Principles of Hydroacoustics

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Cambridge Scholars Publishing

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This book first published 2024

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-0364-0453-6
ISBN (13): 978-1-0364-0453-6

CONTENTS

PREFACE

The textbook “Principles of Hydroacoustics” is intended for graduate students, postgraduates and research related to hydroacoustics by the nature of their scientific activates or studies, match attention is paid in the textbook to the reflection and scattering of sound by bodies in a liquid medium. The author considers them as scatterers of a simple analytical form (sphere, infinite cylinders of elliptical and circular sections, prolate and oblate spheroids), both and non-analytical shape (finite cylinder with hemispheres at the ends). The textbook provides correlation functions of reflected signals, first obtained by the author of the textbook for randomly oriented bodies. When calculating diffraction characteristics both harmonic and pulsed signals were used.

A separate Chapter of the textbook is devoted to the study of reflected signals for bodies located near interfaces between media, in an underwater sound channel or a plane waveguide with an isotropic elastic bottom. In the third Chapter of the textbook such modern methods of the theory of sound diffraction are studied as a method of integral equations, finite element method and a boundary element method. In the fourth Chapter, we will study the use of equations and diffraction theory methods for problems related to the synthesis of hydroacoustic antennas. The fifth Chapter of the textbook describes the experimental technique in hydroacoustic basin conditions, presents and analyzes the results of a model experiment on low-frequency sonar.

The research into the characteristics of sound scattering by spheroidal bodies was performed by S. A. Bespalova, E. Sedola, and A. P. Eglaya, under the direction of the Doctor of Physical and Mathematical Sciences, Yu. A. Klokov.

The calculations for the phase velocities in cylindrical bodies were carried out by S. L. Il’menkov and K. A. Klyubina. E. I. Kuznetsova performed the calculations for diffracted and radiated pulses.

B. Ivanov and M. Moshchuk actively participated in the experiment to find the amplitude-phase characteristics of the scattering field of models in the Fresnel zone.

The author thanks V. I. Gromova, M. A. Kleshchev, S. M. Kleshchev, A. S. Klimenkov, and M. M. Pavlov. I would also like to thank the CSP team, including Helen Edwards, Alison Duffy, Adam Rummens, Sophie Edminson, Courtney Dixon, Amanda Millar, and Joanne Parsons, for editing, proofreading, and publishing of my textbook.

CHAPTER ONE

DIFFRACTION OF STATIONARY AND NON-STATIONARY SOUND IN A LIQUID BY BODIES OF VARIOUS FORMS

1.1. Diffraction Amplitude – Phase Characteristics of Sound Scattering by Bodies in Liquid

In this section, the resonances of prolate and oblate spheroidal bodies are investigated. Hydroacoustics, as a rule, studies the distant scattered fields of various bodies. Pressure in a scattered wave p_s (scattered pressure) of the circular frequency ω in the Fraunhofer zone (far field) in an unbounded liquid for the final body can be represented in the form:

$$p_s(r,\theta,\varphi) = (A/r)D(\theta,\varphi)exp[-i(\omega t - kr)], \quad (1.1)$$

where r, θ, φ represent spherical coordinates of the observation point; $D(\theta; \varphi)$ represents the angular characteristic of the scatterer; $k=2\pi/\lambda$ represents the number of waves in a liquid; λ represents the length of a sound wave in a liquid medium; and A represents the amplitude factor. In other words, it is possible to separate the characteristics of a scattering body, the wave size of the scatterer, and its shape, material, and orientation relative to the source and the observer, thereby determining $D(\theta; \varphi) = 1$ the characteristics of the source (amplitude A of the wave created at the location of the scatterer) and the distance r to the observation point.

We assume that the distance r from the source (the irradiating antenna) to the scatterer is such that, in the incident wave, the pressure p_i obeys a relation of the type (1.1):

$$p_i(r_1;\theta_1;\varphi_1)=(B/r_1)D(\theta_1;\varphi_1)exp[-i(\omega t-k r_1)], \qquad (1.2)$$

where r, θ, and φ represent the scatter's coordinates in a system of the source's spherical coordinates; $D(\theta;\varphi)$ represents the angular characteristic of a radiation system; B represents the amplitude of the pressure of the wave created by the source at a distance of 1m from $A=B/r$.

For the intensity I_s of the reflected signal at the observation point, we can write the following:

$$I_s=\left(A^2|D(\theta;\varphi)|^2\right)/\left(r^2 2\rho_0 c\right)=K\left(I_0/r^2\right), \qquad (1.3)$$

where $K=|D(\theta;\varphi)|^2; I_0=A^2/2\rho_0 c$ represents the intensity of the wave incident on the reflecting object (the source is non-directed $D(\theta;\varphi)=1$).

As a rule, the distance from the source to the scatterer is such that the wave incident on the body, at least within its geometric dimensions, can be considered to be flat.

For the logarithm (1.3), we obtain the following:

$$10\lg I_s=10\lg K+10\lg I_0-20\lg r. \qquad (1.4)$$

In real conditions, the intensity I_s of the reflected wave decreases more rapidly than r^2 does, due to the sound attenuation in the marine environment. Therefore, the total attenuation due to the spacing of the front of the spherical wave and the attenuation is denoted by *2H*.

