The Resolvent Family for Evolutionary Processes with Memory and Applications

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By

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Contents

Preface

Quite a few things in the world are relevant to time and hence are called evolutionary processes. Among these processes, further development of some is not only related to the current state but also closely linked to their history. This phenomenon is known as the hereditary effect or memory effect. Such an evolutionary process with memory (non-Markovian) can be observed in various systems, including human society where the effects of historical civilization's inheritance or memory are reflected in every aspect of the current society. Additionally memory effects exist in many physical systems, ecologic systems, economic systems etc.

The presence of memory poses significant challenges in understanding, analyzing, modeling, predicting and synthesizing evolutionary processes. For example, when modeling physical systems with smart materials, accurately describing and approximating the process requires incorporating and modeling the memory effect. Investigating issues such as well-posedness, sensitivity to initial conditions and historical functions of differential equations with memory is a difficult task due to mathematical complexity caused by memory. In engineering, designing controllers for evolutionary processes with memory that achieve desired performance have practical significance. Additionally, developing novel intelligent technology using system memory presents a vast challenge in artificial intelligence.

Due to the significant importance and universality of the memory process, this book presents a fundamental investigation of evolutionary processes with memory in general Banach spaces. The purpose is to provide some elementary results for further investigation. The contents mainly include state representation, second-order evolutionary processes, dual processes and intertwining, invariant subspaces, and spectral problems.

As a preliminary investigation, this book primarily focuses on the basic features of linear evolutionary processes with memory. In terms of applications, it includes some results on the series representation of evolutionary families for certain evolutionary processes with memory, as well as the existence of solutions for time-varying memory and semi-linear evolutionary processes with memory.

In this book, there are several aspects which are different from the earlier researches:

(i) The phase space settings (also known as the historical function space) do not adhere to the Phase Space Hypothesis.

(ii) The resolvent family method is used to distinguish the effects of the current state and its historical function in the evolutionary process with memory.

(iii) A dual process for an evolutionary process with memory is defined and examined, which differs from the dual process obtained through the semigroup approach.

(iv) The issues of invariant subspace and spectral issues of linear evolutionary processes with memory are discussed, highlighting differences between eigenvectors and eigenfunctions.

This book is written for students in mathematics and engineering, engineers in control theory, and mathematical workers interested in the application of mathematical physics. Scholars devoted to pure analysis or spectral theory of linear operators will find novel issues to tackle. Readers absorbed in mathematical physics and dynamics will also gain an innovative theory with practical importance.

I am grateful to my former students and collaborators for their work on the stability analysis of specific models, which complements and enhances our theoretical results, although this book does not include stability as a part of its contents. I owe them a debt of gratitude. At the same time, I apologize for any unintentional omission mentioning researchers who have contributed to this aspect of investigation.

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March 2024

Chapter 1 Researches on the Evolutionary Process with Memory Overview

In this chapter, we briefly overview the research progress of the evolutionary process with memory from several aspects: recalling modeling procedure of physical systems with the hereditary effects through differential equation model in finite-dimensional space and partial differential equation models with spacial variable; considering time delay in physical processes and control as well as the Phase Space issue; discussing existing results, methods, and problems. Finally we present the purpose and goal of this book.

§1.1 Hereditary effect and memory process

The hereditary effects or memory effects generally exist in various fields, such as viscoelastic mechanics, nuclear reactors, heat flow, combustion, species interaction, microbiology, epidemiology, physiology, social science and so on. In this section, let us review some classical models in earlier investigation on materials with memory.

Earlier in the second half of the nineteenth century, due to industrial development, extensive studies were conducted on using glass as dielectrics in capacitors. In 1863, F. Kohlrausch proposed that glasses are viscoelastic materials with a, memory effect [72, (F. Kohlrausch,1863)]. In 1866, J. Hopkinson discovered that pure silica glass used as dielectrics in capacitors (Leiden jars) exhibited a discharge law characterized by exponential relaxation over time, represented by the equation $N(t) = \beta e^{-\eta t}$ (e.g., see [57, 58, (John Hopkinson,1876,1877)]). In 1869, J. C. Maxwell introduced a device known as the Maxwell Model to simulate viscoelastic fluids by considering each element of the fluids as a cell containing both a spring and a damper in series. This model described stress through an equation

$$\sigma'(x,t) + \frac{1}{\tau}\sigma(x,t) = \kappa \nabla w(x,t),$$

where $\tau > 0$ and $\kappa > 0$ [77, James Clerk Maxwell. (1867)]. Later on, this model became widely accepted and is now considered as a standard description of viscoelastic materials in engineering. Inspired by Maxwell Model and after certain mathematical treatment, an integro-differential equation was developed to describe systems with viscoelasticity (see [57, 58, (F. Hopkinson,1876,1877)]):

$$w'(t) = \alpha A w(t) + \int_0^t N(t-s) A w(s) ds + h(t) \tag{1.1.1}$$

where $\alpha > 0$ and A is a linear operator in some Banach space, $N(t)$ is a positive real function called kernel function, and $h(t)$ is a source term. Later on when it was realized that most glasses are mixture, the kernel function was extended to more complicated form

$$N(t) = \sum_k \beta_k e^{-\eta_k t}, \quad \beta_k,\ \eta_k > 0.$$

The kernel functions of the above form are usually referred to as **relaxation kernel**, which are called **Prony Sum** in engineering literature. In 1889, M. J. Curie observed that when studying piezoelectricity, the best description for the discharge law of capacitors with certain crystals as dielectrics is $N(t) \sim \frac{M}{t^\gamma}, \gamma \in (0,1)$. For example, please see [18, (Marie Curie,1889)] for more details. This leads to a fractional integral law for the current released by a capacitor with crystalline dielectric and later became known as Curie's Law. During the same time period, L. Boltzmann [10, 11, (Ludwig Boltzmann. 1874,1878)] investigated more general memory in viscoelasticity. However, E. Schmitt in [100, (E. Schmitt,1911)] provided several examples from geometry and number theory that also included memory effect.

