An Introduction to Graph Theory and Combinatorics and their Applications

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By

Mukesh Kumar and Mohammad Tamsir

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PRELIMINARY

Floor and Ceiling Functions:

The **floor function** is the function that takes as its input a real number and gives as its output the greatest integer less than or equal to x, denoted as floor(x)=$\lfloor x \rfloor$.

Similarly, the **ceiling function** maps x to the least integer greater than or equal to x, denoted as ceil(x) = $\lceil x \rceil$.

Absolute Value:

For any real number, *x*, the **absolute value** or **modulus** of *x* is denoted by |*x*| (a vertical bar on each side of the quantity) and is defined as

$$f(x) = |x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

Set Notations:

Z = {0, 1, -1, 2, -2, ...} (the set of integers)

N = {0, 1, 2, 3, ...} (the set of non-negative integers)

Z^+ = {1, 2, 3, ...} (the set of positive integers)

Q = {a/b | a, b∈Z ∧ b≠0}

R = the set of real numbers...

Mathematical Induction:

A formal statement of principle of Mathematical Induction is defined as follows:

Let the statement be denoted by S(n); n = 1, 2, 3, …

Suppose the statement S(n) is true for all positive integers, n, provided that

(a) S(1) is true.
(b) S(k + 1) is true if S(k) is true.

Thus, the following three steps prove the given statement using the principle of Mathematical induction:

Step 1: Induction base: To verify that S(1) is true.

Step 2: Inductive hypothesis: Suppose S(k) is true for an arbitrary value of k.

Step 3: Inductive step: We want to prove that S(k + 1) is true on the basis of the inductive hypothesis.

Principle of Strong Mathematical Induction:

For a given statement, S(n), involving a natural number, n, is true for all positive integers $n \geq m$ if

(a) S(1) is true.

(b) S(k + 1) is true if S(k) is true for $1 \le n \le k$.

The three steps for proofs by principle of strong mathematical induction are as follows:

Step 1: Inductive Base: To prove S(1) is true.

Step 2: Strong Inductive Hypothesis: Suppose S(n) is true for all integers $1 \le n \le k$.

Step 3: Inductive Step: We want to show that S(k + 1) is true on the basis of step 2.

CHAPTER 1

INTRODUCTION TO GRAPH THEORY

1.1 Introduction

The graphs are very useful in everyday life. Suppose a salesman wants to visit five cities, A, B, C, D and E. The cities are connected by roads, R_1, R_2, R_3, R_4, R_5, R_6, R_7 and R_8, where R_1 is the road between cities A and B, R_2 between cities B and C, R_3 between cities C and D, R_4 between cities B and D, R_5 between cities B and E, R_6 between cities A and E, R_7 between cities C and E, R_8 between cities E and D, then this situation can be described by means of a diagram as given in Fig. 1.1.

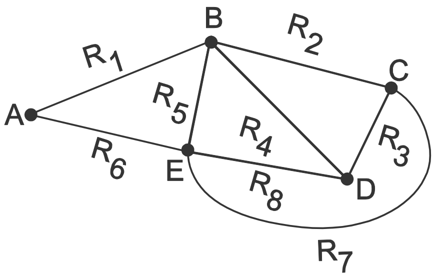


Fig. 1.1

If we denote the cities by vertices and roads by edges, then this forms a graph. Graphs can represent many real-world situations. In mathematics, a graph is a way of describing a network.

The concept of the graph was raised in the eighteenth century when Euler represented the situation of the Königsberg Bridge problem using a graph. The Königsberg Bridge problem was that there were two islands, C and D, formed by the Pregel river in Königsberg city. The islands were connected to each other and to the banks, A and B, with seven bridges, as shown in Fig. 1.2.

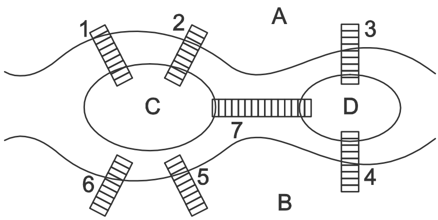


Fig. 1.2 The Königsberg Seven Bridge Problem

The problem was whether, starting from any land area, A, B, C, or D, one could walk over each of the seven bridges exactly once and return back to their starting point. In 1736, Leonhard Euler represented this problem graphically, as shown in Fig. 1.3.