The quantity, *10lgK*, is called the strength of the target. To calculate it, one can write that it proceeds from (1.4):

$$T = E - S + 2H, \tag{1.5}$$

where $E = 10\,lg\,I_s$; $S = 10\,lg\,I_0$.

However, since $K = |D(\theta;\varphi)|^2$, then:

$$T = 10\,lg\,K = 20\,lg|D(\theta;\varphi)|. \tag{1.6}$$

For an ideally reflective sphere with a radius *R* and large wave dimensions *kR* we can, with a good approximation, use θ and φ for all angles except for the shadow direction:

$$|D(\theta;\varphi)| = const = R/2. \tag{1.7}$$

But for a scatterer with arbitrary parameters, we need to introduce the concept of the equivalent radius $\mathrm{Re}\,q$ in accordance with the formula:

$$\mathrm{Re}\,q = 2|D(\theta;\varphi)|. \tag{1.8}$$

When the radiating antenna is the same as the receiving antenna, the system is called a combined (or single-position). We can only measure one value of the scattered pressure corresponding to the coordinates of the combined system. Since the total diagram of the scattering of the sound is not considered and is usually unknown, it is convenient to assume that the obstacle acts as an isotropic scatterer which creates, in all directions, the same scattered pressure as in the combined system. Under this assumption, one can introduce a new characteristic for the reflectivity of bodies. This is a relative backscattering cross-section σ_0, which is the ratio of the total power P_s dissipated by the fictitious isotropic scatterer to the power P_i incident on the reflecting object of the wave from the source:

$$\begin{aligned}\sigma_0 &= Ps/P_i = \lim_{r\to\infty}\left[\left(4\pi r^2/A_0\right)I_s/I_0\right]= \\ &= \lim_{r\to\infty}\left[\left(4\pi r^2/A_0\right)\left|p_s\right|^2 2\rho_0 c/2\rho_0 c\left|p_i\right|^2\right]= \\ &= \lim_{r\to\infty}\left[4\pi r^2 A^2\left|D(\theta;\varphi)\right|^2/A_0 r^2 A^2\right]=\left(4\pi/A_0\right)\left|D(\theta;\varphi)\right|^2,\end{aligned} \tag{1.9}$$

where A_0 represents the area of the geometrical shadow of a scatterer (its projection on a plane of a wave front) for a given direction of incidence of the wave from a source.

In so-called spaced (two-position) systems, a receiving antenna is located in an arbitrary direction with respect to a radiating antenna. For such a system, we introduce the concept of a two-position scattering cross-section, which is determined by the said formula (1.9). In this case, I_s represents the intensity of a sound in the direction of a receiving antenna. The total scattering cross-section σ can be calculated only based on the known angular characteristic $D(\theta;\varphi)$ of a reflecting object. It is defined as the ratio of a total scattering power P_s in a solid angle 4π in relation to the intensity I_0 of a wave irradiating on an obstacle from a source.

$$\sigma = P_s / I_0 =$$
$$= \lim_{r\to\infty}\left[\left(A^2\, 2\rho_0 c / r^2\, 2\rho_0 c A^2\right) \iint_S \left|D(\theta;\varphi)\right|^2 dS \right] = \qquad (1.10)$$
$$= \int_0^{2\pi}\int_0^{\pi} \left|D(\theta;\varphi)\right|^2 \sin\theta\, d\theta\, d\varphi ,$$

where $dS = r^2 \sin\theta\, d\theta\, d\varphi$ represents the surface element of a sphere of radius *r*.

The relative cross-section of the scattering σ_r is expressed in terms of a total cross-section σ and an area of the geometric shadow A_0: $\sigma_r = \sigma / 2A_0$.

With the help of formulas (1.6), (1.8), and (1.9), one can establish relations between the named characteristics of the reflectivity of scatterers:

$$\left.\begin{array}{ll} \sigma_0 = \pi \operatorname{Re} q / A_0 ; & \operatorname{Re} q = 2\left|D(\theta;\varphi)\right| ; \\ T = 20 \lg\left(\operatorname{Re} q / 2\right); & T = 10 \lg\left(\sigma_0 A_0 / 4\pi\right); \\ T = 20 \lg\left|D(\theta;\varphi)\right| . & \end{array}\right\} \qquad (1.11)$$

To calculate the scattered field and the characteristics of a sound reflection by bodies in the liquid, an important role is played by the boundary

conditions on their surface. We can formulate them in connection with the following three-dimensional problem.

On the boundary ξ_0 of a liquid body with a vacuum (absolutely soft medium) or an elastic body with a vacuum, an homogeneous Dirichlet condition must be satisfied:

1. For a liquid body

$$\Phi_0\big|_{\xi_0} = 0, \qquad (1.12)$$

where Φ_0 represents the potential of a displacement in a liquid medium

a) For an elastic body

$$\sigma_n\big|_{\xi_0} = 0; \qquad \tau_{n\alpha}\big|_{\xi_0} = \tau_{n\beta}\big|_{\xi_0} = 0, \qquad (1.13)$$

here σ_n and $\tau_{n\alpha}$, $\tau_{n\beta}$ represent normal and tangential stresses within an elastic body.