Different from the physical systems, however, due to the abundance of models with memory effects in both physical systems and mathematics, E. Picard emphasized the importance of the consideration of hereditary effect from mathematical viewpoint at the International Conference of Mathematicians in 1908 [93, (E. Picard,1908)]. Since the hereditary effect or memory primarily involve time lag and integration of the evolutionary process [100, (E. Schmitt,1911)], most actual problems are modeled using differential equation or integro-differential equation, and more generally by functional differential equations.

During the period from 1909 to 1931, V. Volterra studied the existence of solutions to linear integro-differential equation derived from the viscoelasticity model. For example, please refer to [118, 119, (Vito Volterra,1909,1912)] [120, (Vito Volterra,1928)]. In 1931, he wrote a fundamental book [121, (Vito Volterra,1931)] and explored the role of hereditary effects on species interaction models. During the same period, D. Graffi investigated the memory effect of ferromagnetic materials [41, 42, (Dario Graffi, 1928,1936)].

In 1949, C. Cattaneo introduced a hyperbolic model for heat diffusion with a finite propagation speed in thermodynamics [12, (C. Cattaneo,1949)]:

$$w'' = -\eta w' + \beta \Delta w, \quad \beta > 0, \eta > 0. \tag{1.1.2}$$

This model is known as **Cattaneo Equation** and can be considered as a special case of (1.1.1) with $N(t) = \beta e^{-\eta t}$ and $A = \Delta$ being the Laplace operator. Later, B. D. Coleman, M. E. Gurtin [16, (Bernard D. Coleman,1964)], [17, (Bernard D. Coleman and Morton E. Gurtin,1967)], along with D. De Kee et al. [70, (D. De Kee et al. 2005)], extended (1.1.2) to a more general equation with memory to model heat diffusion processes. It is well-known that solutions of (1.1.2) exhibit a wave front – a discontinuity in the solution's direction of wave propagation – but do not have a backward wave. This forward wave front is observed in solute diffusion within solvents having complex molecular structure. For this reason, both (1.1.2) and its extended version have been applied to model non-Fickian diffusion in biology [15, (RM Christensen,1982)] [67, (D D Joseph and Luigi Preziosi,1989)].

In 1968, M. E. Gurtin and A. C. Pipkin [43, (M E Gurtin and A C Pipkin, 1968)] utilized the Cattaneo equation to describe the diffusion of solutes in polymers as the prototype of the heat equation with memory is given by

$$w'(t) = b \int_{-\infty}^{t} e^{-a(t-s)} \Delta w(s) ds \tag{1.1.3}$$

where $w(t) = w(x,t)$ represents the temperature at time t and position $x \in$

$\Omega \subset \mathbb{R}^n$. Shortly after, W. A. Day in [28, 29, 30, (W. A. Day,1970,1971)] discussed various types of relaxation kernels and their properties for linear viscoelastic materials.

So far, a differential equation model has been obtained, which provides a complete description of evolutionary process with memory. An important characteristic is the hereditary effect, which means that the historical memory as a whole affects the state change. From this perspective, it is completely different from (1.1.1). In actual physical systems, there are many scenarios where persistent memory exist. For example, smart material systems [110, (Ralph Smith,2005)], electromagnetic theory of dielectrics [27, (Paul L Davis, 1975)], porous-viscoelastic materials [1, (Adel M. Al-Mahdi et al. 2021)], heat transfer processes [111, (Renato Spigler, 2020)] [90, L Pandolfi, 2020)] etc, require modeling of persistent memory. Summarizing the modeling of systems with memory, essentially speaking, there are two types of evolutionary processes with memory: diffusion process, whose dynamic behaviour is governed by an abstract evolutionary equation in some Banach space:

$$\begin{cases} \frac{dw(t)}{dt} = Aw(t) + \int_{-\infty}^{0} N(\theta)Aw(t+\theta)d\theta + f(t,w,w_t), & t > 0, \\ w(0) = w_0, & \\ w(\theta) = \varphi(\theta), & \theta < 0. \end{cases} \tag{1.1.4}$$

where $w_t(\theta) = w(t+\theta)$ with $\theta < 0$ and $f(t,w,w_t)$ represents the nonlinear dynamics of systems, and A is a closed and densely defined linear operator. The other is non-Fickian diffusion process, whose dynamic behaviour is governed by

$$\begin{cases} \frac{d^2w(t)}{dt^2} = Aw(t) + \int_{-\infty}^{0} N(\theta)Aw(t+\theta)d\theta + f(t,w,w_t), & t > 0, \\ w(0) = w_0, \quad w'(0) = w_1, & \\ w(\theta) = \varphi(\theta), \quad w.(\theta) = \psi(\theta), & \theta < 0, \end{cases} \tag{1.1.5}$$

where A is a self-adjoint and negative definite operator in Hilbert space. The equation (1.1.4) is now widely applied to describe the diffusion processes in systems with complex molecular structure.

