Command and Diagnosis of Dynamic Systems

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By

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Cambridge
Scholars
Publishing

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This book first published 2024

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN: 978-1-0364-0876-3
ISBN (Ebook): 978-1-0364-0877-0

Preface

Until recent years and still today, the primary objective of automatic control has been, and rightly still is, the improvement of dynamic and static performance in systems. This has led to the emergence of new, highly efficient control methods, albeit with more cumbersome and complex implementations than traditional PIDs.

It is well known that an increase in the complexity of a system is accompanied by a greater vulnerability to various failures that can occur in each of its components, generally resulting in a loss of installation availability. This loss of availability has obvious economic consequences. On the other hand, enhancing dynamic performance leads to greater and more frequent demands on various installation components, thereby increasing the risk of failure.

Therefore, it appears that merely increasing the performance of a system is insufficient; it is also necessary to minimize its vulnerability to failures to preserve availability—ensuring it can fulfill its intended function.

Taking into account both control and availability aspects leads to a two-level vision of the system:

- A control level ensuring proper behavior of the controlled system.
- A level of preventive maintenance allowing for the monitoring of the system.

The control level aims to modify the natural dynamic and static behavior of the system to achieve satisfactory performance for the task at hand. The level of preventive maintenance ensures installation availability, maintaining performance despite potential faults.

That being said, the problem boils down to synthesizing a control law to ensure that the controlled system achieves the desired dynamic and static performances, along with designing preventive maintenance to ensure installation availability.

It becomes apparent that the subjects covered in this work are of considerable practical importance and interest to all individuals involved in improving the performance of production machines and significantly enhancing installation safety from the perspectives of people, the environment, and property.

With this work, we aim to provide the reader with the main tools necessary to address a control and preventive maintenance problem for a given system. While an exhaustive presentation of these aspects in a single book is hardly realistic, the fundamental principles are present, and the provided bibliography should enable the reader to explore any aspect they find useful.

To this end, the book will include numerous examples and numerical applications, as well as exercises with answers.

This book is intended for engineering students, as well as master's and doctoral students, enabling them to assimilate the primary methods needed to undertake a control and preventive maintenance project for a given system.

This work is also valuable to engineers responsible for the design or implementation of a production site, and generally to all individuals involved in the fields of modeling, control, and maintenance of industrial systems, seeking to refresh or augment their knowledge.

Rosario Toscano
La Ricamarie, France
November 2023

Contents

Notation and Acronyms

Sets

$\mathbf{R}$	set of real numbers
$\mathbf{R}^n$	set of real column vectors with n entries
$\mathbf{R}^{n\times m}$	set of real matrices of dimension $n \times m$
$\mathbf{C}$	set of complex numbers
$\mathbf{C}^-$	open left-half plane
$\mathbf{C}^n$	set of complex column vectors with n entries
$\mathbf{C}^{n\times m}$	set of complex matrices of dimension $n \times m$
$\mathcal{H} \subset \mathbf{R}^{n\times n}$	set of real Hurwitz matrices
$\mathbf{S}^n$	set of real symmetric matrices of size n, *i.e.*, $\mathbf{S}^n = \{S \in \mathbf{R}^{n\times n} : S = S^T\}$
$\mathcal{P}_{\mathcal{H}}$	set of Hurwitz polynomials
$\mathbf{L}_1$	set of absolute-value integrable signals
$\mathbf{L}_2$	set of square integrable signals
$\mathbf{L}_\infty$	set of signals bounded in amplitude
$\mathbf{RH}_2^{n\times m}$	set of strictly proper and stable transfer matrices of dimension $n \times m$
$\mathbf{RH}_\infty^{n\times m}$	set of proper and stable transfer matrices of dimension $n \times m$