2. On the boundary of a liquid body with an absolutely hard medium or elastic bodies with an absolutely hard medium, the homogeneous Neumann condition must be satisfied:

a) for a liquid body

$$(\partial\Phi_0/\partial n)\big|_{\xi_0} = 0,$$

(1.14)

where n represents a normal border;

b) for an elastic body

$$\vec{u}\big|_{\xi_0} = 0, \qquad (1.15)$$

where $\vec{u}$ represents the displacement vector of elastic body particles.

3. At an interface between an elastic body and a medium (rigidly connected/fused), the continuity of normal and tangential stresses and displacements must be observed.

4. At the liquid-elastic interface, normal displacements are continuous; normal stresses in an elastic body are equal to the boundary pressures in a liquid medium; and tangential stresses in an elastic body at the boundary with a liquid are absent.

5. At the interface between two liquid media pressures, where normal displacements are continuo-us.

In most books and articles on wave processes, the problems presented are generally concerned with the diffraction of electromagnetic waves rather than acoustic waves. Therefore, it is necessary to be able to interpret the results of the solutions to electromagnetic problems with respect to acoustic waves. Methods for the mathematical description of electromagnetic and acoustic waves are, formally, very close, as both are subject to the wave equation. Therefore, many, although by no means all, of the results obtained for electromagnetic waves are valid for sound waves.

1.2.1. Characteristics of Ideal and Elastic Spherical Scatterers

An ideally reflecting sphere of radius r_0 is irradiated by a plane harmonic wave from a source placed at infinite (see Fig. 1-1). Scalar potential $\Phi_i(r;\theta;\varphi) = exp[-i(\omega t - \vec{k}\vec{r})$ of a plane mo-nochromatic wave of unit amplitude can be written in a spherical coordinate system:

$\Phi_i(r;\theta;\varphi) = exp(i\vec{k}\vec{r})\,exp(-i\omega t) = exp[ikr\,cos(\pi-\theta)]\,exp(-i\omega t) = exp(-ikr\,cos\,\theta)\,exp(-i\omega t)$. (1.16)

Due to axisymmetric nature of the problem, there is no dependence on the second (azimuthal) coordinate. We will omit the time multiplier $exp(-i\omega t)$ to shorten the record in what follovs.

Let us expand the potential Φ_i in terms of the eigenfunctions of the Helmholtz equation in a spherical coordinate system:

$\Phi_i(r;\theta) = exp(-ikr\,cos\,\theta) = \sum_{m=0}^{\infty} A_m P_m(cos\,\theta)$, (1.17)

Where $P_m(cos\,\theta)$ is the Legendre polynomial, which in the axisymmetric case is a solution in the second order separated differential equation for the coordinate θ.

Legendre polynomials are orthogonal to each other in the range of angles θ from 0^0 to 180^0, i.e.

$\int_{-1}^{+1} P_n\,cos(\theta) P_m(cos\,\theta) d(cos\,\theta) =$ [0 by $m \neq n; 2/(2m+1)$ by m=n] .
(1.18)

Using the orthogonality of the Legendre polynomials, we find the unknown coefficients A_m of the expansion (1.17):

$A_m = [(2m+1)/2] \int_{-1}^{+1} P_m(cos\,\theta)\,exp(-ikr\,cos\,\theta) d(cos\,\theta) = (2m+1) i^{-m} j_m(kr)$.
(1.19)

The final expression for potential Φ_i will have the form:

$\Phi_i(r;\theta) = exp(-ikr\,cos\,\theta) = \sum_{m=0}^{\infty} i^{-m}\,(2m+1) P_m(cos\,\theta) j_m(kr)$.

(1.20)

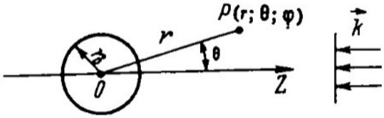

Figure 1-1: Ideally reflecting sphere in the field of a plane sound wave

Let us represent the scattered wave potential Φ_s as a series of Hankel eigenfunctions 1st kind $h_m^{(1)}(kr)$ satisfying the radiation condition (Sommerfeld condition):

$$\Phi_s(r;\theta) = \sum_{m=0}^{\infty} a_m P_m(\cos\theta) h_m^{(1)}(kr), \tag{1.21}$$

where a_m- are the unknown expansion coefficients, which are found from the boundary condition on the surface of the sphere.

Substituting (1.20) and (1.21) into the homogeneous Dirichlet and Neumann boundary conditions and using the orthogonality of the Legendre polynomials, we find the unknown expansion coefficients a_m

$$a_m = i^{-m}(2m+1)\Omega j_m(kr_0)/\Omega h_m^{(1)}(kr_0), \tag{1.22}$$

where $\Omega = 1 -$ for an ideally soft sphere and $\Omega = \partial/\partial r|r = r_0$- for an ideally hard sphere.

