \left\{ \frac{1}{2n} : n \in \mathbb{N} \right\} \cup \{0\}$ *and* $t \in \mathbb{T}$ *be arbitrarily chosen.*

1. Let $t = \frac{1}{2}$. *Then*

$$\begin{aligned} \sigma\left(\frac{1}{2}\right) &= \inf\left\{ \frac{1}{2l}, 0 : \frac{1}{2l}, 0 > \frac{1}{2}, l \in \mathbb{N} \right\} \\ &= \inf \emptyset \\ &= \sup \mathbb{T} \\ &= \frac{1}{2}, \\ \rho\left(\frac{1}{2}\right) &= \sup\left\{ \frac{1}{2l}, 0 : \frac{1}{2l}, 0 < \frac{1}{2}, l \in \mathbb{N} \right\} \\ &= 0 \end{aligned}$$

$$< \quad \frac{1}{2},$$

i.e., the point $\frac{1}{2}$ *is left-scattered.*

2. Let $t = \frac{1}{2n}$**,** $n \in \mathbb{N}$**,** $n \geq 2$**.** *Then*

$$\begin{aligned}
\sigma\left(\frac{1}{2n}\right) &= \inf\left\{\frac{1}{2l} : \frac{1}{2l} > \frac{1}{2n}, l \in \mathbb{N}\right\} \\
&= \frac{1}{2(n-1)} \\
&> \frac{1}{2n},
\end{aligned}$$

$$\begin{aligned}
\rho\left(\frac{1}{2n}\right) &= \sup\left\{\frac{1}{2l}, 0 : \frac{1}{2l}, 0 < \frac{1}{2n}, l \in \mathbb{N}\right\} \\
&= \frac{1}{2(n+1)} \\
&< \frac{1}{2n}.
\end{aligned}$$

Therefore all points $\frac{1}{2n}$*,* $n \in \mathbb{N}$*,* $n \geq 2$*, are right-scattered and left-scattered, i.e., all points* $\frac{1}{2n}$*,* $n \in \mathbb{N}$*,* $n \geq 2$*, are isolated.*

3. Let $t = 0$**.** *Then*

$$\begin{aligned}
\sigma(0) &= \inf\{s \in \mathbb{T} : s > 0\} \\
&= 0, \\
\rho(0) &= \sup\{x \in \mathbb{T} : s < 0\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup \emptyset \\
&= \inf \mathbb{T} \\
&= 0.
\end{aligned}
$$

Example 1.10. *Let* $\mathbb{T} = \left\{ \frac{n}{3} : n \in \mathbb{N}_0 \right\}$ *and* $t = \frac{n}{3}$*,* $n \in \mathbb{N}_0$*, be arbitrarily chosen.*

1. Let $n \in \mathbb{N}$**.** *Then*

$$
\begin{aligned}
\sigma\left(\frac{n}{3}\right) &= \inf \left\{ \frac{l}{3}, 0 : \frac{l}{3}, 0 > \frac{n}{3}, l \in \mathbb{N}_0 \right\} \\
&= \frac{n+1}{3} \\
&> \frac{n}{3}, \\
\rho\left(\frac{n}{3}\right) &= \sup \left\{ \frac{l}{3}, 0 : \frac{l}{3}, 0 < \frac{n}{3}, l \in \mathbb{N}_0 \right\} \\
&= \frac{n-1}{3} \\
&< \frac{n}{3}.
\end{aligned}
$$

Therefore all points $t = \frac{n}{3}$*,* $n \in \mathbb{N}$*, are right-scattered and left-scattered, i.e., all points* $t = \frac{n}{3}$*,* $n \in \mathbb{N}$*, are isolated.*

2. Let $n = 0$**.** *Then*

$$
\begin{aligned}
\sigma(0) &= \inf \left\{ \frac{l}{3}, 0 : \frac{l}{3}, 0 > 0, l \in \mathbb{N}_0 \right\} \\
&= \frac{1}{3} \\
&> 0,
\end{aligned}
$$

$$\begin{aligned}\rho(0) &= \sup\left\{\frac{l}{3} : \frac{l}{3}, 0 < 0, l \in \mathbb{N}_0\right\} \\ &= \sup \emptyset \\ &= \inf \mathbb{T} \\ &= 0,\end{aligned}$$

i.e., $t = 0$ is right-scattered.