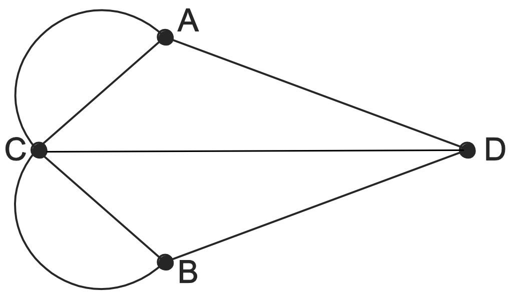
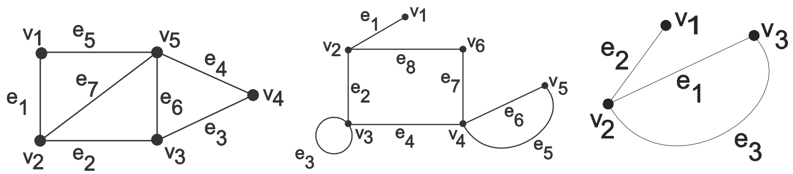


Fig. 1.3 A Graphical Representation of the Königsberg Seven Bridge Problem

The vertices represent the land areas, A, B, C and D, and the bridges are represented by the edges. So, every graph consists of two sets. One set is the vertices, V, and the other is the edges, E.

1.2 Graph*

A Graph, $G(V, E)$, consists of a set of pairs, (V, E), where $V = \{v_1, v_2, v_3, \ldots\}$ is the set of vertices and $E = \{e_1, e_2, e_3, \ldots\}$ is the set of edges, such that each edge, e_k, is connected with a pair of vertices, (v_i, v_j). The vertices v_i and v_j are called the end points of the edge e_k. For instance, some examples of the graphs are shown in Fig. 1.4 (a), 1.4 (b) and 1.4 (c).



(a) (b) (c)

Fig. 1.4 Some Examples of Graphs

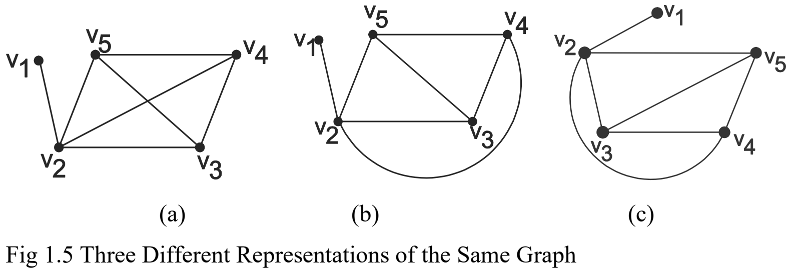
If two or more edges have the same end points, i.e., are connected by the same pair of vertices (v_i, v_j), then such edges are called parallel edges or multiple edges. In Fig. 1.4 (b), edges e_5 and e_6 are parallel edges because these edges are connected by same pair of vertices, v_4 and v_5. In Fig. 1.4 (c), edges e_1 and e_3 are parallel edges.

* A graph with a finite number of vertices, as well as the finite number of the edges, is called a finite graph; otherwise, it is called an infinite graph. In this book, we study only finite graphs, and so the term "graph" will always mean a finite graph.

An edge having the same vertex, v_i, as both its end vertices is called a self-loop. In Fig. 1.4 (b), edge e_3 is a self-loop because it has the same end vertex, v_3. In other words, an edge connected with the vertex pair, (v_i, v_j), is called self-loop if $i = j$.

The edge, e_k, joining the vertices (v_i, v_j) is called an incident to the vertices v_i and v_j. In Fig. 1.4 (a), the edge e_1 is incident with the vertices v_1 and v_2. If two or more edges join the common vertex v_i, then the edges are incident on the vertex v_i. Two non-parallel edges are said to be adjacent if they are incident on a common vertex v_i. In Fig. 1.4 (a), the edges e_1 and e_5 are adjacent edges since both are incident on common vertex v_1. Two vertices are said to be adjacent to each other if they are connected by an edge. In Fig. 1.4 (a), the vertices v_1 and v_2 are adjacent vertices since both are connected by an edge, e_1.