The final expression for the potential Φ_s of the scattered wave a taking into account (1.22) will be:

$$\Phi_s(r;\theta) = -\sum_{m=0}^{\infty} i^{-m}\,(2m+1)P_m(\cos\theta)h_m^{(1)}(kr)\Omega j_m(kr_0)/\Omega h_m^{(1)}(kr_0). \quad (1.23)$$

In the far field (Fraunhofer zone) potential $\Phi_s(r;\theta)$ can be represented (taking into account the radiation condition):

$$\Phi_s(r;\theta) = D(\theta)\,exp(\,ikr)/r, \quad (1.24)$$

where $D(\theta) -$ angular characyeristic of sound scattering by sphere.

Hankel function $h_m^{(1)}(kr)$ for large kr (in the Fraunhofer *zo*ne), in turn, has an asymptotic representation:

$$h_m^{(1)}(kr) \approx i^{-m-1}\,exp(\,ikr)/kr \text{ by } kr \to \infty. \quad (1.25)$$

Substituting (1.25) into (1.23) and using (1.24), we obtain the following expression for the angular scattering characteristic $D(\theta)$

$$D(\theta) = (i/k)\sum_{m=0}^{\infty}(-1)^m\,(2m+1)P_m(\cos\theta)\Omega j_m(kr_0)/h_m^{(1)}(kr_0). \quad (1.26)$$

Modules of angular characteristic of a hard sphere for four wave sizes kr_0 *a*re presented in Figure 1-2.

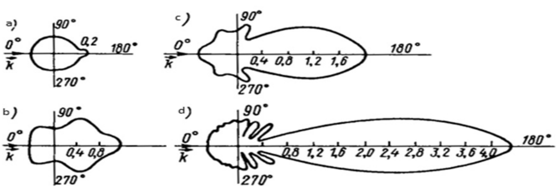

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Figure 1-2: Modules of angular characteristics of an absolutely hard spheres: *a* - kr_0= 1,0; *b* - kr_0= 3; *c* - kr_0= 5; *d* - kr_0= 10..

Let us turn to an elastic isotropic scatterer in the form of a solid sphere. We will assumg that for such a sphere of radius r_0 with Lame coefficients λ and μ and density ρ, placed to an ideal liquid medium with density ρ_0 and velocity of sound c_0, a plane monochromatic wave of unit amplitude fales along its polar axis (see Fig. 1-1).

The displacement vector in the elastic sphere obeys the Lame equation and the scalar Φ and vector $\vec{\Psi}$ potentials - to the scalar and vector Helmholtz equations respectively. Due to the axial symmetry of the

problem, the vector potential $\vec{\Psi}$ has one, different from zero component Ψ_φ and the vector equation transforms into the scalar Helmholtz equation for the function Ψ_φ:

$$\Delta\Psi_\varphi + k_t^2\Psi_\varphi = 0. \qquad (1.27)$$

Potentials Φ and $\Psi = \Psi_\varphi$ are expanded into series in terms of eigenfunctions Helmholtz equation in spherical coordinate system:

$$\Phi(r;\theta) = \sum_{n=0}^{\infty} b_n\, j_n(k_l r) P_n(\cos\theta); \qquad (1.28)$$

$$\Psi(r;\theta) = \sum_{n=1}^{\infty} c_n j_n\,(k_t r) P_n^{(1)}(\cos\theta), \qquad (1.29)$$

where k_l and k_t - are the wave numbers of longitudinal and transverse waves respectively in the sphere material, b_n and c_n – are unknown expansion coefficients. The potentials of the incident and scattered waves, as before, are taken in the form expansions (1.20) and (1.21).

Boundary conditions for finding unknown coefficients are:

1) continuity of normal displacements at the boundary of the liquid and elastic sphere;
2) equality of normal stress in the sphere to diffracted pressure p_Σ in a liquid mediumв;
3) absence of tangential stresses in this boundary.

We will be written boundary condition in the form:

$$-(\partial\Phi/\partial r)+r^{-1}ctg\theta(\Psi)+r^{-1}(\partial\Psi/\partial\theta)=-(\partial/\partial r)(\Phi_i+\Phi_s)|r=r_0; \quad (1.30)$$

$$\lambda k_l^2\Phi+2\mu[-(\partial^2\Phi/\partial r^2)+r^{-1}ctg\theta(\partial\Phi/\partial r)-r^{-2}ctg\theta(\Psi)+$$
$$+r^{-1}(\partial^2\Psi/\partial\theta\partial r)-r^{-2}(\partial\Psi/\partial\theta)=\rho_0\omega^2(\Phi_i+\Phi_s)|r=r_0; \quad (1.31)$$

$$\mu[r^{-2}ctg\theta(\partial\Phi/\partial\theta)+r^{-2}(\partial^2\Psi/\partial\theta^2)-(\partial^2\Psi/\partial r^2)-$$
$$2r^{-2}(\partial\Phi/\partial\theta)-r^{-2}(\sin\theta)^{-2}\Psi]|r=r_0=0. \quad (1.32)$$

1.2.2. The Green's Functions Method

Initially, the Green's function method was used to solve the problem of the sound scattering from ideal scatterers in mixed boundary conditions. It was later applied to the sound diffraction studies of ideal and elastic bodies of a non-analytical form. The first part of section 3 sets out a detailed illustration for the use of the Green's function method to solve diffraction problems regarding simple bodies (sphere; spheroid) in mixed boundary conditions. Analytical solutions are complemented by the results of calculations of similar bodies in the Fresnel and Fraunhofer zones of the scattered sound field. In future, the Green's function method will be extended to ideal and elastic scatterers of a non-analytical form.