Relational Operators

$=$	equal to
$\approx$	approximately equal to
$<$	less than
$\leq$	less than or equal to
$\ll$	much less than
$>$	greater than
$\geq$	greater than or equal to
$\gg$	much greater than
$\prec$, $\preceq$	component-wise inequalities between vectors
$<_e$, $\leq_e$	component-wise inequalities between matrices
$\Rightarrow$	implies
$\Leftrightarrow$	is equivalent to

Miscellaneous

s	the complex Laplace variable
j	the imaginary unit $\sqrt{-1}$
π	the ratio of a circle's circumference to its diameter $\pi \approx 3.1416$
exp(.)	exponential of the quantity passed in argument also denoted $e^{(.)}$
ln(.)	the natural (or Napierian) logarithm of the quantity passed in argument
$\in$	belongs to
$\subset$	subset of
$\cup$	union
$\exists$	there exists
$\forall$	for all
:	such that
Re(.)	real part of the complex number passed in argument
Im(.)	imaginary part of the complex number passed in argument
$x \in [a,\, b]$	$a \leq x \leq b$, where $a, x, b \in \mathbf{R}$
$\lim_{x \to a} f(x)$	the value of $f(x)$ in the limit as x tends to a
$\nabla f(x)$	gradient vector of $f(x)$, $\nabla f(x) = \left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right)^T$, $x = (x_1, \cdots, x_n)^T$
$\nabla_x f(x, y)$	gradient vector of $f(x, y)$ with respect to the vector x
$[G,\, K]$	standard feedback interconnection of system G and controller K

Matrix Operations

0	zero matrix of compatible dimension
I	identity matrix of compatible dimension
I_n	identity matrix of dimension $n \times n$
A^T	transpose of matrix A
A^*	conjugate transpose of the complex matrix A
A^{-1}	inverse of matrix A
A^{-T}	denotes $(A^{-1})^T$ or equivalently $(A^T)^{-1}$)
$\det(A)$	determinant of matrix A
$\text{diag}(v)$	diagonal matrix with the elements of the vector v on the main diagonal
$\text{diag}(A_1, \cdots, A_n)$	block-diagonal matrix with matrices A_i on the main diagonal
$\text{rank}(A)$	rank of matrix A
$\text{trace}(A)$	trace of matrix A
$\text{vect}(A)$	vector of the column vectors of the matrix A
$\text{vect}^d(A)$	vector of the diagonal elements of the square matrix A
$\mathcal{F}_l(G, K)$	lower linear fractional transformation of matrices G and K
$\mathcal{F}_u(G, K)$	upper linear fractional transformation of matrices G and K
$A \succ 0$	symmetric matrix $A = A^T$ with strictly positive eigenvalues
$A \succeq 0$	symmetric matrix $A = A^T$ with non-negative eigenvalues
$A \prec 0$	symmetric matrix $A = A^T$ with strictly negative eigenvalues
$A \preceq 0$	symmetric matrix $A = A^T$ with non-positive eigenvalues
$A \prec B$	denotes $(A - B) \prec 0$

Measure of Size

$\lambda_i(A)$	i^{th} eigenvalue of the matrix A
$\bar{\lambda}(A)$	largest eigenvalue of the symmetric matrix A
$\underline{\lambda}(A)$	smallest eigenvalue of the symmetric matrix A
$\sigma_i(A)$	i^{th} singular value of the matrix A
$\bar{\sigma}(A)$	largest singular value of the matrix A
$\underline{\sigma}(A)$	smallest singular value of the matrix A
$\lvert x \rvert$	modulus (or magnitude) of $x \in \mathbf{C}$
$\lVert x \rVert$	Euclidean norm of the real or complex vector x, also denoted $\lVert x \rVert_2$
$\lVert u \rVert_1$	1-norm of the signal u
$\lVert u \rVert_2$	two-norm of the signal u
$\lVert u \rVert_\infty$	infinity-norm of the signal u
$\lVert G \rVert_2$	two-norm of transfer matrix $G \in \mathbf{RH}_2$
$\lVert G \rVert_\infty$	infinity-norm of the transfer matrix $G \in \mathbf{RH}_\infty$
$\mu_\Delta(M)$	structured singular value of the matrix M with respect to a given uncertainty structure Δ
$b_{G,K}$	robust stability margin for system G and controller K

