Scattering of Elastic Waves by Obstacles in Solids

Scattering of Elastic Waves by Obstacles in Solids

By

Anders Boström

Cambridge
Scholars
Publishing

Scattering of Elastic Waves by Obstacles in Solids

By Anders Boström

This book first published 2024

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

Copyright © 2024 by Anders Boström

All rights for this book reserved. No part of this book may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior permission of the copyright owner.

ISBN (10): 1-0364-0631-8
ISBN (13): 978-1-0364-0631-8

Contents

Preface

It is almost 50 years since I started as a PhD student in mathematical physics and was introduced to the area of scattering of elastic waves in solids by my supervisor Staffan Ström. This has remained my main area of research, and being professor emeritus since a few years, it now seems appropriate to summarize my views on the subject.

This book is thus concerned with the scattering of elastic waves by various types of obstacles of simple shape, mostly in an infinite solid, but also in a half-space, layered medium, or waveguide. Isotropic media is mostly assumed, but some anisotropic ones are also considered. The focus is on more or less analytical approaches, in most cases leading to "exact" results. Two distinctive features are that the transition matrix (relating the scattered wave coefficients to those of the incident wave coefficients) is calculated when relevant and that this matrix is given to leading order at low frequencies.

There are, of course, useful methods of scattering that are not considered. This includes high frequency asymptotic methods, like ray theory, geometrical theory of diffraction, and Kirchhoff theory. Low frequency asymptotic methods are not considered either, although it should be noted that the methods employed in the book work very well for low frequencies and, as noted above, in a number of cases the leading order T-matrix elements are given for low frequencies. Purely numerical methods, like finite-element methods, are not taken up either, although such methods are of course also useful.

The book is rather unique in its relatively broad coverage of scattering by obstacles in elastic solids, other books being more limited in scope. Pao and Mow [112] use separation of variables for some simply shaped obstacles, mostly in 2D. Zhang and Gross [147] employ integral equation methods, mostly of the same types as here, for scattering by some types of cracks and provide many references. Martin [88] considers multiple scattering for

acoustic, electromagnetic, elastic, and surface water waves and this includes a review of scattering by a single obstacle. He also gives an extensive list of references.

The mathematics used in the book is more or less standard, like vector analysis, special functions, separation of variables, analytic function theory, Green functions. Vectors are used as much as possible, although sometimes it is more convenient to use tensors.

I have been influenced by and cooperated with many persons and I thank all these and others that have helped me during the years. I also thank Paul Martin for providing many useful comments on two versions of the manuscript.

1 Elastodynamics

The topic of this book is the scattering of elastic waves (also called acoustic waves [9]) by obstacles in solids. Only linear and time-harmonic problems are considered. In this first chapter the basic equations describing propagation and scattering of linear waves in elastic solids are given. Isotropic media are in particular treated and the relevant potentials are introduced for different coordinate systems. Some of the simpler types of anisotropy are described. As an introduction to the rest of the book the chapter ends with a simple scattering problem.

1.1 Basic equations

In this section the basic equations governing wave propagation in an elastic solid are stated. This includes the equation of motion (Newton's second law), the constitutive relation between stress and strain, and the definition of strain. Throughout the book only linear problems are considered, so all relations are taken in their linear form. The boundary conditions on an elastic body and the interface conditions between two elastic bodies in contact are stated. Energy densities and flow are defined and the energy balance is derived. The Fourier transform between time and frequency is introduced; however, throughout only problems with a fixed frequency are taken up. Reciprocity is a useful concept in elastodynamics and the integral form of reciprocity is derived at fixed frequency. Finally, the equations of motion and the strain are given in cylindrical and spherical coordinates. The material in this section is standard and can be found in many textbooks, for example de Hoop [41], Auld [9], Achenbach [1], Langenberg *et al.* [84], and Aki and Richards [5].

1.1.1 The wave equation

The basic field variable in an elastic solid is the displacement $\boldsymbol{u}(\boldsymbol{r}, t)$, which is a function of position $\boldsymbol{r}$ and time t. The displacement is given components u_x, u_y, u_z in a Cartesian coordinate system (x, y, z) as

$$\boldsymbol{u} = \hat{\boldsymbol{x}} u_x + \hat{\boldsymbol{y}} u_y + \hat{\boldsymbol{z}} u_z$$

As much as possible vectors are used and these are always bold-faced. Unit vectors are denoted with a hat so that $\hat{\boldsymbol{x}}$ is the unit vector in the x-direction. However, sometimes it is convenient to use Cartesian tensors. Then the coordinates are (x_1, x_2, x_3) and the displacement is written

$$\boldsymbol{u} = \hat{\boldsymbol{x}}_1 u_1 + \hat{\boldsymbol{x}}_2 u_2 + \hat{\boldsymbol{x}}_3 u_3 = \hat{\boldsymbol{x}}_i u_i$$

where Einstein's summation convention is used so that a repeated index is summed over $i = 1, 2, 3$.

Both displacements and strains are assumed small and the linear strain tensor is then

$$\varepsilon_{ij} = \frac{1}{2} \left(\partial_j u_i + \partial_i u_j \right)$$

As usual in tensor notation the free indices i and j take on the values 1, 2, 3, so that nine components of strain are defined. Derivatives are sometimes indicated with a short-hand notation

$$\partial_j u_i = \frac{\partial u_i}{\partial x_j}$$

The strain tensor is obviously symmetric. The definition of the strain tensor can also be made in dyadic notation [99].