Ideal scatterers, ones that have—on different parts of the surface area—dissimilar boundary conditions that are not the same (Dirichlet or Neumann conditions) refer to bodies with mixed boundary conditions. Sound diffraction problems regarding such scatterers are solved by means of one or two methods. The first method, which was proposed by A. Sommerfeld is called the variational method (or the method of least squares). The second method, the Green's function method, is based on the use of the corresponding Green's function for each part of the surface of the scatterer. We will now look at the use and characteristics of both methods at an example of a sphere, with radius *R*, in mixed boundary conditions (one half of a sphere is ideally soft; another is ideally hard). On the surface area of a sphere $S_1\left(\theta = 0 \div 90^\circ\right)$, the Dirichlet condition is perfomed and in the area $S_2\left(\theta = 90 \div 180^\circ\right)$, the Neumann condition is perfomed (see Fig. 1-3). In accordance with the given boundary conditions, and by using the variational method, a functional G^N of the following form is made:

$$G^N = k^2 \int_{S_1} \left| \Phi_i + \Phi_s \right|^2 dS + \int_{S_2} \left| \frac{\partial \Phi_i}{\partial n} + \frac{\partial \Phi_s}{\partial n} \right|^2 dS . \qquad (1.33)$$

where *k* represents the wave number of the incident plane wave.

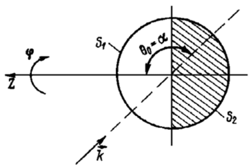

Figure 1-3: The sphere with mixed boundary conditions

In general, ($\theta_0 = \alpha \neq 0^\circ$ or $\theta_0 = \alpha \neq 180^\circ$) the problem is three-dimensional and the potentials of incident (Φ_i) and scattered (Φ_s) waves are found in the following form:

$$\Phi_i(r;\theta;\varphi) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} i^{-n}(2n+1)\varepsilon_m[(n-m)!/(n+m)!]\times \quad (1.34)$$
$$\times P_n^m(\cos\alpha)P_n^m(\cos\theta)\,j_n(kr)\cos m\varphi\,;$$

$$\Phi_s(r;\theta;\varphi) = \sum_{v=0}^{N}\sum_{q=0}^{v} A_q^v P_q^v(\cos\theta)h_q^{(1)}(kr)\cos v\varphi\,, \quad (1.35)$$

where A_q^v represents the unknown coefficients of expansions,

$$\varepsilon_m = \begin{cases} 1, & m = 0; \\ 2, & m \neq 0\,. \end{cases}$$

In expanded form, a functional G^N is:

$$G^N = k^2R^2\times$$
$$\times\int_0^{2\pi}\int_0^{\pi/2}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{m} i^{-n}(2n+1)\varepsilon_m P_n^m(\cos\alpha)[(n-m)!/(n+m)!]P_n^m(\cos\theta)\,j_n(kR)\times\right.$$
$$\left.\times\cos m\varphi + \sum_{v=0}^{N}\sum_{q=0}^{v} A_q^v P_q^v(\cos\theta)h_q^{(1)}(kr)\cos v\varphi\right\}\times$$
$$\times\left\{\sum_{m_1=0}^{\infty}\sum_{n_1=0}^{m_1} i^{n_1}(2n_1+1)\varepsilon_{m_1}[(n_1-m_1)!/(n_1+m_1)!]P_{n_1}^{m_1}(\cos\alpha)P_{n_1}^{m_1}(\cos\theta)\,j_{n_1}(kR)\times\right.$$
$$\left.\times\cos m_1\varphi + \sum_{v_1=0}^{N}\sum_{q_1=0}^{v_1} \overline{A}_{q_1}^{v_1} P_{q_1}^{v_1}(\cos\theta)h_{q_1}^{(2)}(kR)\cos v_1\varphi\right\}\sin\theta\,d\theta\,d\varphi +$$
$$+R^2\int_0^{2\pi}\int_{\pi/2}^{\pi}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{m} i^{-n}(2n+1)\varepsilon_m P_n^m(\cos\alpha)[(n-m)!/(n+m)!]P_n^m(\cos\theta)\,j_n'(kR)k\times\right.$$
$$\left.\times\cos m\varphi + \sum_{v=0}^{N}\sum_{q=0}^{v} A_q^v P_q^v(\cos\theta)h_q^{(1)'}(kR)k\cos v\varphi\right\}\times$$
$$\times\left\{\sum_{m_1=0}^{\infty}\sum_{n_1=0}^{m_1} i^{n_1}(2n_1+1)\varepsilon_{m_1}[(n_1-m_1)!/(n_1+m_1)!]P_{n_1}^{m_1}(\cos\alpha)P_{n_1}^{m_1}(\cos\theta)\,j_{n_1}'(kR)k\times\right. \quad (1.36)$$
$$\left.\times\cos m_1\varphi + \sum_{v_1=0}^{N}\sum_{q_1=0}^{v_1} \overline{A}_{q_1}^{v_1} P_{q_1}^{v_1}(\cos\theta)h_{q_1}^{(2)'}(kR)k\cos v_1\varphi\right\}\sin\theta\,d\theta\,d\varphi\,,$$