Note: In drawing a graph, it does not matter whether straight lines or curves join two vertices, or if the lengths of the edges are short or long, but the incidence between the edges and vertices is essential. For example, the graphs shown in Fig. 1.5 are the same because the incidence between the edges and vertices is the same in all three cases.



(a) (b) (c)

Fig 1.5 Three Different Representations of the Same Graph

Example 1.1: Find self-loops and parallel edges in the following graph:

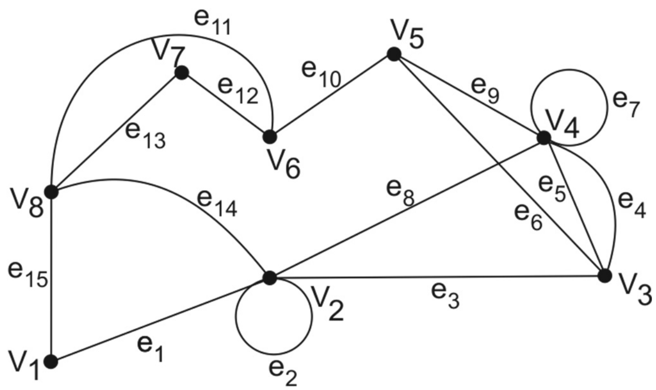


Fig. 1.6

Solution: Here, the edges e_2 and e_7 are self-loops because e_2 is connected with vertex pair (v_2, v_2) and e_7 is connected with vertex pair (v_4, v_4). The edges e_4 and e_5 are parallel edges since both have same end vertices, v_4 and v_3.

1.2.1 Subgraphs

A graph $H(V', E')$ is said to be a subgraph of graph $G(V, E)$ if $V' \subseteq V$, $E' \subseteq E$ and each edge of H have the same end vertices in H as in G. For example, the graphs given in Fig. 1.8 (a) and Fig. 1.8 (b) are subgraphs of graph G given in Fig. 1.7.

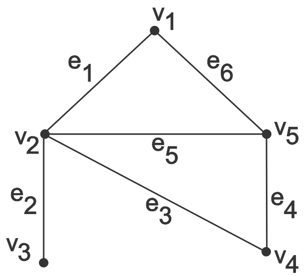


Fig 1.7 Graph G

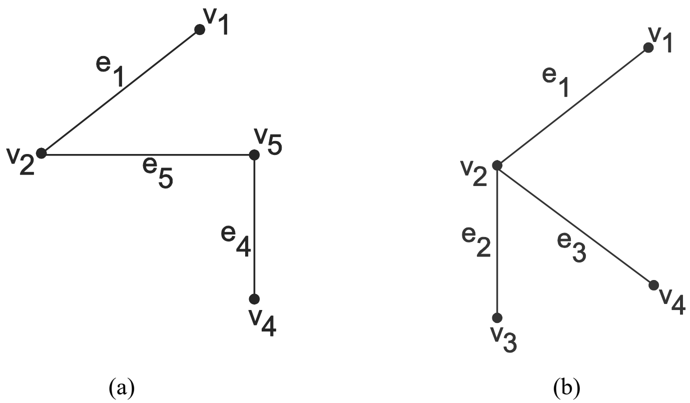


Fig. 1.8 Two Subgraphs of Graph G

Remarks:

1. Every graph is its own a subgraph.
2. A single vertex of graph G is also a subgraph of G.

1.3 Degree of a Vertex

The number of edges incident on a vertex, v_i, is called the degree of a vertex, v_i. In other words, the degree of a vertex is equal to the number of the edges connected by that vertex (note that self-loops are counted twice). It is denoted by deg(v_i), d(v_i) or d_i. The minimum possible degree of a vertex in a graph is zero. For instance, the degrees of the vertices v_2, v_3 and v_5 in the graph given in Fig. 1.9 are 4, 4 and 2, respectively.

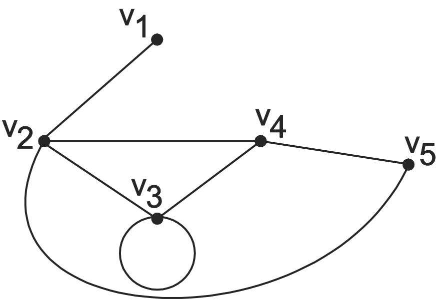


Fig. 1.9

1.3.1 Degree Sequence

The list of degrees of all vertices of a graph in ascending or descending order is called the degree sequence. "For instance, the degree sequence of the graph given in Fig. 1.9 is {1, 2, 3, 4, 4} or {4, 4, 3, 2, 1}."