Exercise 1.11. *Classify each point $t \in \mathbb{T} = \{\sqrt[3]{2n-1} : n \in \mathbb{N}_0\}$ as left-dense, left-scattered, right-dense, or right-scattered.*

Answer. The points $\sqrt[3]{2n-1}$, $n \in \mathbb{N}$, are isolated, the point -1 is right-scattered.

Definition 1.12. *The numbers*

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N},$$

will be called harmonic numbers.

Exercise 1.13. *Let*

$$\mathbb{H} = \{H_n : n \in \mathbb{N}_0\}.$$

Prove that $\mathbb{H}$ is a time scale. Find $\sigma(t)$ and $\rho(t)$ for $t \in \mathbb{T}$.

Answer. $\sigma(H_n) = H_{n+1}, \quad \rho(H_n) = H_{n-1}, \quad n \in \mathbb{N}, \quad \rho(H_0) = H_0.$

Definition 1.14. *The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\mu(t) = \sigma(t) - t, \quad t \in \mathbb{T}.$$

Example 1.15. *Let $\mathbb{T} = \left\{2^{n+1} : n \in \mathbb{N}\right\}$. Let also, $t = 2^{n+1} \in \mathbb{T}$ for some $n \in \mathbb{N}$. Then*

$$\begin{aligned}\sigma(t) &= \inf\left\{2^{l+1} : 2^{l+1} > 2^{n+1}, l \in \mathbb{N}\right\} \\ &= 2^{n+2}\end{aligned}$$

$$= \quad 2t.$$

Hence,

$$\mu(t) \quad = \quad \sigma(t) - t = 2t - t = t$$

or

$$\mu\left(2^{n+1}\right) = 2^{n+1}, \quad n \in \mathbb{N}.$$

Example 1.16. *Let* $\mathbb{T} = \left\{\sqrt{n+1} : n \in \mathbb{N}\right\}$. *Let also,* $t = \sqrt{n+1}$ *for some* $n \in \mathbb{N}$. *Then* $n = t^2 - 1$ *and*

$$\begin{aligned}
\sigma(t) &= \left\{\sqrt{l+1} : \sqrt{l+1} > \sqrt{n+1}, l \in \mathbb{N}\right\} \\
&= \sqrt{n+2} \\
&= \sqrt{t^2+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu(t) &= \sigma(t) - t \\
&= \sqrt{t^2+1} - t
\end{aligned}$$

or

$$\mu(\sqrt{n+1}) = \sqrt{n+2} - \sqrt{n+1}, \quad n \in \mathbb{N}.$$

Example 1.17. *Let* $\mathbb{T} = \left\{\frac{n}{2} : n \in \mathbb{N}_0\right\}$. *Let also,* $t = \frac{n}{2}$ *for some* $n \in \mathbb{N}_0$. *Then* $n = 2t$ *and*

$$\begin{aligned}
\sigma(t) &= \inf\left\{\frac{l}{2} : \frac{l}{2} > \frac{n}{2}, l \in \mathbb{N}_0\right\} \\
&= \frac{n+1}{2} \\
&= t + \frac{1}{2}.
\end{aligned}$$

Hence,

$$\mu(t) \quad = \quad \sigma(t) - t$$

$$\begin{aligned} &= t+\frac{1}{2}-t \\ &= \frac{1}{2} \end{aligned}$$

or

$$\mu\left(\frac{n}{2}\right)=\frac{1}{2}.$$

Example 1.18. *Suppose that* $\mathbb{T}$ *consists of finitely many different points:* $t_1,t_2,\ldots,t_k$. *Without loss of generality, we can assume that*

$$t_1<t_2<\ldots<t_k.$$

For $i=1,2,\ldots,k-1$, *we have*

$$\begin{aligned} \sigma(t_i) &= \inf\{t_l\in\mathbb{T}: t_l>t_i, l=1,2,\ldots,k\} \\ &= t_{i+1}. \end{aligned}$$

Hence,

$$\mu(t_i)=t_{i+1}-t_i,\quad i=1,2,\ldots,k-1.$$

Also,

$$\begin{aligned} \sigma(t_k) &= \inf\{t_l\in\mathbb{T}: t_l>t_k, l=1,2,\ldots,k\} \\ &= \inf\emptyset \\ &= \sup\mathbb{T} \\ &= t_k. \end{aligned}$$

Therefore

$$\begin{aligned} \mu(t_k) &= \sigma(t_k)-t_k \\ &= t_k-t_k \end{aligned}$$

$$= \quad 0.$$

From here,

$$\begin{aligned}
\sum_{i=1}^{k}\mu(t_i) &= \sum_{i=1}^{k-1}\mu(t_i)+\mu(t_k)\\
&= \sum_{i=1}^{k-1}(t_{i+1}-t_i)\\
&= t_k-t_1.
\end{aligned}$$

Exercise 1.19. *Let* $\mathbb{T}=\left\{\sqrt[3]{n+2} : n\in\mathbb{N}_0\right\}$. *Find* $\mu(t)$, $t\in\mathbb{T}$.