where a line written over unknown coefficients is a sign of a complex conjugation.

The minimization condition of a functional G^N ensures the best satisfaction of boundary conditions on the surface of a scatterer:

$$\partial G^N / \partial \overline{A}_i^i = 0. \qquad (1.37)$$

Substituting (1.36) in (1.37), we obtain equations for the determination of unknown coefficients A_q^v:

$$\sum_{v=0}^{N} \sum_{q=0}^{M} A_q^v C_{qq_1}^{vv} = -\sum_{n=0}^{N} \sum_{q=0}^{M} d_{nq_1}^{vv} ,$$

where M represents the integer index whose value depends on the size of the wave kR;

$$d_{nq_1}^{vv} = 2i^n (2n+1)[(n-v)!/(n+v)!] P_n^v(\cos\alpha) \times$$
$$\times j_n(kR) h_{q_1}^{(2)}(kR) \int_0^{\pi/2} P_n^v(\cos\theta) P_{q_1}^v(\cos\theta) \sin\theta \, d\theta +$$
$$+ j_n'(kR) h_{q_1}^{(2)'}(kR) \int_{\pi/2}^{\pi} P_n^v(\cos\theta) P_{q_1}^v(\cos\theta) \sin\theta \, d\theta ;$$

$$\Delta v = \begin{cases} 2, & v = 0 ; \\ 1, & v \neq 0 , \end{cases}$$

In accordance with the Green's function method, the potential of a scattered wave Φ_s from a sphere with mixed boundary conditions can be represented by a one-term Huygens integral as follows:

$$\Phi_s(P)=\Phi_s(r;\theta;\varphi)= \\ =(1/4\pi)\left\{-\int_{S_1}\Phi_i(Q)\left[\partial G_1(P,Q)/\partial r'\right]dS_1+\right. \\ \left.+\int_{S_2}\left[\partial\Phi_i(Q)/\partial r'\right]G_2(P,Q)\,dS_2\right\}, \qquad (1.38)$$

where P represents the point observation, with the spherical coordinates r, θ, φ; Q represents the surface point with angular coordinates φ'', θ'' and a radial coordinate $r'=R$; G_1 represents the Green's function which vanishes on the surface of a scatterer; and G_2 represents the Green's function which has a zero derivative along the normal of this surface:

$$G_1(r;\theta;\varphi;r';\theta';\varphi')=ik\sum_{m=0}^{\infty}\sum_{n=0}^{m}\varepsilon_m(2n+1)P_n^m(\cos\theta')\times \\ \times P_n^m(\cos\theta)[(n-m)!/(n+m)!]\cos[m(\varphi-\varphi')]\times \\ \times[j_n(kr')h_n^{(1)}(kr)-h_n^{(1)}(kr')(kr)\,j_n(kR)/h_n^{(1)}(kR)]; \qquad (1.39)$$

$$G_2(r;\theta;\varphi;r';\theta';\varphi')=ik\sum_{m=0}^{\infty}\sum_{n=0}^{m}\varepsilon_m(2n+1)P_n^m(\cos\theta'')\times \\ \times P_n^m(\cos\theta)[(n-m)!/(n+m)!]\cos[m(\varphi-\varphi')]\times \\ \times\left[j_n(kr')h_n^{(1)}(kr)-h_n^{(1)}(kr')h_n^{(1)}(kr)\,j_n'(kR)/h_n^{(1)'}(kR)\right]. \qquad (1.40)$$

The formula (1.38) for the potential $\Phi_s(r;\theta;\varphi)$ of a scattered wave is approximated as a formula (1.36) of the variational method. However, there are special cases in which the Green's function method method can be used because it gives accurate results. We will consider these special cases using the example of the homogeneous (soft) sphere, visualizing it being broken into two halves by a plane XOZ (see Fig. 1-3). The wave vector $\vec{k}$ of the incident plane wave is put in a plane XOZ ($\theta_0=90^\circ$) in the same as the observation point P (on a contour of the border of hemispheres S_1 and S_2). We will find $\Phi_s(P)$ in this point by using Green's function G_1 for the left hemisphere (S_1) and using Green's function G_2 for the right hemisphere (S_2).