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\nabla \boldsymbol{u} + \boldsymbol{u} \nabla)$$

where the convention is that $(\nabla \boldsymbol{u})_{ij} = (\boldsymbol{u} \nabla)_{ji} = \partial_i u_j$. According to the identity

$$\boldsymbol{I} \times (\nabla \times \boldsymbol{u}) = \boldsymbol{u} \nabla - \nabla \boldsymbol{u}$$

the strain can be written as

$$\boldsymbol{\varepsilon} = \nabla \boldsymbol{u} + \frac{1}{2} \boldsymbol{I} \times (\nabla \times \boldsymbol{u})$$

where $\boldsymbol{I}$ is the unit dyadic with components δ_{ij}, where δ_{ij} is the Kronecker symbol ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$).

The material behaviour is also assumed to be linear and the linear stress tensor σ_{ij} is thus related to the linear strain tensor by the constitutive relation

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} \tag{1.1}$$

Here the stiffness tensor c_{ijkl} is a tensor of rank four. The stress tensor is also symmetric and together with the assumption of the existence of a strain energy function this means that the stiffness tensor satisfies the following symmetry properties

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{klij}$$

This reduces the number of independent components of the stiffness tensor to 21. Most materials have much fewer independent stiffness components, all the way down to an isotropic material which has only two. Some of the simpler examples of constitutive relations are taken up later in the chapter.

The equation of motion in the linear case is

$$\partial_k \sigma_{ki} + \rho f_i = \rho \ddot{u}_i \tag{1.2}$$

where ρ is the density and f_i is the body force (per unit mass). Time derivatives are denoted with a dot above a variable, thus $\ddot{u}_i$ is the second time derivative of u_i. In vector notation this equation is

$$\nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f} = \rho \ddot{\boldsymbol{u}} \tag{1.3}$$

where $\boldsymbol{\sigma}$ is the stress dyadic. In most cases the body force is abscent, $\boldsymbol{f} = \boldsymbol{0}$. Eliminating the stress and strain the equation of motion becomes

$$\partial_k \left(c_{kilj} \partial_l u_j \right) = \rho \ddot{u}_i \tag{1.4}$$

This is the most general form of the linear elastodynamic wave equation that is valid for general anisotropy and inhomogeneous materials (for which the stiffness tensor is dependent on position).

1.1.2 Boundary conditions

In all scattering problems at least two different domains in space are considered, although one of them may be a domain where no wave propagates,

typically vacuum. Then some conditions at the surface of a domain or at an interface between two domains are required. On the other hand no initial conditions are needed as only time harmonic problems are considered.

If a domain where waves are propagating has a boundary to another domain where no waves can propagate then one (vector) boundary condition is required. This can be of different kinds:

- The displacement is specified. Typically the displacement is zero and this corresponds to a boundary to a rigid body where also the tangential components are restrained from moving.
- The traction vector $\boldsymbol{t}^{(n)} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}$, where $\hat{\boldsymbol{n}}$ is the unit normal to the boundary and $\boldsymbol{\sigma}$ is the stress dyadic, is specified. In tensor notation this is $t_i^{(n)} = n_j \sigma_{ji}$, where n_j are the components of the normal $\hat{\boldsymbol{n}}$. In most cases the traction is zero as this is a good model for a boundary to air (because of the great mismatch in impedance between a solid and air very little energy leaks from a solid into air).
- Normal displacement $u_n = \hat{\boldsymbol{n}} \cdot \boldsymbol{u}$ and tangential traction $\boldsymbol{t}_{tan}^{(n)}$ are specified. If these quantities are both zero this is a model for a boundary to a smooth, rigid body.
- Spring boundary conditions $\boldsymbol{t}^{(n)} = \boldsymbol{\alpha}\boldsymbol{u}$ ($t_j^{(n)} = \alpha_{ji} u_i$), where $\boldsymbol{\alpha}$ is a dyadic, where usually only the diagonal elements are nonzero. This can be used to model a thin surface layer.

There are other possibilities, but there should be one condition in each of three perpendicular directions and they should involve displacement and traction and possibly tangential derivatives thereof.

At the surface between two solids both supporting waves some interface conditions must likewise be satisfied:

- Welded contact where both the displacement $\boldsymbol{u}$ and traction $\boldsymbol{t}^{(n)}$ are continuous. This is by far the most common condition and is usually assumed if two solids are in contact.
- Slip conditions where u_n and $t_n^{(n)} = \hat{\boldsymbol{n}} \cdot \boldsymbol{t}^{(n)}$ are continuous and $\boldsymbol{t}_{tan}^{(n)} = \boldsymbol{0}$ from both sides.
- If a solid has a boundary to a fluid then u_n and $t_n^{(n)} = -p$, where p is the pressure in the fluid, are continuous and $\boldsymbol{t}_{tan}^{(n)} = \boldsymbol{0}$ in the solid.

- Spring interface conditions where the traction $\boldsymbol{t}^{(n)}$ is continuous and where $\boldsymbol{t}^{(n)} = \boldsymbol{\alpha}[\boldsymbol{u}]$, where $[\boldsymbol{u}]$ denotes the jump in displacement and $\boldsymbol{\alpha}$ is a dyadic, where usually only the diagonal elements are nonzero. This condition is widely used to model a thin interface layer or a damaged interface.

Other interface conditions are also used, for instance to obtain a better model of a thin interface layer (for which the spring interface conditions give a somewhat poor model). These conditions can involve tangential derivatives of displacement and traction.