The presence of persistent memory presents several additional challenges in analysis and synthesis of the evolutionary process, as pointed out by Hale and Kato [48, (Jack K. Hale and Jumji Kato,1978)]. These challenges include limitations of available analysis tools, the sensitivity of solutions to initial condition and historical functions, as well as mathematical complexity caused by the lack of boundedness of integral region. Although these models originate from practice problems, issues regarding well-posed-ness and stability are crucial. In this regard, we refer to some earlier results by b Hale [46, 47, 49, (Jack K Hale,1974, 1977, Jack K Hale and Sjoerd M Verduyn Lunel,1993)], Dafermos [19, 20, (Constantine M Dafermos,1970,1971)], Dassios et al. [22, George Dassios and Filareti Zafiropoulos,1990)] and Rivera [95, (Jaime E. Munoz Rivera,1994)] on well-posed-ness and stability. The well-posed-ness and stability of linear evolutionary equations with monotone relaxation kernels have been discussed for specific models, such as the transmission problem of viscoelastic waves [96, (Jaime E. Munoz Rivera et al. 2000)], thermo-elastic equation [123, (Jun Min Wang and Bao Zhu Guo, 2007)], porous elasticity system [36, (Baowei Feng et al. 2018)], Kortewegde Vries-Burgers and Kuramoto-Sivashinsky equations [14, (Boumediene Chentouf and Aissa Guesmia,2022)], and the energy decay rate for viscoelastic-type Timoshenko and laminated beam systems [38, (Claudio Giorgi et al. 2004)] [2, 3, (Adel M. Al-Mahdi,2021)]. Recently, the well-posed-ness and stability of linear memory systems with non-monotone relaxation kernels have been obtained, please refer to the latest literature including references therein by Mu et al (2021) [82, (Rong Sheng Mu and Gen QI Xu, 2021)] [133, (Gen Qi Xu and Min Li, 2021)] [91, (L. Pandolfi, 2021)] [140, 141, 142, (Hai-E Zhang, et al. 2022, 2023)] [134, (Gen Qi Xu, 2023)]. However, when certain nonlinear dynamics are involves in an equation, ill-posed issue may arise; see ill-posed problems for integro-differential equations in mechanics and electromagnetic theory by Bloom (1981) who provided some results on both well-posed-ness as well as ill-posed-ness [9, Frederick Bloom,1981)]. Even linear evolutionary equations with memory may also be ill-posed, please see a model presented in Shang et al.'s work (2011) [103, (Ying Feng Shang et al. 2011)].

Herein we mention some recent works about the memory processes. In the papers [59, 60, 61, (Jordan ristov, 2018, 2019)], Hristov studied the different strain effect in solid and liquids and discussed the functional representation of the viscoelastic material responses. This includes: 1) the stress relaxation function $R(k,t)$ which represents the stress history due to a shear step of size ε, and 2) the creep function $C(t)$ (shear history) due to unit stress σ applied. By applying fading memory concept introduced by [115, Nicholas W. Tschoegl, Christopher A. Tschoegl,1989)], instantaneous responses G_∞ and J_∞ corresponding to equilibrium states (for long times) when the effects of memory terms (convolution integrals) fade out are introduced as follows:

$$\sigma(t) = G_\infty + \int_0^t G(t-\tau)d\varepsilon(\tau)$$

$$\varepsilon(t) = J_\infty - \int_0^t J(t-\tau)d\sigma(\tau).$$

Here $G(t)$ is referred to as linear stress relaxation modulus and $J(t)$ is known as linear creep compliance. Based on different background in physical system, corresponding linear viscoelastic constitutive equations and response functions are derived. Specially for the unknown functions $G(t)$ and $J(t)$ an approximate approach is proposed using a series fractional operators with the Prony kernels. Such a treatment allows for approximating solutions using fractional operator similar to those found in Prony's series. For more relevant works about fractional calculus and Prony's series approximation please refer to [62, 63, 64, 65, (Jordan Hristov, 2022,2023)] along with their references.

§1.2 Delay differential equation and control

Before stating problem, let us call up a definition of the differential-difference equation.

Definition 1.2.1 [7, Definition, p.49. (Richard Bellman and Kenneth L. Cooke,1963)] Let $x(t)$ be a scalar function. Let us consider a differential-difference equation

$$a_0x'(t) + a_1x'(t-h) + b_0x(t) + b_1x(t-h) = f(t),$$

where $h > 0$ is the delay time.

(1) The equation is called a retarded type if $a_0 \neq 0$, $a_1 = 0$ and $b_1 \neq 0$;

(2) The equation is called a neutral type if $a_0 \neq 0$ and $a_1 \neq 0$;

(3) The equation is called an advanced type if $a_0 = 0$, $a_1 \neq 0$ and $b_0 \neq 0$.

A retarded type equation also is called delay differential equation, or hystero-differential equation.

Time delays are natural phenomena and exist extensively in the actual word. So-call time delay means that there exists a time delay in state $x(t-\tau)$. A general form of delay differential equation is

$$\frac{dx(t)}{dt} = g(t, x(t), x(t-\tau_1), x(t-\tau_2), \cdots, x(t-\tau_n)) \tag{1.2.1}$$

where $0 < \tau_1 < \tau_2 < \cdots < \tau_n$ are the delay times, the linear delay differential equation is the simplest one. Such a kind of time delay is also called discrete delay since the variable τ takes its value in a finite set $\{\tau_1, \tau_2, \cdots, \tau_n\}$. If the τ is a continuous variable and taken its value in a finite interval of positive real axis or whole positive real axis, it is called distributed delay (bounded delay or unbounded delay, respectively). In this situation, the delayed state is denoted by $x_t = x(t-\tau)$ where $\tau \in I \subset \mathbb{R}_+$. For the systems with distributed delay, usually whose dynamic behaviour is described as an abstract evolutionary equation in a Banach space

$$\frac{dx(t)}{dt} = Ax(t) + L(x_t) + f(t, x, x_t) \tag{1.2.2}$$

where A is a linear operator, $L(x_t)$ is called delay operator that describes the delay effect on the system, it has a representation

$$L(z) = \int_{-h}^{0} d\eta(s) z(s)$$

where $\eta : [-h, 0] \to \mathscr{B}(\mathbb{X})$ is a bounded variation function [①] and f is a nonlinear operator which is called nonlinear dynamics.

[①]From theory of bounded variation function (cf. for instance, E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer Verlag, Berlin 1969), a bounded variation function η can be split into three parts $\eta = \eta_1 + \eta_2 + \eta_3$, where η_1 is a saltus function of bounded variation with at most countable many jumps on $[-h, 0]$, η_2 is an absolutely continuous function on $[-h, 0]$ and η_3 is either zero or a singular function of bounded variation on $[-h, 0]$, i.e., η_3 is non constant, continuous and have derivative $\eta_3' = 0$ almost everywhere on $[-h, 0]$. In most situations it is sufficient to consider the special case where $\eta_3 = 0$ and η_1 has only a finite number of jumps.