Answer. $\mu\left(\sqrt[3]{n+2}\right)=\sqrt[3]{n+3}-\sqrt[3]{n+2}$.

Definition 1.20. *If* $f:\mathbb{T}\to\mathbb{R}$ *is a function, then we define the function* $f^\sigma:\mathbb{T}\to\mathbb{R}$ *by*

$$f^\sigma(t)=f(\sigma(t)) \quad \textit{for any} \quad t\in\mathbb{T}, \quad \textit{i.e.}, \quad f^\sigma=f\circ\sigma.$$

Example 1.21. *Let* $\mathbb{T}=\left\{t=2^{n+2} : n\in\mathbb{N}\right\}$, $f(t)=t^2+t-1$. *Then*

$$\begin{aligned}
\sigma(t) &= \inf\left\{2^{l+2} : 2^{l+2}>2^{n+2}, \quad l\in\mathbb{N}\right\}\\
&= 2^{n+3}\\
&= 2t.
\end{aligned}$$

Hence,

$$\begin{aligned}
f^\sigma(t) &= f(\sigma(t))\\
&= (\sigma(t))^2+\sigma(t)-1\\
&= (2t)^2+2t-1=4t^2+2t-1, \quad t\in\mathbb{T}.
\end{aligned}$$

Example 1.22. *Let* $\mathbb{T}=\{t=\sqrt{n+3} : n\in\mathbb{N}\}$, $f(t)=t+3$, $t\in\mathbb{T}$. *Then* $n=t^2-3$ *and*

$$\sigma(t) = \inf\{\sqrt{l+3} : \sqrt{l+3}>\sqrt{n+3}, l\in\mathbb{N}\}$$

$$= \sqrt{n+4}$$

$$= \sqrt{t^2+1}.$$

Hence,

$$\begin{aligned} f(\sigma(t)) &= \sigma(t)+3 \\ &= \sqrt{t^2+1}+3. \end{aligned}$$

Example 1.23. *Let* $\mathbb{T} = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$, $f(t) = t^3 - t$, $t \in \mathbb{T}$.

1. Let $t = \frac{1}{n}$**,** $n \geq 2$**.** *Then* $n = \frac{1}{t}$ *and*

$$\begin{aligned} \sigma(t) &= \inf\left\{ \frac{1}{l}, 0 : \frac{1}{l}, 0 > \frac{1}{n}, l \in \mathbb{N} \right\} \\ &= \frac{1}{n-1} \\ &= \frac{t}{1-t}. \end{aligned}$$

Hence,

$$\begin{aligned} f(\sigma(t)) &= (\sigma(t))^3 - \sigma(t) \\ &= \left(\frac{t}{1-t} \right)^3 - \frac{t}{1-t} \\ &= \frac{t^3}{(1-t)^3} - \frac{t}{1-t} \\ &= \frac{t^3 - t(1-t)^2}{(1-t)^3} \\ &= \frac{t^3 - t(1-2t+t^2)}{(1-t)^3} \end{aligned}$$

$$
\begin{aligned}
&= \frac{t^3 - t + 2t^2 - t^3}{(1-t)^3} \\
&= \frac{t(2t-1)}{(1-t)^3}.
\end{aligned}
$$

2. Let $t = 1$**.** *Then*

$$
\begin{aligned}
\sigma(1) &= \inf\left\{\frac{1}{l}, 0 : \frac{1}{l}, 0 > 1, l \in \mathbb{N}\right\} \\
&= \inf \emptyset \\
&= \sup \mathbb{T} \\
&= 1, \\
f(\sigma(1)) &= (\sigma(1))^3 - \sigma(1) \\
&= 1 - 1 \\
&= 0.
\end{aligned}
$$

3. Let $t = 0$**.** *Then*

$$
\begin{aligned}
\sigma(0) &= \inf\left\{\frac{1}{l}, 0 : \frac{1}{l}, 0 > 0\right\} \\
&= 0, \\
f(\sigma(0)) &= (\sigma(0))^3 - \sigma(0) \\
&= 0.
\end{aligned}
$$

Exercise 1.24. *Let* $\mathbb{T} = \left\{t = \sqrt[3]{n+2} : n \in \mathbb{N}\right\}$, $f(t) = 1 - t^3$, $t \in \mathbb{T}$. *Find* $f(\sigma(t))$, $t \in \mathbb{T}$.