If a solid is of infinite extent, which is the normal situation in scattering problems, then a "boundary condition", called a radiation condition, is needed at infinity. If there is damping in the material this is that the fields decay exponentially towards infinity, and the case without damping can be taken as the limit of this. For a fixed frequency in an isotropic solid that is infinite in all directions this means that the behaviour of the field far away is $\exp(\mathrm{i}kr)/r$ in 3D and $\exp(\mathrm{i}kr)/\sqrt{r}$ in 2D (where r is the radial coordinate in 3D or 2D, respectively) with appropriate wavenumber k and polarization. In other cases without damping, like waves in an infinite plate or for general anisotropy, the situation is more complicated. Generally speaking, the energy flow must always be directed outwards.

1.1.3 Energy densities and flow

It is sometimes of interest to calculate the energy flow or energy densities in waves. To this end the equation of motion is multiplied by $\dot{u}_i$ and summed over i

$$\dot{u}_i \partial_j \sigma_{ji} = \rho\, \ddot{u}_i \dot{u}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho\, \dot{u}_i \dot{u}_i\right)$$

Manipulate the left-hand side

$$\dot{u}_i \partial_j \sigma_{ji} = \partial_j\left(\dot{u}_i \sigma_{ji}\right) - \left(\partial_j \dot{u}_i\right)\sigma_{ji} = \partial_j\left(\dot{u}_i \sigma_{ji}\right) - \dot{\varepsilon}_{ij}\sigma_{ji}$$

where the symmetry of σ_{ij} is used in the last step. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon_{ij}\sigma_{ij}) = \dot{\varepsilon}_{ij}\sigma_{ij} + \varepsilon_{ij} c_{ijkl} \dot{\varepsilon}_{kl} = 2\dot{\varepsilon}_{ij}\sigma_{ij}$$

Introduce the kinetic energy density

$$e_T = \frac{1}{2}\rho\, \dot{u}_i \dot{u}_i$$

the strain energy density

$$e_U = \frac{1}{2}\varepsilon_{ij}\sigma_{ij}$$

and the energy flow density

$$j_i = -\dot{u}_j \sigma_{ji} \tag{1.5}$$

then all the above can be summarized as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e_T + e_U\right) + \partial_i j_i = 0$$

This is the differential form of energy conservation. The physical meaning becomes clearer if this equation is integrated over a fixed volume V with boundary S and outward unit normal $\hat{\boldsymbol{n}}$ and Gauss' law is employed

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(e_T + e_U\right)\mathrm{d}V = -\int_S j_n \,\mathrm{d}S$$

where $j_n = -\dot{u}_i \sigma_{ij} n_j$ and n_j are the components of $\hat{\boldsymbol{n}}$. In words this equation says that "change in energy inside V equals influx of energy through S".

1.1.4 Time harmonic fields

The Fourier transform between time t and angular frequency ω is here defined as

$$\tilde{\boldsymbol{u}}(\omega) = \int_{-\infty}^{\infty} \boldsymbol{u}(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t \tag{1.6}$$

with the inverse

$$\boldsymbol{u}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\boldsymbol{u}}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{1.7}$$

where the space dependence has been left out. Switching from the time domain to the frequency domain has the great advantage of removing all time derivatives and replacing them by multiplication by $-\mathrm{i}\omega$, which of course greatly facilitates the solution of the wave equation. As the physical displacement field in the time domain is a real function it follows that $\tilde{\boldsymbol{u}}(-\omega) = \tilde{\boldsymbol{u}}(\omega)^\star$ (where the star denotes the complex conjugate) and

$$\boldsymbol{u}(t) = \frac{1}{\pi} \operatorname{Re}\left[\int_0^\infty \tilde{\boldsymbol{u}}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\right]$$

Thus the angular frequency ω can always be regarded as a positive real quantity in the following. This is important when propagation direction and damping are discussed.

A note on the definition of the Fourier transform used here is in order. In some areas, particularly in electrical engineering, the sign in the exponential is switched so that instead of the implicit time factor $\exp(-\mathrm{i}\omega t)$ the time factor is $\exp(\mathrm{j}\omega t)$ (where the imaginary unit is called "j"). For wave propagation problems the present definition is more natural as a plane wave like $\exp(+\mathrm{i}kx)$ then propagates in the $+x$ direction. Also the factor $1/2\pi$ in front of the inverse transform can be moved to the transform instead, or a factor $1/\sqrt{2\pi}$ can be put in front of both the transform and the inverse transform (and this makes the Fourier transform a Hermitian operator).

Another approach is to say from the start that a time harmonic problem is considered and put

$$\boldsymbol{u}(t) = \operatorname{Re}\left(\tilde{\boldsymbol{u}}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right)$$

In the following only time harmonic problems with a fixed angular frequency ω are considered. The tilde on the displacement field is then skipped as is the dependence on ω. The basic field variable is thus written $\boldsymbol{u}(\boldsymbol{r})$. In principle the field in the time domain can be synthesized by taking an inverse Fourier transform.