A physical process can have an inherent time delay as it is built, can have delays in input or output. Time-delays emerge frequently in industrial processes, economical [21, (Hippolyte D'albis et al.2014)], physiological and biological systems and mechanical applications, some of them are modeled as partial differential equations, as to the concrete models please see [45, (K. P. Hadeler, 1979. 136-156)] [112, (Jeyaraman Srividhya et al. 2006)] [89, (Silviu-Iulian Niculescu, 2001)] and references therein. In practice engineering, delays are applied to model the physical system, for instance, the transcription of genetic materials [68, (Kresimir Josic et al. 2011)], the cell dynamics [34, (Walid Djema et al. 2018)], the traffic flow systems [109, (Rifat Sipahi et al. 2007)] [79, (Wim Michiels et al. 2009)], coupled oscillators [5, (FM Atay, 2003)], neural networks [40, (K. Gopalsamy and Xue-Zhong He, 1994)], wireless communication networks [97, (O. Roesch and H. Roth,2005)] etc. Such a class equations arose from geometric, physical, engineering and economic sources have been studied extensively, for example, [46, (Jack K. Hale,1974)] [52, (Daniel Henry,1974)] [102, (George Seifert, 1982)] [84, (Shin-ichi Nakagiri,1987)] [98, (Aldo Rustichini,1989)] [56, (Yoshiyuki Hino et al, 1981)] [49, (Jack K. Hale and Sjoerd M. Verduyn Lunel, 1993)] [78, (Cornelis van der Mee,2008)] for general functional differential equations, [24, (Richard Datko,1978)] [69, (Junji Kato,1978))] [99, (Kensuke Sawano, 1979)] [55, (Yoshiyuki Hino, 1983)] for stability and representation of solution [83, (Shin-ichi Nakagiri,1981)]. More relevant problems please see [4, (O Arino et al. 2006)].

In a control system, the purpose of control is to design a control rule so as to suppress disturbance and to realize the desired performance of the controlled system. In engineering, most engineers were well aware of the delay effects occurring in physical systems. According to the position of time delay occurring in the control system, the delays are divided into the following classes: state delay (delay in state, or called interior delay), input delay (control delay) and output delay (measurement or observation delay). After linearization the physic system, the linearization systems have merely one of three-kind delays, which is of the form:

(1) State delay

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + L(x_t) + Bu(t), \quad t > 0 \\ x(0) = x_0 \quad x(s) = \varphi(s), \quad s \in [-h, 0); \end{cases} \tag{1.2.3}$$

(2) Input delay

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + L(u_t), \quad t > 0 \\ x(0) = x_0 \\ u(s) = v(s), \quad s \in [-h, 0]; \end{cases} \tag{1.2.4}$$

(3) Output delay

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = L(x_t), \quad t > 0 \\ y(s) = \psi(s), \; s \in [-h, 0]; \end{cases} \tag{1.2.5}$$

where

1) $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{Y}$ are Banach spaces, $x(t) \in \mathbb{X}$, $u(t) \in L^p_{loc}(\mathbb{R}_+, \mathbb{U})$ and $y(t) \in L^p_{loc}(\mathbb{R}_+, \mathbb{Y})$;

2) $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is an unbounded and linear operator with dense domain. Usually we suppose that A generates a C_0 semigroup on $\mathbb{X}$;

3) The spaces of the form $M([-h, 0], \mathbb{V})$ are called phase space , $L : M([-h, 0], \mathbb{X}) \to \mathbb{X}$ ($M([-h, 0], \mathbb{U}) \to \mathbb{X}$, $M([-h, 0], \mathbb{X}) \to \mathbb{Y}$ respectively) is the delay operator;

4) $x(t) : [-h, \infty) \to \mathbb{X}$ is a $\mathbb{X}$-valued function, which is called a state of the system, and $x_t : [-h, 0] \to \mathbb{X}$ defined by $x_t(s) = x(t + s)$, $s \in [-h, 0)$ is called the delay function (where h can be taken as $+\infty$)). Similar notation for the control function $u(t)$;

5) B is the control operator, $\varphi(s) \in M([-h, 0], \mathbb{X})$, $v(s) \in M([-h, 0], \mathbb{U})$ and $\psi(s) \in M([-h, 0], \mathbb{Y})$ are delayed functions.

When $\eta(s) = \alpha e(s) + \sum_{j=1}^n \beta_j e(\tau_j + s)$ where $e(s)$ is the Heaviside function [①] and $0 < \tau_1 < \tau_2 \cdots < \tau_n$ are delay times, and α and β_j are constants or linear operators, the delay operator is of the form

$$L(z) = \alpha z(0) + \sum_{j=1}^{n} \beta_j z(-\tau_j). \tag{1.2.6}$$

[①] $e(s)$ defined on $\mathbb{R}$ is said to be a Heaviside function if $e(s) = 1$ for all $s \geq 0$ and $e(s) = 0$ if $s < 0$.

Since

$$L(x_t) = \alpha x(t) + \sum_{j=1}^{n} \beta_j x(t - \tau_j),$$

time delays result only in the signal time-shifts, but do not affect the signal character. However, the time delay influences dynamic behaviour of the system. From the Laplace transform of function $x(t)$ we see that a time-shift in the time domain becomes an exponential increasing in the frequency domain. Hence a time shift can affect the stability of a system, and occasionally can destroy the stability of a system. At the same time, time-delay issue always is an infinite-dimensional problem whatever h is how much smaller.