Answer. $-t^3$.

Definition 1.25. *We define the set*

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \textit{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} \quad \text{otherwise.} \end{cases}$$

Example 1.26. *Let* $\mathbb{T} = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$. *Then* $\sup \mathbb{T} = 1$ *and*

$$\begin{aligned} \rho(1) &= \sup \left\{ \frac{1}{l}, 0 : \frac{1}{l}, 0 < 1, l \in \mathbb{N} \right\} \\ &= \frac{1}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{T}^{\kappa} &= \mathbb{T} \backslash \left(\frac{1}{2}, 1 \right] \\ &= \left\{ \frac{1}{n} : n \in \mathbb{N}, n \geq 2 \right\} \cup \{0\}. \end{aligned}$$

Example 1.27. *Let* $\mathbb{T} = \{2n : n \in \mathbb{N}\}$. *Then* $\sup \mathbb{T} = \infty$ *and* $\mathbb{T}^{\kappa} = \mathbb{T}$.

Example 1.28. *Let* $\mathbb{T} = \left\{ \frac{1}{n^2+3} : \quad n \in \mathbb{N} \right\} \cup \{0\}$. *Then* $\sup \mathbb{T} = \frac{1}{4} < \infty$,

$$\begin{aligned} \rho\left(\frac{1}{4}\right) &= \sup \left\{ \frac{1}{l^2+3}, 0 : \frac{1}{l^2+3}, 0 < \frac{1}{4}, l \in \mathbb{N} \right\} \\ &= \frac{1}{7}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{T}^{\kappa} &= \mathbb{T} \backslash \left(\frac{1}{7}, \frac{1}{4} \right] \\ &= \left\{ \frac{1}{n^2+3} : n \geq 2 \right\} \cup \{0\}. \end{aligned}$$

Definition 1.29. *We assume that* $a \leq b$. *We define the interval* $[a,b]$ *in* $\mathbb{T}$ *by*

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals, half-open intervals and so on, are defined accordingly.

Example 1.30. *Let $[a,b]$ be an interval in $\mathbb{T}$ and b be a left-dense point. Then $\sup[a,b]=b$ and since b is a left-dense point, we have that $\rho(b)=b$. Hence,*

$$\begin{aligned}[a,b]^\kappa &= [a,b]\backslash(b,b] \\ &= [a,b]\backslash\emptyset \\ &= [a,b].\end{aligned}$$

Example 1.31. *Let $[a,b]$ be an interval in $\mathbb{T}$ and b be a left-scattered point. Then $\sup[a,b]=b$ and since b is a left-scattered point, we have that $\rho(b)<b$. We assume that there is $c\in(\rho(b),b]$, $c\in\mathbb{T}$. Then $\rho(b)<c\leq b$, which is a contradiction. Therefore*

$$\begin{aligned}[a,b]^\kappa &= [a,b]\backslash(\rho(b),b] \\ &= [a,b).\end{aligned}$$

Exercise 1.32. *Let* $\mathbb{T}=\left\{\frac{1}{2n+1}:n\in\mathbb{N}\right\}\cup\{0\}$. *Find* $\mathbb{T}^\kappa$.

Answer. $\left\{\frac{1}{2n+1}:n\in\mathbb{N},n\geq 2\right\}\cup\{0\}$.

1.2 Differentiation

Definition 1.33. *Assume that $f:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}^\kappa$. We define $f^\Delta(t)$ to be the number, provided it exists, as follows: for any $\varepsilon>0$ there is a neighbourhood U of t, $U=(t-\delta,t+\delta)\cap\mathbb{T}$ for some $\delta>0$, such that*

$$|f(\sigma(t))-f(s)-f^\Delta(t)(\sigma(t)-s)|\leq\varepsilon|\sigma(t)-s| \quad \textit{for all} \quad s\in U,$$

$s\neq\sigma(t)$. We say $f^\Delta(t)$ is the delta or Hilger derivative of f at t.