It must be remembered that when quadratic quantities are calculated the time factor $\exp(-\mathrm{i}\omega t)$ and the real part must be taken before multiplying together the parts. For the energy flow vector defined in Eq. (1.5) this means that the instantaneous (time dependent) value is

$$\begin{aligned}\boldsymbol{j} &= -\dot{\boldsymbol{u}} \cdot \boldsymbol{\sigma} = -\operatorname{Re}\left(-\mathrm{i}\omega \boldsymbol{u}\,\mathrm{e}^{-\mathrm{i}\omega t}\right) \cdot \operatorname{Re}\left(\boldsymbol{\sigma}\,\mathrm{e}^{-\mathrm{i}\omega t}\right) \\ &= \omega\left(\operatorname{Re}\boldsymbol{u}\sin\omega t - \operatorname{Im}\boldsymbol{u}\cos\omega t\right) \cdot \left(\operatorname{Re}\boldsymbol{\sigma}\cos\omega t + \operatorname{Im}\boldsymbol{\sigma}\sin\omega t\right)\end{aligned}$$

where no notational difference is made between the displacement and stress in the time and frequency domain. Taking the mean in time this becomes

$$\langle \boldsymbol{j} \rangle = \frac{\omega}{2} \left(\operatorname{Re} \boldsymbol{u} \cdot \operatorname{Im} \boldsymbol{\sigma} - \operatorname{Im} \boldsymbol{u} \cdot \operatorname{Re} \boldsymbol{\sigma} \right) = -\frac{\omega}{2} \operatorname{Im} \left(\boldsymbol{u} \cdot \boldsymbol{\sigma}^{\star} \right) \tag{1.8}$$

Damping in a material is easier to model in the frequency domain by allowing the stiffness constants to be complex. The real part then corresponds to the elasticity of the material while the imaginary part determines the damping. In the time domain damping is more difficult to model, in general there should be memory in the material so that a time integral is needed, considerably complicating the analysis (Carcione [35]). In the frequency domain this corresponds to taking the stiffness constants as functions of frequency, and in practice almost any frequency dependence can be taken. To have damping the imaginary part of the stiffness constants must be negative with the present sign convention for the Fourier transform. It is noted that the negative imaginary part of the stiffness constants gives a positive imaginary part of the wavenumbers.

1.1.5 Reciprocity

A reciprocity relation gives a relation between two elastodynamics states. Here it is assumed that the two states occur in the same body. Consider the two states A and B and write the equation for each state (assuming a fixed frequency)

$$\partial_k \sigma_{ki}^{\mathrm{A}} + \rho f_i^{\mathrm{A}} = -\rho \omega^2 u_i^{\mathrm{A}}$$
$$\partial_k \sigma_{ki}^{\mathrm{B}} + \rho f_i^{\mathrm{B}} = -\rho \omega^2 u_i^{\mathrm{B}}$$

Multiply the first of these with u_i^{B} and the second by u_i^{A} (and sum over i) and subtract the second equation from the first

$$u_i^{\mathrm{B}} \partial_k \sigma_{ki}^{\mathrm{A}} - u_i^{\mathrm{A}} \partial_k \sigma_{ki}^{\mathrm{B}} = u_i^{\mathrm{A}} \rho f_i^{\mathrm{B}} - u_i^{\mathrm{B}} \rho f_i^{\mathrm{A}}$$

The left-hand side is written

$$u_i^{\mathrm{B}} \partial_k \sigma_{ki}^{\mathrm{A}} - u_i^{\mathrm{A}} \partial_k \sigma_{ki}^{\mathrm{B}} = \partial_k \left(u_i^{\mathrm{B}} \sigma_{ki}^{\mathrm{A}} - u_i^{\mathrm{A}} \sigma_{ki}^{\mathrm{B}} \right) - \left(\sigma_{ki}^{\mathrm{A}} \partial_k u_i^{\mathrm{B}} - \sigma_{ki}^{\mathrm{B}} \partial_k u_i^{\mathrm{A}} \right)$$

Using the symmetry of the stress tensor, the stress-strain relation Eq. (1.1), and the symmetry of the stiffness constants the second term on the right-hand side is seen to vanish

$$\sigma_{ki}^{\mathrm{A}} \partial_k u_i^{\mathrm{B}} - \sigma_{ki}^{\mathrm{B}} \partial_k u_i^{\mathrm{A}} = \sigma_{ki}^{\mathrm{A}} \varepsilon_{ki}^{\mathrm{B}} - \sigma_{ki}^{\mathrm{B}} \varepsilon_{ki}^{\mathrm{A}} = c_{kijl} \varepsilon_{jl}^{\mathrm{A}} \varepsilon_{ki}^{\mathrm{B}} - c_{kijl} \varepsilon_{jl}^{\mathrm{B}} \varepsilon_{ki}^{\mathrm{A}} = 0$$

Thus the local form of the reciprocity relation becomes

$$\partial_k \left(\sigma^{\mathrm{A}}_{ki} u^{\mathrm{B}}_i - \sigma^{\mathrm{B}}_{ki} u^{\mathrm{A}}_i\right) = \rho \left(u^{\mathrm{A}}_i f^{\mathrm{B}}_i - u^{\mathrm{B}}_i f^{\mathrm{A}}_i\right)$$

Integrate this equation over a volume V with boundary S and employ Gauss' law to obtain the global form of the reciprocity relation

$$\int_S n_k \left(\sigma^{\mathrm{A}}_{ki} u^{\mathrm{B}}_i - \sigma^{\mathrm{B}}_{ki} u^{\mathrm{A}}_i\right) \mathrm{d}S = \int_V \rho \left(u^{\mathrm{A}}_i f^{\mathrm{B}}_i - u^{\mathrm{B}}_i f^{\mathrm{A}}_i\right) \mathrm{d}V$$

where n_k are the components of the outward pointing normal $\hat{\boldsymbol{n}}$ to S. In vector and dyadic notation this equation is

$$\int_S \hat{\boldsymbol{n}} \cdot \left(\boldsymbol{\sigma}^{\mathrm{A}} \cdot \boldsymbol{u}^{\mathrm{B}} - \boldsymbol{\sigma}^{\mathrm{B}} \cdot \boldsymbol{u}^{\mathrm{A}}\right) \mathrm{d}S = \int_V \rho \left(\boldsymbol{u}^{\mathrm{A}} \cdot \boldsymbol{f}^{\mathrm{B}} - \boldsymbol{u}^{\mathrm{B}} \cdot \boldsymbol{f}^{\mathrm{A}}\right) \mathrm{d}V \quad (1.9)$$

In the derivation it is assumed that the body has a linear stress-strain relation but the body can be both anisotropic and inhomogeneous. Reciprocity relations are very useful in elastodynamics, e.g. in the derivation of the orthogonality properties of the spherical vector wavefunctions and in the derivation of integral representations. For more about reciprocity relations see, for instance, Achenbach [3], Schmerr [119], Aki and Richards [5], de Hoop [41], and Carcione [35].