1.4 Isolated Vertex and Pendant Vertex

A vertex, v_i, having degree zero is called an isolated vertex and a vertex, v_j, having degree one is called a pendant vertex. For instance, the vertices v_1

and v_6 are isolated vertices in the graph given in Fig. 1.10, while the vertex v_5 is a pendant vertex.

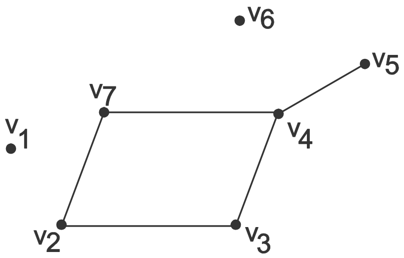


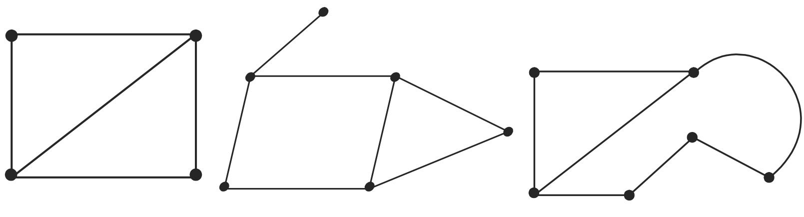
Fig. 1.10

1.5 Types of Graphs

In this section, we will describe the various types of graphs.

1.5.1 Simple Graph

A graph with no self-loops nor parallel edges is called a simple graph, i.e., a graph is called simple if it has no self-loops and no two of its edges join the same pair of vertices. For instance, the graphs shown in Fig. 1.11 (a), (b) and (c) are simple graphs.

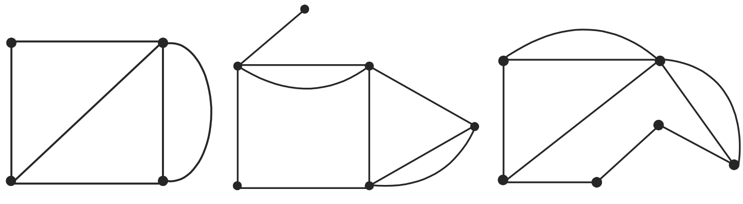


(a) (b) (c)

Fig. 1.11 Examples of Simple Graphs

1.5.2 Multi Graph

A graph that has some parallel edges is called a multigraph. For instance, the graphs shown in Fig. 1.12 (a), (b) and (c) are multigraphs.

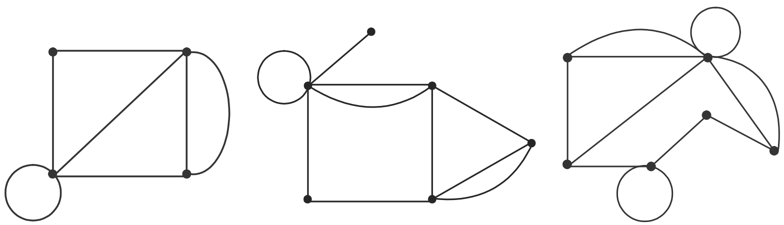


(a) (b) (c)

Fig. 1.12 Some Examples of Multi Graphs

1.5.3 Pseudo Graph

A graph that has a self-loop(s) as well as a parallel edge(s) is called a pseudo graph. For instance, the graphs shown in Fig. 1.13 (a), (b) and (c) are examples of pseudo graphs.



(a) (b) (c)

Fig. 1.13 Examples of Pseudo Graphs

1.5.4 Null Graph

A graph, $G(V, E)$, is called null if each vertex, v_i, of the graph G is an isolated vertex, i.e., the degree of each vertex of G is zero. In other words, a graph, G, containing no edges is called a null graph. For instance, the graph given in Fig. 1.14, with five vertices, v_1, v_2, v_3, v_4 and v_5, is a null graph because the degree of each vertex is zero or there is no edge in the graph.