We say that f is delta or Hilger differentiable, shortly differentiable, in $\mathbb{T}^\kappa$ if $f^\Delta(t)$ exists for all $t\in\mathbb{T}^\kappa$. The function $f^\Delta:\mathbb{T}\to\mathbb{R}$ is said to be the delta derivative or the Hilger derivative, shortly the derivative, of f in $\mathbb{T}^\kappa$.

Remark 1.34. *If $\mathbb{T}=\mathbb{R}$, then the delta derivative coincides with the classical derivative.*

Theorem 1.35. *The delta derivative is well defined.*

Proof. Let $t \in \mathbb{T}^\kappa$ and $f_i^\Delta(t)$, $i = 1, 2$, be such that

$$\begin{aligned} |f(\sigma(t)) - f(s) - f_1^\Delta(t)(\sigma(t) - s)| &\leq \frac{\varepsilon}{2}|\sigma(t) - s|, \\ |f(\sigma(t)) - f(s) - f_2^\Delta(t)(\sigma(t) - s)| &\leq \frac{\varepsilon}{2}|\sigma(t) - s|, \end{aligned}$$

for any $\varepsilon > 0$ and any s belonging to a neighbourhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, $s \neq \sigma(t)$. Hence,

$$\begin{aligned} |f_1^\Delta(t) - f_2^\Delta(t)| &= \left| f_1^\Delta(t) - \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} + \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f_2^\Delta(t) \right| \\ &\leq \left| f_1^\Delta(t) - \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \right| + \left| \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f_2^\Delta(t) \right| \\ &= \frac{|f(\sigma(t)) - f(s) - f_1^\Delta(t)(\sigma(t) - s)|}{|\sigma(t) - s|} \\ &\quad + \frac{|f(\sigma(t)) - f(s) - f_2^\Delta(t)(\sigma(t) - s)|}{|\sigma(t) - s|} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$f_1^\Delta(t) = f_2^\Delta(t),$$

which completes the proof. □

Remark 1.36. *Let us assume that* $\sup \mathbb{T} < \infty$ *and* $f^\Delta(t)$ *is defined at a point* $t \in \mathbb{T} \backslash \mathbb{T}^\kappa$ *with the same definition as it is given in Definition 1.33. Then the unique point* $t \in \mathbb{T} \backslash \mathbb{T}^\kappa$ *is* $\sup \mathbb{T}$.

Hence, for any $\varepsilon > 0$ *there is a neighbourhood* $U = (t - \delta, t + \delta) \cap (\mathbb{T} \backslash \mathbb{T}^\kappa)$*, for some* $\delta > 0$*, such that*

$$f(\sigma(t)) = f(s)$$

$$
\begin{aligned}
&= f(\sigma(\sup \mathbb{T}))\\
&= f(\sup \mathbb{T}), \quad s \in U, \quad s \neq \sigma(t).
\end{aligned}
$$

Therefore, for any $\alpha \in \mathbb{R}$ and $s \in U$, we have

$$
\begin{aligned}
|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| &= |f(\sup \mathbb{T}) - f(\sup \mathbb{T}) - \alpha(\sup \mathbb{T} - \sup \mathbb{T})|\\
&\leq \varepsilon|\sigma(t) - s|,
\end{aligned}
$$

i.e., any $\alpha \in \mathbb{R}$ is the delta derivative of f at the point $t \in \mathbb{T} \backslash \mathbb{T}^{\kappa}$.

Example 1.37. *Let $f(t) = \alpha \in \mathbb{R}$. We will prove that $f^{\Delta}(t) = 0$ for any $t \in \mathbb{T}^{\kappa}$.*

Really, for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0$, there is a $\delta > 0$ such that $s \in (t - \delta, t + \delta) \cap \mathbb{T}$, $s \neq \sigma(t)$, implies

$$
\begin{aligned}
|f(\sigma(t)) - f(s) - 0(\sigma(t) - s)| &= |\alpha - \alpha|\\
&\leq \varepsilon|\sigma(t) - s|.
\end{aligned}
$$

Example 1.38. *Let $f(t) = t$, $t \in \mathbb{T}$. We will prove that $f^{\Delta}(t) = 1$ for any $t \in \mathbb{T}^{\kappa}$.*

Really, for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0$ there is a $\delta > 0$ such that $s \in (t - \delta, t + \delta) \cap \mathbb{T}$, $s \neq \sigma(t)$, implies

$$
\begin{aligned}
|f(\sigma(t)) - f(s) - 1(\sigma(t) - \sigma(s))| &= |\sigma(t) - s - (\sigma(t) - s)|\\
&\leq \varepsilon|\sigma(t) - s|.
\end{aligned}
$$