1.1.6 Cylindrical and spherical coordinates

The formulas above are given in Cartesian tensor notation and are thus directly valid in Cartesian coordinates (x, y, z). In many scattering problems it is natural to use cylindrical (polar in 2D) and spherical coordinates instead. Cylindrical coordinates (r, φ, z) are defined by

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = z$$

The equations of motion are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\varphi}}{\partial \varphi} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = \rho \ddot{u}_r$$

$$\frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{\partial \sigma_{\varphi z}}{\partial z} + \frac{2\sigma_{r\varphi}}{r} = \rho \ddot{u}_\varphi$$

$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\varphi z}}{\partial \varphi} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} = \rho \ddot{u}_z \tag{1.10}$$

and the strain-displacement relations are

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\varphi\varphi} &= \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\varepsilon_{\varphi z} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \varphi} + \frac{\partial u_\varphi}{\partial z}\right) \\
\varepsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\
\varepsilon_{r\varphi} &= \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \varphi}\right)
\end{aligned} \tag{1.11}$$

Spherical coordinates (r, θ, φ) are defined by

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta$$

The radius is denoted r in both the spherical and cylindrical systems, but is of course not the same. This is common practice and there should be no confusion as both coordinate systems are seldom used in the same problem. The azimuthal angle φ, on the other hand, is the same. The equations of motion in spherical coordinates are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\varphi r}}{\partial \varphi} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \cot\theta\,\sigma_{r\theta}\right) = \rho\ddot{u}_r \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\varphi}}{\partial \varphi} + \frac{1}{r}\left(\cot\theta(\sigma_{\theta\theta} - \sigma_{\varphi\varphi}) + 3\sigma_{r\theta}\right) = \rho\ddot{u}_\theta \\
&\frac{\partial \sigma_{\varphi r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\varphi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{1}{r}\left(3\sigma_{\varphi r} + 2\cot\theta\,\sigma_{\theta\varphi}\right) = \rho\ddot{u}_\varphi
\end{aligned} \tag{1.12}$$

and the strain-displacement relations are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$
\begin{aligned}
\varepsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
\varepsilon_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\cot\theta}{r}u_\theta + \frac{u_r}{r} \\
\varepsilon_{\theta\varphi} &= \frac{1}{2r}\left(\frac{\partial u_\varphi}{\partial \theta} - \cot\theta\, u_\varphi + \frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \varphi}\right) \\
\varepsilon_{\varphi r} &= \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) \\
\varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)
\end{aligned}
\tag{1.13}
$$

1.2 Isotropic solids

In many cases it may be assumed that the solid is isotropic and this leads to great simplifications. It becomes, in particular, possible to introduce potentials so that the solution can proceed via scalar Helmholtz equations, thus simplifying the solution procedure. For a comprehensive discussion about potentials and the separability of the elastodynamic wave equation in 3D Morse and Feshbach [99, Chapter 13] should be consulted.

1.2.1 P and S waves

For an isotropic solid the stress-strain relation simplifies in that the number of stiffness constants is reduced from 21 to two, so that this relation now becomes

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} = \lambda\delta_{ij}\partial_k u_k + \mu(\partial_i u_j + \partial_j u_i) \tag{1.14}$$

where δ_{ij} is the Kronecker symbol and λ and μ are the Lamé constants. The Lamé constants can be expressed in terms of the engineering constants as

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = G = \frac{E}{2(1+\nu)}$$

where E is Young's modulus, ν Poisson's number, and G the shear modulus. In vector notation the constitutive equation can be written

$$\boldsymbol{\sigma} = \lambda \boldsymbol{I}\,\nabla\cdot\boldsymbol{u} + \mu[2\nabla\boldsymbol{u} + \boldsymbol{I}\times(\nabla\times\boldsymbol{u})] \tag{1.15}$$

From now on it is assumed that the solid is homogeneous so that λ and μ are constant. It is also assumed that there are no body forces. Inserting the stress from Eq. (1.14) into the equation of motion Eq. (1.2) gives

$$(\lambda + \mu)\partial_i \partial_j u_j + \mu \partial_j \partial_j u_i = -\rho\omega^2 u_i$$

where, as noted above, only time harmonic conditions are considered. In vector notation this can be written

$$k_{\mathrm{p}}^{-2}\nabla(\nabla \cdot \boldsymbol{u}) - k_{\mathrm{s}}^{-2}\nabla \times (\nabla \times \boldsymbol{u}) + \boldsymbol{u} = \boldsymbol{0} \tag{1.16}$$

Here the following property of the nabla operator is used

$$\nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u})$$

In Cartesian coordinates this is straightforward to show and for other orthogonal coordinates it is the definition of the operator ∇^2 operating on a vector field. Two wavenumbers are introduced

$$k_{\mathrm{p}} = \omega/c_{\mathrm{p}}, \qquad k_{\mathrm{s}} = \omega/c_{\mathrm{s}}$$

where the two wave velocities are

$$c_{\mathrm{p}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad c_{\mathrm{s}} = \sqrt{\frac{\mu}{\rho}}$$

When there is damping in the material the wavenumbers are complex, and as noted above the imaginary part must be positive. The wavenumbers are thus located in the first quadrant of the complex plane (or on the positive real axis if there is no damping). Much of this book is concerned with the solution of Eq. (1.16) in various scattering configurations.