Engineers have been of great interesting in control issues of the system with delay. Early in 1960s, L. Weiss [127, (Leonard Weiss,1967)] discussed controllability of systems described by delay differential equation. After that, the subject gained much momentum due to consideration of meaningful models of engineering systems and control. In past several decades, there were great deal works on the controllability, observability and feedback stabilization of delay systems, for example, [75, A Manitius and R Triggiani, 1978)] [108, (Gabi Ben-Simon,1984)][85, (Shin-Ichi Nakagiri and Masahiro Yamamoto, 1989)] for controllability and observability, [71, Farid Ammar Khodja et al. 2014)], [113, (Sukavanam Nagarajan et al. 2011)] for null and approximate controllability respectively, and [13, (Shinn-Horng Chen et al. 2012)] for the robust controllability of delay systems, and [107, (Felipe W. Chaves-Silva et al. 2017)] for the null controllability of a heat equation with memory, [66, (Fa Lun Huang,1986)], [51, (El Mustapha Ait Ben Hassi et al.2009)] for feedback stabilization and [117, (Firdaus E Udwadia, 1991)] for using time delay as non-collocated point control.

However, R. Datko et al. [25, 26, (Richard Datko, et al. 1986,1988)] observed that not all feedback stabilized systems are robust with respect to small delays. So the control problem of systems with input delay becomes an interesting topic in the control theory. Han and Xu et al. [50, (Peng cheng Han et al. 2016)], [132, (Gen Qi Xu et al. 2017)] discussed boundary feedback stabilization of Euler-Bernoulli beam and Timoshenko beam with interior delay respectively, and Tian et al. [114, (Zong Fei Tian et al. 2017)] discussed stability of a Timoshenko beam with interior damping and boundary delay. Xu et al. [131, (Gen Qi Xu et al. 2006)], [104, 105, (Ying

Feng Shang et al. 2012)] and S. Nicaise et al. [86, 87, 88, (Serge Nicaise et al. 2006, 2014, 2015)] considered the stabilization problem of distributed parameter systems with difference-type delay in control, Liu et al. [74, (Xiu Fang Liu et al. 2013)] and Shang et al. [106, (Ying Feng Shang et al. 2015)] investigated the case of control involving distributed delay.

All works mentioned above are highlight the controller design for the distributed parameter system with input discrete delay. When $\mathbb{X}$ is a finite dimensional space, for instance $\mathbb{R}^n$, there are plenty of literature on the controller design and stabilization of linear and semi-linear system with unbounded delay via Lyapunov function method. Here we refer to some papers, for example, [135, 136, 137, (Xiang Xu et al. 2018, 2019, 2020)]) for stabilization of linear unbounded delay system, and [138, 139, (Xiang Xu et al. 2020, 2021)] for stability of input to state (ISS) of unbounded delayed systems.

Obviously the delayed term in delay differential equations always includes the history information, so time delay is a special form of memory. When the delay variable τ is a continuous variable, the delayed state $x(t-\tau)$ is consistent with its historical function. In this sense, the delay issue is equivalent to the memory issue. So the bounded delay and unbounded delay of continuous variable are respectively called finite memory and persistent memory. In the sequel, only when the delay differential equation with discrete delays is called delay differential equations.

§1.3 About phase space for evolutionary process with memory

In the study of evolutionary process with memory, the choice of phase space (or historical function space) is crucial. This is in contrast to ordinary differential equation, as pointed out by Hale and Kato [48, (Jack K. Hale, and Jumji Kato,1978)], where the state equations does not include a memory effect. From a mathematical perspective, an evolutionary process living on a Banach space $\mathbb{X}$ can be described as a $\mathbb{X}$-valued continuous function defined on $\mathbb{R}_+$, the state of the process at time $t \in \mathbb{R}_+$ is denoted by $x(t)$. The historical function of a state $x(t)$ is another function with an additional variable θ, defined on $\mathbb{R}_- = (-\infty, 0)$. It can be written as $x_t(\theta) = x(t+\theta)$

with $\theta \in \mathbb{R}_-$.

(I) What is the function of historical memory?

From a mathematical perspective, the historical function $x_t(\theta)$ represents an accurate record of the state $x(t)$ over time, specifically defined as $x_t(\theta) = x(t + \theta), \forall \theta \in (-\infty, 0)$. A historical memory function of $x(t)$ denoted by $\varphi_x(\theta)$ serves as a summary of $x_t(\theta)$, capturing major events but not specific details, in other words, $\varphi_x(\theta)$ reflects the historical information contained in our knowledge structure (or information we possess), while also potentially including oral legends and informal information about $x_t(\theta)$ or niose/disturbances. Such phenomena are commonly observed in sociology and studies on historical evolution. In most cases, $x_t(\theta) \neq \varphi_x(\theta)$. When referring to a function of historical memory as a "historical function", it implies that $m\{\theta \in \mathbb{R}_- \mid |\varphi_x(\theta) - x_t(\theta)| \neq 0\} = 0$ under certain measure m on the set $\mathbb{R}_-$.

Let m be a measure on $\mathbb{R}_-$. Let $\mathbb{X}$ be a Banach space and $\varphi(\theta)$ be an abstract $\mathbb{X}$-valued function defined on $I \subseteq \mathbb{R}_-$. Initially, we need to interpret measurability of $\varphi(\theta)$. Let $\{A_j^{(n)}, j = 1, 2, \cdots, n\}$ be a partition of I with $A_j^{(n)}$ being m measurable sets, $A_j^{(n)} \cap A_k^{(n)} = \emptyset$ and $\cup_j A_j^{(n)} = I$. Set

$$S_n(\theta) = \sum_{j=1}^{n} x_j \chi_{A_j^{(n)}}(\theta), \quad x_j \in \mathbb{X}.$$

$S_n(\theta)$ is referred to as an m-measurable step function on I. $\varphi(\theta)$ is said to be strong m-measurable in the sense of Bochner if there exists a sequence of step functions $S_n(\theta)$ such that

$$\lim_{n\to\infty} \|\varphi(\theta) - S_n(\theta)\|_{\mathbb{X}} = 0, \quad \forall \theta \in I, \ a.e.$$

A strong measurable function must have a countable value. $\varphi(\theta)$ is said to be p-power integrable on I in the sense of Bochner if $\varphi(\theta)$ is strongly measurable and $\int_I \|\varphi(\theta)\|_{\mathbb{X}}^p dm < \infty$.