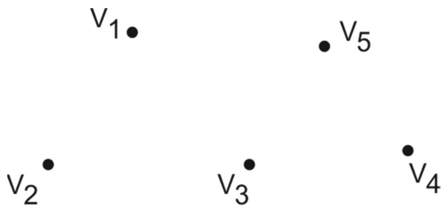


Fig. 1.14

1.5.5 Complete Graph

A simple graph, $G(V, E)$, is called a complete graph if there is an edge between every pair of vertices, (v_i, v_j). In other words, a simple graph, $G(V, E)$, of n vertices is called complete if the degree of each vertex, v_i, is equal to $(n - 1)$. A complete graph is denoted by K_n, where n represents the number of vertices. For instance, the complete graphs K_1, K_2, K_3, K_4, K_5 and K_6 with one, two, three, four, five, and six vertices, respectively, are shown in Fig. 1.15 (a), (b), (c), (d), (e) and (f).

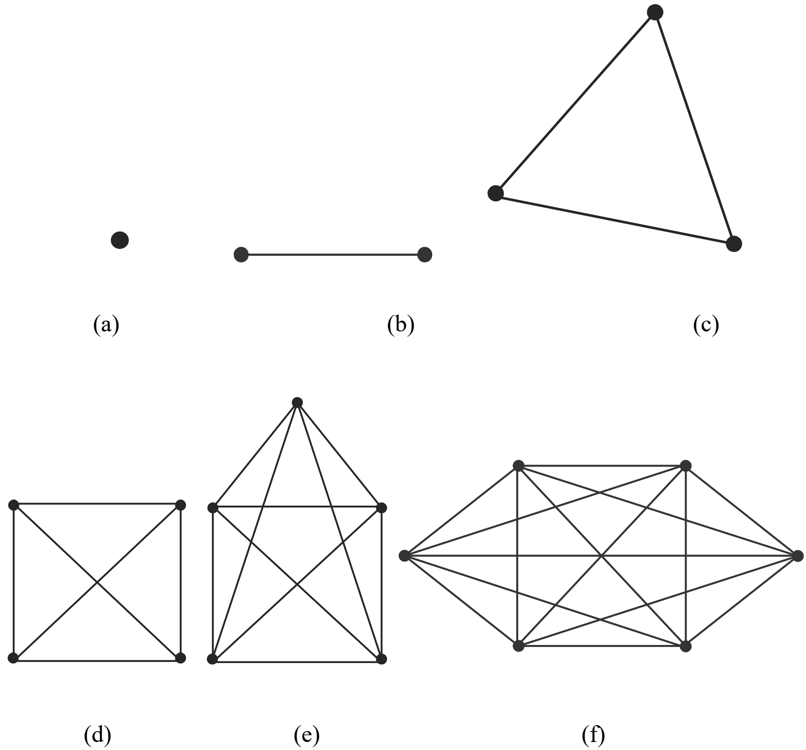
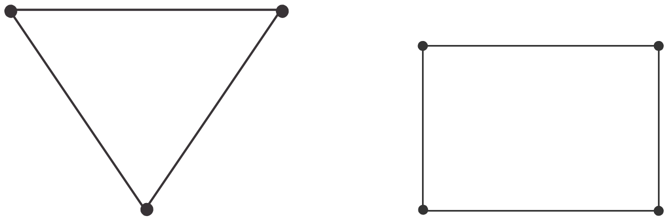


Fig. 1.15

1.5.6 Regular Graph

A simple graph, $G(V,E)$, is called a regular graph if the degrees of all vertices are equal. A graph is k - regular or regular of degree k if every vertex has degree k and is denoted by k -regular. The graphs given in Fig. 1.16 (a) and (b) are 2-regular.



(a) (b)

Fig. 1.16 The 2-regular Graphs with (a) 3 and (b) 4 Vertices

1.5.7 Bipartite Graph

A graph, $G(V, E)$, is called bipartite, if there are exist subsets V_1 and V_2 of vertex set V, such that $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 = V$, and each edge, e_k, has one end point in V_1 and its other end point in V_2. For instance, the graph given in Fig. 1.17 is a bipartite graph because if we suppose two subsets, $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5\}$, of the vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$, then $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 = V$, and each edge of G has one end point in V_1 and its other in V_2.