To see the significance of the two wavenumbers k_{p} and k_{s} it is useful to take the divergence and curl of Eq. (1.16). First, introduce the scalar potential Φ according to

$$\Phi = -k_{\mathrm{p}}^{-2}\nabla \cdot \boldsymbol{u}$$

and take the divergence of Eq. (1.16) to get (using that $\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$ for arbitrary vector fields $\boldsymbol{v}$)

$$\nabla^2 \Phi + k_{\mathrm{p}}^2 \Phi = 0$$

which is a scalar Helmholtz equation. Second, introduce the vector potential $\boldsymbol{\Psi}$ according to

$$\boldsymbol{\Psi} = k_{\mathrm{s}}^{-2} \nabla \times \boldsymbol{u}$$

and take the curl of Eq. (1.16) to get (using that $\nabla \times (\nabla s) = \mathbf{0}$ for arbitrary scalar fields s)

$$\nabla \times (\nabla \times \boldsymbol{\Psi}) - k_{\mathrm{s}}^2 \boldsymbol{\Psi} = \mathbf{0}$$

which is a vector Helmholtz equation. As all solutions to this equation has $\nabla \cdot \boldsymbol{\Psi} = \mathbf{0}$ it can alternatively be written

$$\nabla^2 \boldsymbol{\Psi} + k_{\mathrm{s}}^2 \boldsymbol{\Psi} = \mathbf{0}$$

Using the definition of the two potentials in Eq. (1.16) then gives

$$\boldsymbol{u} = \nabla \Phi + \nabla \times \boldsymbol{\Psi}$$

This is just the well known Helmholtz decomposition, which is, in fact, valid for an arbitrary vector field.

There are thus two types of waves in an isotropic, homogeneous solid. Inside the solid they propagate independently, but in general they couple at the boundaries. One is the wave corresponding to the scalar potential Φ which travels with the speed c_{p} and is a compressional wave. It is also longitudinal, i.e. it is irrotational (meaning that $\nabla \times \boldsymbol{u} = 0$), and the displacement vector in a plane wave is in the same direction as the propagation direction (as is also the case for acoustic waves in a fluid). When this wave was first discovered experimentally in seismology it was called the primary wave because it arrives first. Here compressional waves are denoted P waves for short, with P for primary (or pressure). The other wave type is corresponding to the vector potential $\boldsymbol{\Psi}$ and is a shear wave which travels with the slower velocity c_{s}. This wave is also transverse, i.e. it is equivoluminal ($\nabla \cdot \boldsymbol{u} = 0$), and the displacement vector in a plane wave is perpendicular to the propagation direction (as is also the case for electromagnetic waves). This wave was called secondary in seismology. Here shear waves are denoted S waves for short, with S for secondary (or shear). P and S waves are often denoted L (longitudinal) and T (transverse) waves instead.

The potential for the S waves is a vector with three components, but it is only the curl of the potential that has physical significance. This in effect

takes away one component, so that an auxiliary restriction can be made, and this is often chosen as

$$\nabla \cdot \boldsymbol{\Psi} = 0$$

However, the way in which potentials are soon introduced this form of the restriction is not always fulfilled.

The P waves are solutions to the scalar Helmholtz equation, for which there exist various ways to obtain analytical solutions depending on the coordinate system chosen. For the S waves the situation is more complex as they satisfy a vector Helmholtz equation.

1.2.2 2D potentials

Looking first at 2D situations (meaning that all fields are independent of the z-coordinate) there is an immediate division into out-of-plane waves and in-plane waves. For the out-of-plane waves the displacement is assumed on the form $\boldsymbol{u}(\boldsymbol{r}) = \hat{\boldsymbol{z}} u_z(x, y)$. Insertion into Eq. (1.16) gives that the scalar displacement must satisfy the Helmholtz equation

$$\nabla^2 u_z + k_{\mathrm{s}}^2 u_z = 0$$

These waves are apparently S waves without any coupling to P waves. In the following they are denoted 2D SH waves (the designation SH is explained below). As they are the simplest waves of interest in scattering problems, situations involving these are taken up first in many of the later chapters when various types of scattering configurations are investigated. Often it is useful to use polar coordinates (r, φ) instead of Cartesian coordinates (x, y), for instance when the scattering by a circle is studied.