The measure may vary for different models. For instance, in the functional differential equation, $\mathbb{X}$ is a finite dimensional space and I is a bounded set in $\mathbb{R}_-$, in this case, the measure m is taken as a bounded variation function [52, (Daniel Henry,1974)] [49, (Jack K. Hale and Sjoerd M. Verduyn Lunel,1993)]. On the other hand, in delay differential equation, where

I is a bounded set, the usual Lebesgue measure m is used as the measure m [7, (Richard Bellman and Kenneth L. Cooke, 1963)] [24, (Richard Datko, 1978)]. For more concrete models and space setting please refer to [4, O. Arino, Hbid Moulay Lhassan, and Hoda Ait Dads (Eds),2006)]. However, when dealing with an unbounded delay case where there is a relaxation kernel available, finding a universal measure becomes a challenge; see for example [46, (Jack K. Hale,1974)] [69, (Junji Kato,1978)]. Henceforth the Phase space issue was proposed by Haleand Kato [48, (Jack K. Hale and Jumji Kato, 1978)].

(II) About phase space (the space of possible historical state)

Note that, for the process with persistent memory, the function $x_t(\theta)$ is not continuous at $\theta = 0$ in the sense of Bochner' s measurability. Specifically, $\lim_{\theta\to 0_-} x_t(\theta) \neq x(t)$. Additionally, as a function of variable θ, $x_t(\theta)$ could be discontinuous and unbounded. Therefore, the phase space is a function space that satisfies certain conditions. Let $M(\mathbb{R}_-, \mathbb{X})$ denote the phase space or memory space (also known as historical function space), equipped with some norm $\|\cdot\|_*$. Before delving into detail, let us consider an example.

Example 1.3.1 Let us consider the following scalar differential equation with memory

$$\begin{cases} \frac{dx(t)}{dt} = -ax(t) + b\int_{-\infty}^{0} w(\theta)x(t+\theta)d\theta + f(x(t)), & t > 0 \\ x(0) = x_0, & \\ x(\theta) = \varphi(\theta), & \theta < 0. \end{cases} \tag{1.3.1}$$

where $a > 0$, $b \in \mathbb{R}$, $w(\theta) \geq 0$ is a bounded monotone function and $\int_{-\infty}^{0} w(\theta)d\theta = 1$, and f is a nonlinear function defined on $\mathbb{R}$ with $f(0) = 0$.

It can be observed that $f(x) = x^2$, Eqs. (1.3.1) have a nonzero equilibrium point $x = a - b$. Furthermore, if $f(x) = x^3$, the equation (1.3.1) has two nonzero equilibrium points $x_1 = \sqrt{a-b}, x_2 = -\sqrt{a-b}$.

Such an example suggests that the phase space $M(\mathbb{R}_-, \mathbb{X})$ should include the constant vector x in order to guarantee the existence of an equilibrium point. Therefore, $M(\mathbb{R}_-, \mathbb{X})$ should satisfy the following features:

(i) $(M(\mathbb{R}_-, \mathbb{X}), \|\cdot\|_*)$ is a Banach space.

(ii) The constant function should attach to $M(\mathbb{R}_-, \mathbb{X})$, i.e., $\forall x \in \mathbb{X}$, $x(\theta) \equiv x \in M(\mathbb{R}_-, \mathbb{X})$.

(iii) For any $\varphi \in M(\mathbb{R}_-, \mathbb{X})$, the shifted function $S(t)\varphi = \varphi(t+\theta)$ also belongs to $M(\mathbb{R}_-, \mathbb{X})$, and its norm satisfies $\|S(t)\varphi\|_* \leq \|\varphi\|_*$.

For $\mathbb{X} = \mathbb{R}^n$, in the 1970s and 1980s, some mathematicians proposed the following hypothesis for phase spaces of systems with memory.

Definition 1.3.1 Phase Space Hypothesis [69, (Jumji Kato,1978)] [48, (Jack K. Hale and Jumji Kato,1978))] [55, (Yoshiyuki Hino,1983)] [56, (Yoshiyuki Hino et al, 1991)] Let $(B, \|\cdot\|_B)$ denote a vector-valued function space consisting of functions that map from $[-r, 0]$ to $\mathbb{R}^n$, where $r > 0$ can be taken $+\infty$. For any $0 \leq \sigma < T < \infty$, if $x(t)$ is defined on $(\sigma - r, T)$ and continuous on (σ, T), then the function $x_t : [-r, 0] \to \mathbb{R}^n$ is defined by $x_t(\theta) = x(t+\theta)$ with $\theta \in [-r, 0]$.

A Phase Space $(B, \|\cdot\|_B)$ is defined as a Banach space where for any $t \in (\sigma, T)$,

(1) if $x_\sigma \in B$, then $x_t \in B$;

(2) x_t is continuous with respect to variable t in the sense of norm in $\|\cdot\|_B$;

(3) there exist two positive constants M_0 and K, and a positive continuous function $M(t)$ defined on $\mathbb{R}_+$ such that $\lim_{t\to\infty} M(t) = 0$ and

$$\|x(t)\| \leq M_0 \|x_t\|_B, \quad \forall t \in (\sigma, T) \tag{1.3.2}$$

and

$$\|x_t\|_B \leq K \sup_{s\in[\sigma,t]} \|x(s)\| + M(t-\sigma)\|x_\sigma\|_B \tag{1.3.3}$$

According to the phase space hypothesis, several fundamental theorems for systems with unbounded delay were proven, including the existence and uniqueness of solutions, continuous dependence, and stability theorems. These proofs can be found in [46, (Jack K. Hale, 1974)], [69, (Junji Kato,1978)], [99, (Kensuke Sawano,1979)] and [55, (Yoshiyuki Hino, 1983)]. It was demonstrated in [44, (John R. Haddock,1985)] [102, (George Seifert,1982)] that if the Phase Space Hypothesis is not satisfied, the system may have no solution.