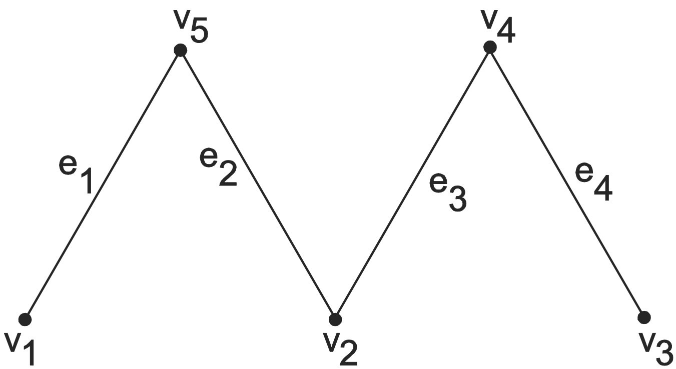


Fig. 1.17 Bipartite Graph

Remark: If a graph, G, is bipartite, then each edge, e_k, of G has one end point in the subset V_1 and its other in V_2. It does not mean that, if v_i is any vertex of the subset V_1 and v_j is any vertex of the subset V_2, there is an edge between v_i and v_j. For instance, the graph in Fig. 1.18 is bipartite because if we suppose two subsets, $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$, of $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, then $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 = V$, and each edge of G has one end point in V_1 and its other in V_2 but there is no edge between vertices v_3 and v_5.

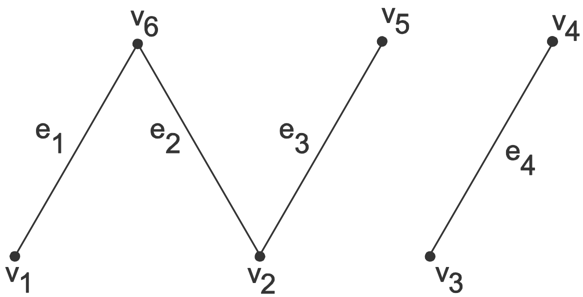


Fig. 1.18

Example 1.2: Show that the graph G, given in Fig 1.19, is not bipartite.

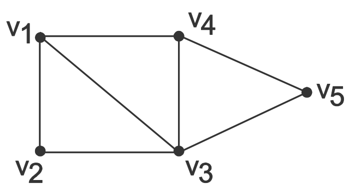


Fig. 1.19

Proof: It can be proved easily arguing by contradiction. If the graph G, given in Fig. 1.19, is bipartite, then the vertex set $V = \{v_1, v_2, v_3, v_4\}$ can be partitioned into two subsets, V_1 and V_2, such that $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 = V$ and each edge of G have one end point in subset V_1 and their other in subset V_2. Now, consider the vertices v_1, v_2 and v_3. Since the vertices v_1 and v_2 are adjacent, one vertex must be in subset V_1 and the other in subset V_2. Let the vertex v_1 be in subset V_1 and the vertex v_2 be in subset V_2. The vertices v_1 and $v_3$are also adjacent. Since vertex v_1 is already in subset V_1, we suppose that vertex v_3 is in subset V_2. Now, the vertices v_2 and v_3 are also adjacent but both are in the subset V_2, which is a contradiction of the definition of a bipartite graph; hence, the given graph is not bipartite.

1.5.8 Complete Bipartite Graph

A bipartite graph, $G(V, E)$, with m vertices in subset V_1 and n vertices in subset V_2 (where V_1 and V_2 are two disjoint subsets of vertex set V) is called a complete bipartite graph if it has all possible edges between all pairs of vertices, v_i and v_j, where $v_i \in V_1$ and $v_j \in V_2$. It is denoted by $K_{m,n}$. For instance, a complete bipartite graph, $K_{3,2}$, is shown in Fig. 1.20.

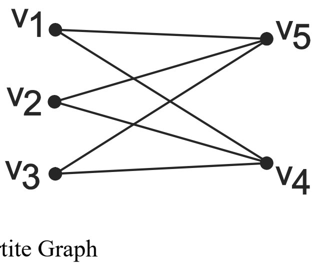


Fig. 1.20 Complete Bipartite Graph

1.5.9 Weighted Graph*

A graph, $G(V, E)$, is called a weighted graph if each edge of $G(V, E)$ is labeled by a numerical weight. For example, the graph given in Fig. 1.21 is a weighted graph.