The in-plane waves involve both P and S waves. The displacement vector lies in the xy-plane and can be assumed on the form

$$\boldsymbol{u} = \nabla \Phi + \nabla \times (\hat{\boldsymbol{z}} \Psi) \tag{1.17}$$

where Φ and Ψ are scalar potentials and any convenient coordinates can be used in the xy-plane; in most scattering problems this is polar coordinates (r, φ). Insertion of the first term into Eq. (1.16) gives that the potential Φ must satisfy the 2D Helmholtz equation

$$\nabla^2 \Phi + k_{\mathrm{p}}^2 \Phi = 0$$

Making a 3D calculation a formula for the nabla operator gives

$$\nabla \times (\hat{\boldsymbol{z}}\Psi) = \Psi(\nabla \times \hat{\boldsymbol{z}}) + (\nabla\Psi) \times \hat{\boldsymbol{z}} = (\nabla\Psi) \times \hat{\boldsymbol{z}}$$

Application of another formula for the nabla operator then gives

$$\begin{aligned}\nabla \times (\nabla \times (\hat{\boldsymbol{z}}\Psi)) &= \nabla\Psi(\nabla \cdot \hat{\boldsymbol{z}}) - \hat{\boldsymbol{z}}\nabla \cdot \nabla\Psi + (\hat{\boldsymbol{z}} \cdot \nabla)\nabla\Psi - (\nabla\Psi \cdot \nabla)\hat{\boldsymbol{z}} \\ &= -\hat{\boldsymbol{z}}\nabla^2\Psi + \frac{\partial}{\partial z}\nabla\Psi\end{aligned}$$

and a final curl thus yields

$$\nabla \times (\nabla \times [\nabla \times (\hat{\boldsymbol{z}}\Psi)]) = -(\nabla\nabla^2\Psi) \times \hat{\boldsymbol{z}}$$

Operating with $\nabla\times$ on Eq.(1.16) it follows that Ψ (or actually $\nabla\Psi$) must satisfy the Helmholtz equation

$$\nabla^2\Psi + k_{\mathrm{s}}^2\Psi = 0$$

It is stressed that the equation for the S potential now is a scalar equation. These S waves are denoted SV waves (explained below). In a homogeneous solid P and SV waves propagate independently, but in general they couple at surfaces of the domain.

If the 2D problem is rotationally symmetric P and SV waves decouple and the displacement can be taken in the form $\boldsymbol{u} = \hat{\boldsymbol{r}}\, u_r(r)$ and $\boldsymbol{u} = \hat{\boldsymbol{\varphi}}\, u_\varphi(r)$, respectively, and no potentials are needed.

1.2.3 3D potentials

Above it is seen that $\hat{\boldsymbol{z}}\Psi$, where Ψ satisfies the scalar Helmholtz equation with the S wavenumber, is a vector potential in 2D. However, as seen by the calculations, it gives a vector potential also in 3D (and it is noticed that its divergence is then nonzero in general). But in 3D there must be another part to the S waves. One way to construct this part is to take a repeated curl so that the displacement has the following representation

$$\boldsymbol{u} = \nabla\Phi + \nabla \times (\hat{\boldsymbol{z}}\Psi_1) + \nabla \times \nabla \times (\hat{\boldsymbol{z}}\Psi_2) \tag{1.18}$$

Here all three potentials satisfy scalar Helmholtz equations. That the last term is a solution for S waves follows directly as the curl of a (S wave)

solution is always also a solution. The unit vector $\hat{\boldsymbol{z}}$ that is used in the construction can be substituted with any constant vector and this still yields a solution to the elastodynamic equation of motion. This representation is useful in Cartesian and cylindrical coordinates because $\hat{\boldsymbol{z}}$ is a constant unit vector in these systems.

In spherical coordinates the representation in Eq. (1.18) is not useful. However, there is a similar representation which works in this case [99]

$$\boldsymbol{u} = \nabla\Phi + \nabla \times (\boldsymbol{r}\Psi_1) + \nabla \times \nabla \times (\boldsymbol{r}\Psi_2) \tag{1.19}$$

where $\boldsymbol{r} = \hat{\boldsymbol{r}}r = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z$ is the radius vector, which points in the direction of one of the unit vectors in spherical coordinates. To see that the second term is a solution the following calculations are performed (using rules for the nabla operator)

$$\nabla \times (\boldsymbol{r}\Psi_1) = \Psi_1(\nabla \times \boldsymbol{r}) + (\nabla\Psi_1) \times \boldsymbol{r} = -\boldsymbol{r} \times \nabla\Psi_1$$

$$\begin{aligned}\nabla\times(\nabla\times(\boldsymbol{r}\Psi_1)) &= -\boldsymbol{r}(\nabla\cdot\nabla\Psi_1)+\nabla\Psi_1(\nabla\cdot\boldsymbol{r})-(\nabla\Psi_1\cdot\nabla)\boldsymbol{r}+(\boldsymbol{r}\cdot\nabla)\nabla\Psi_1 \\ &= -\boldsymbol{r}\nabla^2\Psi_1 + 3\nabla\Psi_1 - \nabla\Psi_1 + r\frac{\partial}{\partial r}\nabla\Psi_1\end{aligned}$$

where the following formula is used

$$(\nabla\Psi_1 \cdot \nabla)\boldsymbol{r} = \nabla\Psi_1$$

Taking an additional curl the middle two terms vanish. The last term is written out in spherical coordinates

$$r\frac{\partial}{\partial r}\nabla\Psi_1 = \hat{\boldsymbol{r}}\, r\frac{\partial^2\Psi_1}{\partial r^2} + \hat{\boldsymbol{\theta}}\left(\frac{\partial^2\Psi_1}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial\Psi_1}{\partial\theta}\right) + \hat{\boldsymbol{\varphi}}\,\frac{1}{\sin\theta}\left(\frac{\partial^2\Psi_1}{\partial r\partial\varphi} - \frac{1}{r}\frac{\partial\Psi_1}{\partial\varphi}\right)$$

Making an explicit calculation in spherical coordinates it is seen that the curl of this expression vanishes. Thus there remains

$$\nabla \times [\nabla \times (\nabla \times (\boldsymbol{r}\Psi_1))] = -\nabla \times (\boldsymbol{r}\nabla^2\Psi_1)$$

and this shows that the scalar potential Ψ_1 must satisfy the scalar Helmholtz equation with the S wavenumber. The third term in Eq. (1.19) is a solution in the same way as in Eq. (1.18).