For a homogeneous soft sphere, formula (1.38) is converted to the following form:

$$\Phi_s(P) = -(1/4\pi)\left\{\int_{S_1}\Phi_i(Q)[\partial G_1(P,Q)/\partial r']dS_1 + \int_{S_2}[\partial\Phi_s(Q)/\partial r']G_2(P,Q)dS_2\right\},$$

(1.41)

where

$$\Phi_s(Q) = -\sum_{m=0}^{\infty}\sum_{n=0}^{m} i^{-n}(2n+1)\varepsilon_m\left[(n-m)!/(n+m)!\right]P_n^m(\cos\alpha)P_n^m(\cos\theta')\cos m\phi' h_n^{(1)}(kr')\times$$
$$\times j_n'(kR)/h_n^{(1)}(kR).$$

Using $\Phi_s(Q)$ and G_1, G_2 from (1.39) and (1.40), we find that the potential of a scattered wave on the surface of a sphere at a point where the contour of the border between the two hemispheres is equal:

$$\Phi_s(P) = -(1/2)\Phi_i(P) - (1/2)\Phi_i(P) = -\Phi_i(P),$$

That is, a boundary condition is fulfilled and the solution is accurate.

If a sphere, θ_0 and $\theta = 90^o$, consists of soft and hard hemispheres (see Fig. 1-3), the contribution of the ideal soft hemisphere in the potential Φ_s at the point of a contour of the border is equal to $\Phi_i(P)/2$, but the contribution of the ideally hard hemisphere $\partial\Phi_s/\partial r$ to a contour of the border is equal to $2^{-1}(\partial\Phi_1(P)/\partial r)$. The potential Φ_s in plane XOZ will be equal to half a sum of the potentials generated by the soft and hard spheres in the same plane.

The arbitrary orientation of a wave vector $\vec{k}$ of the incident wave with respect to our sphere with mixed boundary conditions the potential of a scattered wave $\Phi_s(r;\theta;\varphi)$ will equal approximatly by substituting (1.34) in (1.39) and (1.40) in (1.38):

$$\begin{aligned}
&\Phi_s(r;\theta;\varphi)= \\
&=(1/2)\sum_{m=0}^{\infty}\sum_{n=0}^{m} i^{-n}(2n+1)[(n-m)!/(n+m)!]\varepsilon_m P_n^m(\cos\theta)\cos m\varphi P_n^m(\cos\alpha)\times \\
&\times h_n^{(1)}(kr)\{[j_n(kR)/h_n^{(1)}(kR)]+[j_n'(kR)/h_n^{(1)'}(kR)]\}+(1/2)\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{n_1=0}^{m}(-1)^m\varepsilon_m i^{-n_1}(2n+1)\times \\
&\times(2n_1+1)[(n_1-m)!/(n_1+m)!][(n-m)!/(n+m)!]\cos m\varphi P_n^m(\cos\theta)P_n^m(\cos\alpha)h_n^{(1)}(kr)\times \\
&\times\{[P_{n_1}^{m'}(0)P_n^m(0)-P_n^{m'}(0)P_{n_1}^m(0)]/[n_1(n_1+1)-n(n+1)]\}\times \\
&\times\{[-j_{n_1}(kR)/h_n^{(1)}(kR)]+[j_{n_1}'(kR)/h_n^{(1)'}(kR)]\},
\end{aligned} \tag{1.42}$$

where $n\neq n_1$, $n-n_1$ odd.

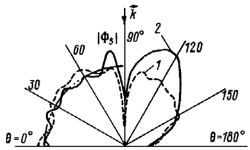

Figure 1-4: The distribution of the module of the scattered wave on the surface of the sphere with mixed boundary conditions: 1, the variational method; 2, the Green's functions method.

In contrast to the variational method, the Green's functions method does not require looking for unknown coefficients of expansions and the computation of the potential Φ_s is a very simple problem, since all quantities in (1.80) are known. Figure 1-4 shows distributions of $|\Phi_s|$ on the surface of a sphere (a half-soft S_1, and a half-hard S_2) by $kR = 5$ and $\alpha = 90^o$. The ideally soft half of a sphere corresponds to the change of angle θ within a range of $0°$ to $90°$. The single value of the potential of a module is shown as a dash-dotted arc; it complies with the strict implementation of a boundary condition on the surface of a sphere a condition, which is approximately satisfied, although the difference is small between the methods themselves.

Figures 1-5 and 1-6 show modules with angular characteristics of soft (see Fig. 1-5, curve 1) and hard (see Fig. 1-6, curve 3) spheroids and the angular characteristic of a spheroid with mixed boundary conditions (see Fig. 1-5 and 1-6, curve 2).

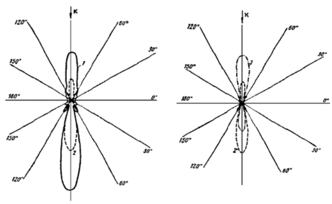

Figure 1-5: Modules angular characteristics

Figure 1-6. Modules angular characteristics of a soft spheroid (curve 1), a hard spheroid (curve 3) and a spheroid with mixed boundary conditions (curve 2).