Example 1.39. *Let $f(t) = t^2$, $t \in \mathbb{T}$. We will prove that $f^{\Delta}(t) = \sigma(t) + t$, $t \in \mathbb{T}^{\kappa}$.*

Really, for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0$, and for any $s \in (t - \varepsilon, t + \varepsilon) \cap \mathbb{T}$, $s \neq \sigma(t)$, we have $|t - s| < \varepsilon$ and

$$
\begin{aligned}
&|f(\sigma(t)) - f(s) - (\sigma(t) + t)(\sigma(t) - s)|\\
&= |(\sigma(t))^2 - s^2 - (\sigma(t) + t)(\sigma(t) - s)|
\end{aligned}
$$

$$
\begin{aligned}
&= |(\sigma(t)-s)(\sigma(t)+s)-(\sigma(t)+t)(\sigma(t)-s)| \\
&= |\sigma(t)-s||t-s| \\
&\le \varepsilon|\sigma(t)-s|.
\end{aligned}
$$

Exercise 1.40. *Let* $f(t)=\sqrt{t}$, $t\in\mathbb{T}, t>0$. *Prove that* $f^{\Delta}(t)=\dfrac{1}{\sqrt{t}+\sqrt{\sigma(t)}}$ *for* $t\in\mathbb{T}^{\kappa}$, $t>0$.

Exercise 1.41. *Let* $f(t)=t^3$, $t\in\mathbb{T}$. *Prove that* $f^{\Delta}(t)=(\sigma(t))^2+t\sigma(t)+t^2$ *for* $t\in\mathbb{T}^{\kappa}$.

Theorem 1.42. *[1] Assume that* $f:\mathbb{T}\to\mathbb{R}$ *is a function and let* $t\in\mathbb{T}^{\kappa}$. *Then we have the following.*

1. *If* f *is differentiable at* t*, then* f *is continuous at* t.
2. *If* f *is continuous at* t *and* t *is right-scattered, then* f *is differentiable at* t *with*
$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}.$$
3. *If* t *is right-dense, then* f *is differentiable if and only if the limit*
$$\lim_{s\longrightarrow t}\frac{f(t)-f(s)}{t-s}$$
exists as a finite number. In this case, we have
$$f^{\Delta}(t)=\lim_{s\longrightarrow t}\frac{f(t)-f(s)}{t-s}.$$
4. *If* f *is differentiable at* t*, then*
$$f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t).$$

Example 1.43. *Let* $\mathbb{T}=\left\{\dfrac{1}{2n+1}:n\in\mathbb{N}_0\right\}\cup\{0\}$, $f(t)=\sigma(t)$, $t\in\mathbb{T}$. *We will find* $f^{\Delta}(t)$, $t\in\mathbb{T}$. *For* $t\in\mathbb{T}$, $t=\dfrac{1}{2n+1}$, $n=\dfrac{1-t}{2t}$, $n\ge 1$, *we have*

$$\sigma(t) = \inf\left\{\frac{1}{2l+1},0:\frac{1}{2l+1},0>\frac{1}{2n+1},l\in\mathbb{N}_0\right\}$$

$$
\begin{aligned}
&= \frac{1}{2n-1} \\
&= \frac{1}{2\frac{1-t}{2t}-1} \\
&= \frac{t}{1-2t} \\
&> t,
\end{aligned}
$$

i.e., any point $t = \dfrac{1}{2n+1}$, $n \geq 1$, *is right-scattered. At these points*

$$
\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\
&= \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} \\
&= 2\frac{(\sigma(t))^2}{(1-2\sigma(t))(\sigma(t)-t)} \\
&= 2\frac{\left(\frac{t}{1-2t}\right)^2}{\left(1-\frac{2t}{1-2t}\right)\left(\frac{t}{1-2t}-t\right)} \\
&= 2\frac{\frac{t^2}{(1-2t)^2}}{\frac{1-4t}{1-2t}\frac{2t^2}{1-2t}} \\
&= 2\frac{t^2}{2t^2(1-4t)} \\
&= \frac{1}{1-4t}.
\end{aligned}
$$