The representation in three terms in Eq. (1.18) and Eq. (1.19) gives a division into one P wave (the first term) and two S waves. The S waves are

denoted SH for the second term and SV for the third term. This nomenclature stems from seismology where the "H" stands for "horizontal" and "V" for "vertical", meaning that the second term only has horizontal components (when $\hat{\boldsymbol{z}}$ is vertical), whereas the third term has a vertical component. For spherical problems the original meaning of "horizontal" and "vertical" is lost, but it is convenient to have a way to refer to the three parts. The SH waves often decouple from the SV and P waves in simple scattering problems, whereas the P and SV waves couple, for example in the scattering by a sphere.

For problems in cylindrical or spherical coordinates that are rotationally symmetric the displacement field is independent of the azimuthal angle φ. This type of problem reduces to a simpler SH (torsional) part where $\boldsymbol{u} = \hat{\boldsymbol{\varphi}}\, u_\varphi(r, z)$ in cylindrical coordinates or $\boldsymbol{u} = \hat{\boldsymbol{\varphi}}\, u_\varphi(r, \theta)$ in spherical coordinates. The P and SV waves couple and it is useful to use the representations in Eq. (1.18) or Eq. (1.19) with the middle term omitted.

When using boundary or interface conditions the traction vector is usually involved. On a surface with normal $\hat{\boldsymbol{n}}$ it is

$$t_i^{(n)} = n_j \sigma_{ji} = n_i \lambda \partial_j u_j + \mu n_j (\partial_j u_i + \partial_i u_j)$$

where n_j are the components of $\hat{\boldsymbol{n}}$. In Cartesian coordinates this can be written in vector notation as

$$\boldsymbol{t}^{(n)} = \hat{\boldsymbol{n}}\, \lambda \nabla \cdot \boldsymbol{u} + \mu \left(\frac{\partial \boldsymbol{u}}{\partial n} + \nabla u_n \right)$$

where $u_n = \hat{\boldsymbol{n}} \cdot \boldsymbol{u}$ is the normal component. In cylindrical or spherical coordinates the following form is more useful [which follows from Eq. (1.15)]

$$\boldsymbol{t}^{(n)} = \hat{\boldsymbol{n}}\, \lambda \nabla \cdot \boldsymbol{u} + \mu \left(2\, \frac{\partial \boldsymbol{u}}{\partial n} + \hat{\boldsymbol{n}} \times (\nabla \times \boldsymbol{u}) \right) \tag{1.20}$$

Introducing potentials this expression can be further simplified. Using that $\lambda = \mu(k_{\mathrm{s}}^2/k_{\mathrm{p}}^2 - 2)$ the traction for a P wave with potential Φ becomes

$$\boldsymbol{t}^{(n)} = \mu \left(\hat{\boldsymbol{n}} (2k_{\mathrm{p}}^2 - k_{\mathrm{s}}^2) \Phi + 2\, \frac{\partial \boldsymbol{u}}{\partial n} \right) \tag{1.21}$$

because the last term vanishes for a P wave. For an S wave the first term vanishes so that

$$\boldsymbol{t}^{(n)} = \mu \left(2\, \frac{\partial \boldsymbol{u}}{\partial n} + \hat{\boldsymbol{n}} \times (\nabla \times \boldsymbol{u}) \right) \tag{1.22}$$

Here the second term can be simplified because the curl of one S wave gives the other S wave multiplied by k_{s}.

The way potentials are introduced and traction vectors are calculated for isotropic solids means that the stress components need not be calculated by the equations given for cylindrical and spherical coordinates in Section 1.1.6, and this leads to simpler calculations.

1.3 Anisotropic solids

Many materials, like pure crystals, composites, and biological materials, are anisotropic. Using analytical approaches, scattering problems for such materials can only be solved in special cases, and some of these are taken up in later chapters. Here a few of the simplest classes of anisotropy are introduced; all different classes of anisotropy are defined by Auld [9] and Ting [129]. van der Hijden [132] and Nayfeh [100] extensively investigates the propagation of waves in stratified anisotropic media.

1.3.1 Orthotropic and transversely isotropic solids

An orthotropic material is characterized by three mutually orthogonal symmetry planes. The stress-strain relations for such a material are

$$\begin{aligned}
\sigma_{xx} &= C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} \\
\sigma_{yy} &= C_{12}\varepsilon_{xx} + C_{22}\varepsilon_{yy} + C_{23}\varepsilon_{zz} \\
\sigma_{zz} &= C_{13}\varepsilon_{xx} + C_{23}\varepsilon_{yy} + C_{33}\varepsilon_{zz} \\
\sigma_{yz} &= 2C_{44}\varepsilon_{yz} \\
\sigma_{zx} &= 2C_{55}\varepsilon_{zx} \\
\sigma_{xy} &= 2C_{66}\varepsilon_{xy}
\end{aligned} \tag{1.23}$$

Here a reduced notation for the stiffness constants C_{IJ}, $I, J = 1, 2, 3, 4, 5, 6$, is used, where the stress and strains are collected as vectors with six components (in the order the stresses are given above) and the stiffness tensor is represented by a symmetric six-by-six matrix, see Auld [9] or Ting [129] for more details. An orthotropic material is thus characterized by nine independent stiffness constants. A special case is a cubic material where all the three coordinate directions are equivalent so that $C_{11} = C_{22} = C_{33}$,