Acronyms

LTI	Linear Time Invariant
MIMO	Multi Input Multi Output
PID	Proportional Integral Derivative
SOF	Static Output Feedback
SISO	Single Input Single Output

Part I
Introduction to Modeling Analysis and Control of Dynamic Systems

Chapter 1
Modeling of Dynamic Systems

1.1 Notion of System

The notion of system play a central role in all area of engineering sciences and need thus to be clearly defined. We call *system*, any entity having inputs and outputs related by *causality*. By causality, we mean that any action on the inputs causes a reaction on the outputs. This system concept is very general and can be applied to everything that surrounds us. In this book, we shall focus on man-made systems, designed to accomplish a desired function.

Figure 1.1 shows the usual structure of a man-made system, it include *actuators* (and their power interfaces) allowing to act appropriately on the so called *operative part* which is designed to realize the desired operation. This operative part is observed through physical variables, denoted $y_s(t)$, measured by the *sensors* and associated signal conditioning circuits. Thus we can act on the system through the *control input vector* $u(t)$, and we know the situation of the system via the *measured output vector* $y_m(t)$.

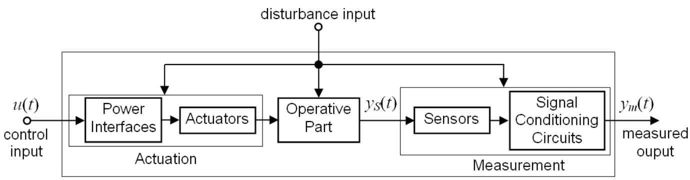

Fig. 1.1 General structure of a system to be controlled. The actuators make it possible to act on the operative part. The sensors give a measurement of the actual situation of this operative part. This measurement can then be used to control the system.

It is important to distinguish the control input vector denoted u which is the manipulated variable and the *disturbance input* (including noise measurement), which represent the undesired influence of the environment on the system. Consequently, if we want to maintain the output to a desired reference value (time varying or not), we have to act on the control input in order to compensate the undesired output deviations caused by the unmeasured disturbances. This is the *closed-loop control* principle which is the subject of the next chapter.

Any design method involved in the field of engineering sciences requires some knowledge about the behavior of the physical system under consideration. This knowledge is generally expressed as a mathematical description of the real system which is called the model of the system. In the next section some guidelines are given to obtain a mathematical model of a system.

- Voluntary actions, also referred to as the control inputs, denoted u.
- Involuntary actions, known as the disturbances, denoted d.
- The responses to these external inputs (u and d), these are the outputs denoted y.

The applied inputs drive the evolution of the system. The resulting output values reflect this evolution. However, inputs and outputs alone are insufficient to fully characterize the system's evolution; states must be considered. States are internal variables that constitute the memory of the system, condensing all information about its past. Knowing the state of the system at a specific initial moment is sufficient to predict its future behavior in response to known applied inputs.

1.2 Model of a System

The design of a control law and/or the implementation of a diagnosis procedure requires having a model of the system. Generally speaking, a model is an abstract representation allowing us to aggregate all the knowledge we have about the system. The models considered in the following are mathematical representations, making it possible to calculate the evolution of the system's outputs, when we know the evolution of its inputs and the initial conditions of this evolution.

1.2.1 State Space Model of a System

A large class of physical models has the property of representing a system with a set of first-order differential equations, together with a set of algebraic equations of the following form, called the *state space model of the system.*

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbf{R}^{n_x}$ is known as the *state vector*, $u(t) \in \mathbf{R}^{n_u}$ represents the *inputs vector*, which includes the control input vector $u_c(t)$ as well as the exogenous disturbances $d(t)$, *i.e.,* $u(t) = [u_c(t), d(t)]^T$, and $y(t) \in \mathbf{R}^{n_y}$ is the *output vector*. The state variables which are the components of the state vector *i.e.*,

$$x(t) = [x_1(t), \ldots, x_n(t)] \tag{1.2}$$

make it possible to determine the future evolution of the system knowing the initial conditions of the state variables and the applied inputs.