Let $n = 0$, *i.e.,* $t = 1$. *Then*

$$
\begin{aligned}
\sigma(1) &= \inf\left\{\frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > 1, l \in \mathbb{N}_0\right\} \\
&= \inf \emptyset
\end{aligned}
$$

$$= \sup \mathbb{T}$$

$$= 1,$$

i.e., $t = 1$ is a right-dense point. Also,

$$\begin{aligned}
\lim_{s \to 1} \frac{f(1) - f(s)}{1 - s} &= \lim_{s \to 1} \frac{\sigma(1) - \sigma(s)}{1 - s} \\
&= \lim_{s \to 1} \frac{1 - \frac{s}{1-2s}}{1 - s} \\
&= \lim_{s \to 1} \frac{1 - 3s}{(1 - s)(1 - 2s)} \\
&= +\infty.
\end{aligned}$$

Therefore $\sigma'(1)$ doesn't exist.

Let now, $t = 0$. Then

$$\begin{aligned}
\sigma(0) &= \inf\left\{ \frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > 0, l \in \mathbb{N}_0 \right\} \\
&= 0.
\end{aligned}$$

Consequently $t = 0$ is right-dense. Also,

$$\begin{aligned}
\lim_{h \longrightarrow 0} \frac{\sigma(h) - \sigma(0)}{h} &= \lim_{h \longrightarrow 0} \frac{\frac{h}{1-2h} - 0}{h} \\
&= \lim_{h \longrightarrow 0} \frac{1}{1 - 2h} \\
&= 1.
\end{aligned}$$

Therefore $\sigma'(0) = 1$.

Example 1.44. *Let $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$, $f(t) = t^2$, $g(t) = \sigma(t)$, $t \in \mathbb{T}$. We will find $f^\Delta(t)$ and $g^\Delta(t)$ for $t \in \mathbb{T}^\kappa$. For $t \in \mathbb{T}^\kappa$, $t = n^2$, $n = \sqrt{t}$, $n \in \mathbb{N}_0$, we have*

$$\sigma(t) = \inf\{l^2 : l^2 > n^2, l \in \mathbb{N}_0\}$$

$$
\begin{aligned}
&= (n+1)^2 \\
&= (\sqrt{t}+1)^2 \\
&> t.
\end{aligned}
$$

Therefore any points of $\mathbb{T}$ are right-scattered. We note that f and g are continuous functions in $\mathbb{T}$. Hence,

$$
\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t))-f(t)}{\sigma(t)-t} \\
&= \frac{(\sigma(t))^2-t^2}{\sigma(t)-t} \\
&= \sigma(t)+t \\
&= \left(\sqrt{t}+1\right)^2+t \\
&= t+2\sqrt{t}+1+t \\
&= 1+2\sqrt{t}+2t, \\
g^{\Delta}(t) &= \frac{g(\sigma(t))-g(t)}{\sigma(t)-t} \\
&= \frac{\sigma(\sigma(t))-\sigma(t)}{\sigma(t)-t} \\
&= \frac{(\sqrt{\sigma(t)}+1)^2-\sigma(t)}{\sigma(t)-t} \\
&= \frac{\sigma(t)+2\sqrt{\sigma(t)}+1-\sigma(t)}{\sigma(t)-t} \\
&= \frac{1+2\sqrt{\sigma(t)}}{\sigma(t)-t}
\end{aligned}
$$

$$= \frac{1+2(\sqrt{t}+1)}{(\sqrt{t}+1)^2-t}$$

$$= \frac{3+2\sqrt{t}}{1+2\sqrt{t}}.$$

Example 1.45. *Let* $\mathbb{T} = \{\sqrt[4]{2n+1} : n \in \mathbb{N}_0\}$, $f(t) = t^4$, $t \in \mathbb{T}$. *We will find* $f^{\Delta}(t)$, $t \in \mathbb{T}$. *For* $t \in \mathbb{T}$, $t = \sqrt[4]{2n+1}$, $n = \frac{t^4-1}{2}$, $n \in \mathbb{N}_0$, *we have*

$$\begin{aligned} \sigma(t) &= \inf\{\sqrt[4]{2l+1} : \sqrt[4]{2l+1} > \sqrt[4]{2n+1}, l \in \mathbb{N}_0\} \\ &= \sqrt[4]{2n+3} \\ &= \sqrt[4]{t^4+2} \\ &> t. \end{aligned}$$