If, for example, it is decided to introduce a prolate spheroid consisting of two identical halves making contact with each other in a plane ($\eta = 0 \Leftrightarrow \theta = 90^\circ$,) and if it is also decided to place a sound source in the same plane ($\eta_0 = 0$) then the total potential Φ_s of a scattered field for an observation point located in the same plane (($\eta = 0$) will be determined by means of expressions (1.43) ($\varphi = 0$) and (1.44) ($\varphi = \pi$)

$$\Phi_s(\xi;\eta;0) = \sum_{n \ge m}^{\infty} \sum_{m=0}^{\infty} i^{-n} \in_m \bar{S}_{m,n}(c,\eta_1) \bar{S}_{m,n}(c,\eta) R_{m,n}^{(3)}(c,\xi) \times \\ \times \left[\frac{R_{m,n}^{(1)}(c,\xi_0)}{R_{m,n}^{(3)}(c,\xi_0)} + \frac{R_{m,n}^{(1)'}(c,\xi_0)}{R_{m,n}^{(3)'}(c,\xi_0)} \right], \tag{1.43}$$

$$\Phi_s(\xi;\eta;\pi) = \sum_{n \ge m}^{\infty} \sum_{m=0}^{\infty} i^{2m-n} \in_m \bar{S}_{m,n}(c,\eta_1) \bar{S}_{m,n}(c,\eta) R_{m,n}^{(3)}(c,\xi) \times \\ \times \left[\frac{R_{m,n}^{(1)}(c,\xi_0)}{R_{m,n}^{(3)}(c,\xi_0)} + \frac{R_{m,n}^{(1)'}(c,\xi_0)}{R_{m,n}^{(3)'}(c,\xi_0)} \right]. \tag{1.44}$$

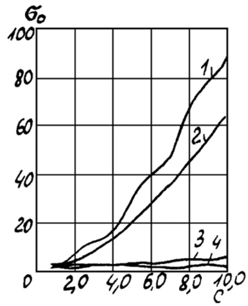

Figure 1-7: Relative backscattering of sound reflected off oblate spheroid cross-sections irradiated along their axis of rotation

Figure 1-7 presents the given values σ_0 of oblate spheroids ($\xi_0 = 0{,}1005$), which are irradiated along the *Z*-axis of rotation ($\theta_0 = 0^\circ$). Curve *1* relates to an ideal hard spheroid, while curve *2* relates to an ideal soft spheroid. Curve *3* relates to a spheroid where *1/3* of the surface corresponds to a Neumann condition and *2/3* of the surface corresponds to a Dirichlet condition. Curve *4* relates to a spheroid where one hemispheroid is hard and the other hemispheroid is soft.

Figure 1-8 shows a module with the angular characteristics $|\psi_s(\eta)|$ of the oblate spheroid with a radial coordinate $\xi_0 = 0{,}1005$. Half of the spheroid is hard, while the other half is soft (see curve *1*). A wave falls along the *Z*-axis ($\theta_0 = 0^\circ \Leftrightarrow \eta_0 = 1{,}0$) and the wave size *C=10*. Figure 1-9 presents the modules $|\psi_s(\eta)|$ for a soft spheroid (see curve *2*) and a hard spheroid (see curve *3*). A comparison of the three curves shows that for a body comprised of combined regions of softness and hardness, the amplitude of the pressure in a wave reflected back from the scatterer is approximately one order of magnitude smaller than for reflections from homogeneous ideal spheroids.

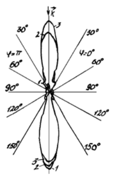

Figure 1-8: A comparison of the modules of the angular scattering characteristics of combined and homogeneous oblate spheroids

1.3.1. Characteristics of Sound Scattering from Infinite Circular Cylinders

We will now turn to the oblique incidence of the plane wave on the elastic hollow cylindrical shell. The geometry of the problem is shown in Figure 1-9.

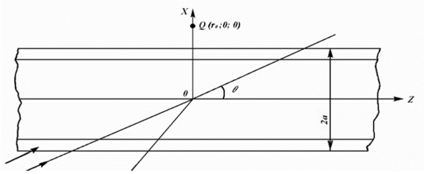

Figure 1-9: Hollow elastic cylindrical shell in the field of the plane sound wave

The scalar potential of an incident wave $\Phi_i(r,\varphi,z)$ for a unit amplitude with wave vector $\vec{k}$ tilted at an angle of θ the axis Z. We can expand this in terms of eigen functions of the scalar Helmholtz equation in a circular cylindrical coordinate system:

$$\Phi_i(r,\varphi,z) = e^{i\gamma z} \sum_{m=0}^{\infty} \varepsilon_m (-i)^m J_m(k_\gamma r) \cos m\varphi, \qquad (1.45)$$

where $\gamma = k\cos\theta;\ k_\gamma = k\sin\theta;\ \varepsilon_m = \begin{cases} 1 \text{ by } m = 0; \\ 2 \text{ by } m \neq 0. \end{cases}$

We can now transform the representation for the vector function, $\vec{A}$ generated by introducing an additional operator *rot* in order that $\vec{A}$ automatically obeys the gauge condition ($div\vec{A} = 0$):