The following example provides a function space that does not satisfy the Phase Space Hypothesis.

Example 1.3.2 Take $w(\theta) = e^{\theta}$ for $\theta \in \mathbb{R}_-$. Clearly $w(\theta)$ is a bounded, monotone, and continuous function defined on $\mathbb{R}_-$, and $\int_{-\infty}^{0} w(\theta)d\theta = 1$. For $p \geq 1$, $L_w^p(\mathbb{R}_-, \mathbb{X})$ denotes the function space composed of all $\mathbb{X}$-valued functions that are p-power integrable with weight w in the sense of Bochner's integral. $L_w^p(\mathbb{R}_-, \mathbb{X})$ is a Banach space equipped with the norm

$$\|\varphi\|_{L_w^p} = \left(\int_{-\infty}^{0} e^{\theta}\|\varphi(\theta)\|_{\mathbb{X}}^p d\theta\right)^{\frac{1}{p}}, \quad \forall \varphi \in L_w^p(\mathbb{R}_-, \mathbb{X}).$$

$M(\mathbb{R}_-, \mathbb{X}) = L_w^p(\mathbb{R}_-, \mathbb{X})$ does not satisfy the Phase Space Hypothesis.

In fact, it is easy to verify that $M(\mathbb{R}_-, \mathbb{X}) = L_w^p(\mathbb{R}_-, \mathbb{X})$ satisfies conditions (i)-(iii). However, inequality (1.3.2) in Phase Space Hypothesis does not hold.

Take a nonzero $y \in \mathbb{X}$, and let $x(t) = yt^n$ for $t \geq 0$. We define $x(\theta) = 0$ for $\theta < 0$. Then $\|x(t)\|_{\mathbb{X}} = \|y\|t^n$ and

$$\|x_t\|_{L_w^p}^p = \int_{-\infty}^{0} e^{\theta}\|(t+\theta)^n y\|_{\mathbb{X}}^p d\theta = \|y\|^p e^{-t}\int_0^t e^s s^{np} ds.$$

Since

$$\lim_{t\to 0^+} \frac{\|x(t)\|^p}{\|x_t\|_{L_w^p}^p} = \lim_{t\to 0_+} \frac{e^t t^{np}}{\int_0^t e^s s^{np} ds} = \lim_{t\to 0^+}(1 + \frac{np}{t}) = \infty,$$

the inequality (1.3.2) does not hold. □

In contrast to the phase space hypothesis, we adopt the following definition of phase space in our study of the evolutionary process with memory [82, (Rong Sheng Mu et al. 2021)] [133, (Gen Qi Xu et al. 2021)] [134, (Gen Qi Xu,2023)]. Let $w(\theta)$ be a bounded and nonnegative monotone function on $\mathbb{R}_-$ with $\int_{-\infty}^{0} w(\theta)d\theta < \infty$, which is referred as a weight function. Let $\mathbb{X}$ be a Banach space. For $p \geq 1$, let $L_w^p(\mathbb{R}_-, \mathbb{X})$ denote the historical function space equipped with the norm $\|\varphi\|_{L_w^p}$:

$$\|\varphi\|_{L_w^p} = \left(\int_{-\infty}^{0} w(\theta)\|\varphi(\theta)\|_{\mathbb{X}}^p\right)^{\frac{1}{p}}, \quad \forall \varphi \in L_w^p(\mathbb{R}_-, \mathbb{X}).$$

It is evident that the following facts are true:

(i) $(L^p_w(\mathbb{R}_-, \mathbb{X}), \|\cdot\|_{L^p_w})$ is a Banach space, $\bigcup_{h>0} C([-h, 0], \mathbb{X})$ is a dense set in $L^p_w(\mathbb{R}_-, \mathbb{X})$;

(ii) For $z \in C(\mathbb{R}_+, \mathbb{X})$, for $t > 0$, $z^e_t(\theta) = e(t+\theta)z(t+\theta)$, where $\theta \in \mathbb{R}_-$, belongs to $L^p_w(\mathbb{R}_-, \mathbb{X})$ and satisfies

$$\|z^e_t\|^p_{L^p_w} = \int_0^t w(s-t)\|z(s)\|^p_{\mathbb{X}} ds \le w(0) \int_0^t \|z(s)\|^p_{\mathbb{X}} ds;$$

(iii) For every $\varphi \in L^p_w(\mathbb{R}_-, \mathbb{X})$, $S(t)\varphi$ also belongs to $L^p_w(\mathbb{R}_-, \mathbb{X})$, and satisfies

$$\|S(t)\varphi\|^p_{L^p_w} = \int_{-\infty}^0 w(\theta - t)\|\varphi(\theta)\|^p_{\mathbb{X}} d\theta \le \|\varphi\|^p_{L^p_w}.$$

Please keep this in mind. In certain models, its weight function $w(\theta)$can be unbounded.

§1.4 Research progress and existing problems

From the preceding brief review we can see that regardless of whatever it is an integro-differential equation or a delay differential equation, their common characteristic is the hereditary effect of the evolutionary process [24, (Richard Datko,1978)], [84, (Shin-ichi Nakagiri,1987)]. We refer to such a process as an evolutionary process with memory, or evolutionary equation with memory. Regardless of the specific models used, the essential issues are solving linear evolutionary equation with memory and to discussing the large time asymptotic behaviour of the evolutionary process.