Therefore every point of $\mathbb{T}$ *is right-scattered. We note that the function* f *is continuous in* $\mathbb{T}$. *Hence,*

$$\begin{aligned} f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\ &= \frac{(\sigma(t))^4 - t^4}{\sigma(t) - t} \\ &= (\sigma(t))^3 + t(\sigma(t))^2 + t^2\sigma(t) + t^3 \\ &= \sqrt[4]{(t^4+2)^3} + t^2\sqrt[4]{t^4+2} + t\sqrt{t^4+2} + t^3. \end{aligned}$$

Exercise 1.46. *Let* $\mathbb{T} = \{\sqrt[5]{n+1} : n \in \mathbb{N}_0\}$, $f(t) = t + t^3$, $t \in \mathbb{T}$. *Find* $f^{\Delta}(t)$, $t \in \mathbb{T}^{\kappa}$.

Answer. $1 + \sqrt[5]{(t^5+1)^2} + t\sqrt[5]{t^5+1} + t^2$.

Theorem 1.47. *[1] Assume that* $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^{\kappa}$. *Then*

1. *the sum* $f + g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

2. *for any constant* α, $\alpha f : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

3. *if* $f(t)f(\sigma(t)) \neq 0$, *we have that* $\frac{1}{f} : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *and*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

4. *if* $g(t)g(\sigma(t)) \neq 0$, *we have that* $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

5. *the product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) \\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).\end{aligned}$$

Example 1.48. *Let* $f, g, h : \mathbb{T} \to \mathbb{R}$ *be differentiable at* $t \in \mathbb{T}^{\kappa}$. *Then*

$$\begin{aligned}(fgh)^{\Delta}(t) &= ((fg)h)^{\Delta}(t) \\ &= (fg)^{\Delta}(t)h(t) + (fg)(\sigma(t))h^{\Delta}(t) \\ &= (f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t))h(t) + f^{\sigma}(t)g^{\sigma}(t)h^{\Delta}(t) \\ &= f^{\Delta}(t)g(t)h(t) + f^{\sigma}(t)g^{\Delta}(t)h(t) + f^{\sigma}(t)g^{\sigma}(t)h^{\Delta}(t).\end{aligned}$$

Example 1.49. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be differentiable at* $t \in \mathbb{T}^{\kappa}$. *Then*

$$\begin{aligned}\left(f^2\right)^{\Delta}(t) &= (ff)^{\Delta}(t) \\ &= f^{\Delta}(t)f(t) + f(\sigma(t))f^{\Delta}(t)\end{aligned}$$

$$= f^{\Delta}(t)(f^{\sigma}(t)+f(t)).$$

Also,

$$\begin{aligned}
(f^3)^{\Delta}(t) &= (ff^2)^{\Delta}(t) \\
&= f^{\Delta}(t)f^2(t)+f(\sigma(t))(f^2)^{\Delta}(t) \\
&= f^{\Delta}(t)f^2(t)+f^{\sigma}(t)f^{\Delta}(t)(f^{\sigma}(t)+f(t)) \\
&= f^{\Delta}(t)(f^2(t)+f(t)f^{\sigma}(t)+(f^{\sigma})^2(t)).
\end{aligned}$$

We assume that

$$(f^n)^{\Delta}(t) = f^{\Delta}(t)\sum_{k=0}^{n-1} f^k(t)(f^{\sigma})^{n-1-k}(t)$$

for some $n \in \mathbb{N}$.

We will prove that

$$(f^{n+1})^{\Delta}(t) = f^{\Delta}(t)\sum_{k=0}^{n} f^k(t)(f^{\sigma})^{n-k}(t).$$

Really,

$$\begin{aligned}
(f^{n+1})^{\Delta}(t) &= (ff^n)^{\Delta}(t) \\
&= f^{\Delta}(t)f^n(t)+f^{\sigma}(t)(f^n)^{\Delta}(t) \\
&= f^{\Delta}(t)f^n(t)+f^{\Delta}(t)(f^{n-1}(t)+f^{n-2}(t)f^{\sigma}(t) \\
&\quad +\cdots+f(t)(f^{\sigma})^{n-2}(t)+(f^{\sigma})^{n-1}(t))f^{\sigma}(t) \\
&= f^{\Delta}(t)\Big(f^n(t)+f^{n-1}(t)f^{\sigma}(t)+f^{n-2}(t)(f^{\sigma})^2(t) \\
&\quad +\cdots+(f^{\sigma})^n(t)\Big) \\
&= f^{\Delta}(t)\sum_{k=0}^{n} f^k(t)(f^{\sigma})^{n-k}(t).
\end{aligned}$$