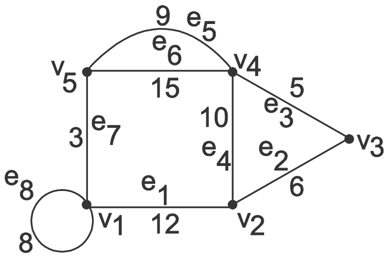


Fig. 1.21 A Weighted Graph

In this graph, the weight of edge e_1 is 12.

1.6 Type of Subgraphs

In this section, we will define some types of subgraphs of a graph.

1.6.1 Vertex Disjoint Subgraph

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two subgraphs of a graph $G(V, E)$. Then, $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are called vertex disjoint subgraphs of $G(V, E)$ if there is no common vertex between $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, i.e., $V_1 \cap V_2 = \phi$, where V_1 and V_2 are sets of vertices of subgraphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, respectively. For instance, the subgraphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, as given in Fig. 1.23 (a) and (b), are vertex disjoint subgraphs of the graph $G(V,E)$ given in Fig. 1.22. Mathematically, $V_1 = \{v_3, v_4, v_5\}$, $V_2 = \{v_1, v_2\}$ and $V_1 \cap V_2 = \phi$.

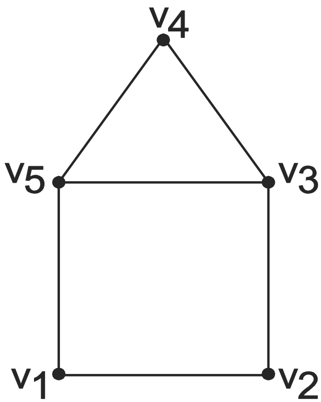


Fig. 1.22 Graph $G(V,E)$

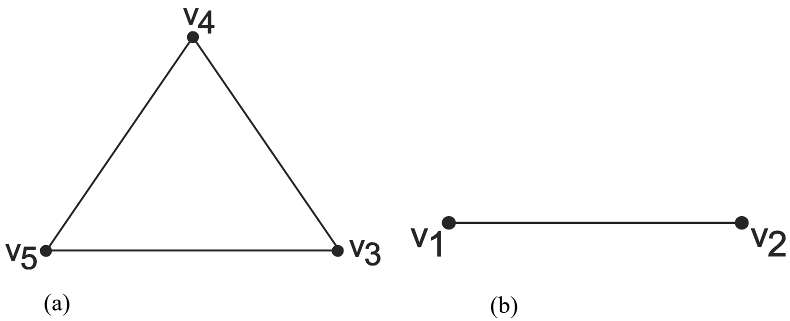


Fig. 1.23 Two Subgraphs, G_1 and G_2, of Graph G

1.6.2 Edge Disjoint Subgraphs

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two subgraphs of graph $G(V,E)$. Then, $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are called edge disjoint subgraphs of $G(V,E)$ if there is no common edge between $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, i.e., $E_1 \cap E_2 =$

ϕ. For instance, let $G(V,E)$ be a graph, as given in Fig. 1.24. If $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ are two subgraphs of $G(V,E)$, as shown in Fig. 1.25 (a) and (b), then $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ are edge disjoint subgraphs of G since $E_1 = \{e_3, e_4, e_5\}$, $E_2 = \{e_1, e_2, e_6\}$ and $E_1 \cap E_2 = \phi$.

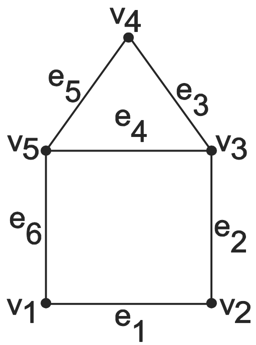
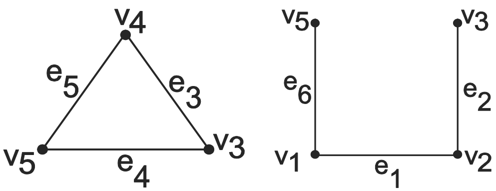


Fig. 1.24 Graph G



(a) $G_1(V_1,E_1)$ (b)$G_2(V_2,E_2)$

Fig. 1.25 Two Edge Disjoint Subgraphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ of Graph G

Theorem 1.1 (Handshaking Lemma): In any graph, the sum of degrees of all vertices is equal to twice the number of edges, i.e., $\sum_{i=1}^{n} d_i = 2 \times \text{Number of edges}$.