Let us briefly recall the research progress of the evolutionary process with memory. The integro-differential equations, after a suitable change, are transformed into integral equations. This class of integral equations was first studied by Volterra [122, (Vito Volterra,1959)], and later this research approach developed into Evolutionary Integral Equation Theory [94, (Jan Pruess,1993)]. The study of differential-difference equation led to Differential-Difference Equations Theory [7, (Richard Bellman and Kenneth L. Cooke,1963)] and Theory of Functional Differential Equations [47, (Jack K. Hale,1977)]. If $\mathbb{X}$ is a finite dimensional space, classical results can be obtained for linear evolutionary equations with discrete delays. These results include solvability, the expansion of solution by Lambert W function [37, (Tryphon Georgiou and Malcolm C Smith, 1989)] which is also referred to as the spectral decomposition approach [124, (Lei Wang et al. 2009)] [125, Siu Ping Wang et al. 2009)] [143, 144, 145, (Ya Xuan Zhang et al. 2009, 2011)] and [126, (Xiao Riu Wang et al. 2019)]. Now the method of Lambert function has become a tool-software in engineering. If $\mathbb{X}$ is an infinite-dimensional Banach space, R. Datko [24, (Richard Datko,1978)] and S. Nakagiri [83, (Shin-ichi Nakagiri,1981)] provided representations of solutions to delay differential equations. Regarding semi-linear systems with memory, whether $\mathbb{X}$ is a finite-dimensional or an infinite-dimensional Banach space, the well-posed-ness,stability, and stabilization of the system are discussed for special models. Due to too much literature we cannot provide a detailed list.

Since most of the works mentioned above focus on studying a specific model, the obtained result does not apply to other models. Specially, some issues of common concerns such as easy of checking solvability conditions, representation of solution of integro-differential equation and evolutionary equation with memory, as well as considering the effects of historical memory and monotonicity of kernel functions, are unclear. It is well-known that the C_0 semigroup theory is associated with the well-posed-ness of Cauchy initial value problems for evolutionary equations in Banach space $\mathbb{X}$ (e.g., see [92, (A. Pazy,1983)] and [35, (Klaus Jochen Engel and Rainer Nagel,2000)]). A natural question is: can one find a family $\{(G(t), F(t)), t \geq 0\}$ that has a property similar to semigroup? This question is referred to as the representation issue of solution. The representation of solutions plays a pivotal role in dealing with semi-linear equation. Additionally, the representation of solutions is crucial in control theory, where it allows for comprehensive discussions on controllability and observability of systems [116, (Marius Tucsnak and George Weiss,2009)].

In the existing literature, the most common form of second-order evolutionary process is given by equation (1.1.5). However, when this process includes friction and memory effects, its outline changes. The questions are: what is the universal form of a second-order evolutionary process with memory? Can we use an operator family to represent its solution?

The issue of well-posedness in evolutionary process with memory can be addressed using different approaches. In the existing literature, the method of full state space is applied to establish the well-posedness through the semigroup theory. For example, refer to [8, (C Bernier and A Manitius,1978)] [32, (Michel C Delfour and Andre Manitius,1980)]. The corresponding dual semigroup is derived using the semigroup method, which corresponds a dual evolutionary process [73, (Karl Kunisch and Miklavz Mastinsek,1990)] [76, (Miklavz Mastinsek, 1994)]. However, we observe that the resulting dual process obtained through the semigroup method does not have a general form of an evolutionary process with memory. The questions are: what is the correct form of the dual process in an evolutionary process with memory? What is spectral problem of an evolutionary process with memory? In existing literature, many scholars have treated solvability of the evolutionary equation with memory using the Laplace transform. Therefore, the set consisting of all complex numbers of without inverse Laplace transform is called the spectrum of the equation and its complement set is referred to as the resolvent set of the equation. On the other hand, from a physics perspective, an eigenvector is a vector under which the physical system remains invariant. From a mathematical viewpoint, an eigenvalue of a linear operator T is a complex number λ, for which $(\lambda I - T)$ is not injective, and corresponding this eigenvalue, there exists a vector x known as an eigenfunction. The question now arises: do eigenvectors coincide with eigenfunctions for an evolutionary equation with memory?

In this book we focus on these questions. Unlike the previous investigation, we utilize the resolvent family to examine the evolutionary process with memory, and demonstrate the distinct effects of both the present state and its historical function. Additionally, we introduce a generation theorem for the resolvent family. Specifically, we seek a sufficient condition for generating the resolvent family that is not only theoretical but also easily verifiable. Since an evolutionary process with memory is fundamentally a physical process, it may possess time-varying properties and nonlinear characteristics, therefore, we investigate both time-varying evolutionary processes and semi-linear evolutionary processes. Furthermore, we discuss invariant subspaces and spectral problems related to evolutionary process with memory.

In this book, we investigate the evolutionary process with memory in a completely different manner. The contents of the present book are organized as follows.

In Chapter 2, we discuss the resolvent family $(G(t), F(t))$ for an evolutionary process with memory and its associated differential equation. Simultaneously, we investigate the generation issue of a resolvent family.

In Chapter 3, we investigate the general form of second-order evolutionary process with memory, and provide sufficient conditions for the generation of a resolvent family corresponding to a second-order evolutionary process with memory.

In Chapter 4, we define a dual process of an evolutionary process with memory, which is also an evolutionary process with memory. We discuss the existence of the dual process and the dual resolvent family in an evolutionary process with memory.

In Chapter 5, we present another approach - the state space method for ensuring the well-posedness of the evolutionary process with memory, through which we derive the memory semigroup. Furthermore we discuss the relationship between the dual process and the dual process resulting from the memory semigroup.

In Chapter 6, we study semi-linear evolutionary processes with memory and the evolutionary processes with a time-varying memory operator as applications of resolvent family. As a special case, we provide some useful results regarding integro-differential equations.

In Chapter 7, we study representation of the evolutionary family $\{G(t), t \geq 0\}$ for certain evolutionary processes with memory. We present a series expression of $G(t)$ that applies to a specific type of memory operator, including discrete delay differential equation.

In Chapter 8, we study invariant subspace and spectral problem of an evolutionary process with memory. We focus on the eigenvector problem from a physics perspective and the eigenvalue issue from a mathematical viewpoint for this evolutionary process with memory. For certain specific equations, we calculate their eigenvectors, eigenvalues, and eigenfunctions.