Proof: If $\boldsymbol{d_i}$ is the degree of vertex $\boldsymbol{v_i}$ then the degree $\boldsymbol{d_i}$ counts the number of edges incident on vertex $\boldsymbol{v_i}$. Since each edge has two end points, i.e., each edge is incident on two vertices, the same $\sum_{i=1}^{n} \boldsymbol{d_i}$ counts each edge twice and, hence, the sum of degrees of all vertices is equal to twice the number of edges.

Example 1.3: A graph, G, has sixteen edges, one vertex of degree 2, two vertices of degree 3 and its other vertices are of degree 4. Find the total number of vertices.

Solution:

If the total number of vertices $= \ n$**,** then the number of vertices of degree 4 $= \ n - (1 + 2)$

$= \ n - 3.$

Now, we know that

$$\sum_{i=1}^{n} d_i = 2 \times Number\ of\ edges$$

$$\Rightarrow 1 \times 2 + 2 \times 3 + \ (n - 3) \ \times 4 \ = \ 2 \times 16,$$

$$\Rightarrow \ 2 + 6 + 4n \ - 12 \ = \ 32,$$

$$\Rightarrow \ 4n \ = 36,$$

$$\Rightarrow \ n \ = \ 9.$$

Theorem 1.2: The number of vertices of odd degree in a graph is always even.

Proof: Let $G(V,E)$ be a graph with n vertices viz. $v_1, v_2, v_3, \ldots, v_n$. Let the vertices v_j, $j = 1,2,3,\ldots,m$ be odd degree and the vertices v_k, $k = m+1, m+2, \ldots, n$ be even degree.

Now, by Handshaking Lemma

$$\sum_{i=1}^{n} d_i = 2 \times \text{Number of edges} \tag{1}$$

The left side of equation (1) can be expressed as a sum of two sums of odd and even degree vertices, respectively, as

$$\underbrace{\sum_{j=1}^{m} d_j}_{\substack{odd \\ degree}} + \underbrace{\sum_{k=m+1}^{n} d_k}_{\substack{even \\ degree}} = 2 \times \text{Number of edges},$$

$$\Rightarrow \sum_{j=1}^{m} d_j = 2 \times \text{Number of edges} - \sum_{k=m+1}^{n} d_k \tag{2}$$

Since $\sum_{k=m+1}^{n} d_k$ is the sum of degrees of even degree vertices, the sum of $\sum_{k=m+1}^{n} d_k$ is always even. Now, 2 × Number of edges is also an even number. Hence, the right side of equation (2) is an even number. Thus, equation (2) can be written as

$$\sum_{j=1}^{m} d_j = \text{an even number} \tag{3}$$

Since, each $d_j, j = 1,2,3,\ldots,m$ is an odd number, the total number of terms on the left side of equation (3) must be even to make the sum even. Hence, the number of odd degree vertices in a graph is always even.

Theorem 1.3: The maximum degree of any vertex in a simple graph with n vertices is $(n-1)$.

Proof: Let $G(V,E)$ be a simple graph with n vertices. Since the degree of any vertex, v_i, is equal to the number of edges incident on the vertex v_i, a simple graph does not contain parallel edges and self-loops. So, there can be at most $(n-1)$ edges incident on the vertex v_i. Hence, the maximum degree of any vertex in a simple graph with n vertices is $(n-1)$.

Theorem 1.4: The maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Proof: If we let $G(V,E)$ be a simple graph with n vertices, we know that the maximum degree of any vertex, v_i, in a simple graph is equal to $(n-1)$.

Therefore, the sum of degrees of n vertices in a simple graph $\leq (n-1)+(n-1)+\ldots+(n-1)$ (n times).

$$\sum_{j=1}^{n} d_j \leq n(n-1), \tag{1}$$

Now, by Handshaking Lemma

$$\sum_{i=1}^{n} d_i = 2 \times \text{Number of edges}. \tag{2}$$

From equations (1) and (2), we get

$$2 \times \text{Number of edges} \leq n(n-1),$$