The non linear vector function $f(x(t), u(t))$ is typically referred to as the evolution function while the non linear function g(x(t),u(t)) is usually called the output equation, we have

$$\begin{aligned} f(x(t), u(t)) &= [f_1(x_1(t), u(t)), \ \ldots, \ f_{n_x}(t), u(t))]^T \\ g(x(t), u(t)) &= [g_1(x(t), u(t)), \ \ldots, \ g_{n_y}(x(t), u(t))]^T \end{aligned} \tag{1.3}$$

If they explicitly depend on time, the system is said to be *time-dependent*. On the contrary, the system is said to be *time-independent*. In what follows, only time-independent systems will be considered.

A special case of great practical importance is that of linear systems. A system is considered linear if it satisfies the principle of superposition[1]. In thise case, whe have

$$\begin{cases} f(x(t), u(t)) = Ax(t) + Bu(t) \\ h(x(t), u(t)) = Cx(t) + Du(t) \end{cases} \tag{1.4}$$

where $A \in \mathbf{R}^{n_x \times n_x}$ is the *state matrix* also known as the *evolution marix*, $B \in \mathbf{R}^{n_x \times n_u}$ is the *input matrix*, $C \in \mathbf{R}^{n_y \times n_u}$ is the *output matrix* and $D \in \mathbf{R}^{n_y \times n_u}$ is the *feedthrough, or feedforward matrix*. This matrix is usually qual to zero because the output vector cannot react instantaneously to the input vector. When this matrice is non-zero this is only at the price of approximation (fast dynamics neglected with respect slow dynamis).

• **Example I.1, Linearizarion of a Robot Arm.** Consider the system depicted in figure 1.2, it is a robot arm, assumed to be rigid, rotating in the plane P, around the axis z which is perpendicular to it. This axis is equipped with a motor making it possible to apply a variable torque u depending on

[1] Consider k input vectors $u_1(t)$, $u_2(t)$,...,$u_k(t)$ and let $y_i(t)$ be the system response to the input $u_i(t)$ $(i = 1, \ldots, k)$. The system is said to satisfy the principle of superposition if any input $u(t)$, which is a linear combination of the $u_i(t)$ *i.e.,* $u(t) = \sum_{i=1}^{k} \alpha_i u_i(t)$, $\forall \alpha_i \in \mathbf{R}$, produce the output $y(t) = \sum_{i=1}^{k} \alpha_i y_i(t)$.

the user, u is the system control variable. The position of the arm, relative to the vertical, is identified by the angle θ. Applying the fundamental principle

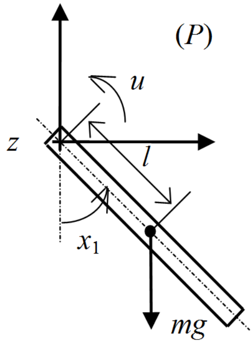

Fig. 1.2 Robot arm.

of dynamics to rotating bodies leads to the following differential equation: $J\ddot{\theta}(t)+b\dot{\theta}(t)+mgl\sin\theta(t) = u(t)$. By setting $\omega(t) = \dot{\theta}(t)$, we obtain a system of two first order differential equations:

$$\begin{cases} \dot{\theta}(t) = \omega(t) \\ \dot{\omega}(t) = \dfrac{1}{J}u(t) - \dfrac{b}{J}\omega(t) - \dfrac{mgl}{J}\sin\theta(t) \end{cases} \tag{1.5}$$

where m is the mass of the arm, J its moment of inertia relative to the axis of rotation, l the distance from the center of gravity to the axis, b the coefficient of viscous friction and g the acceleration due to gravity. The angular position θ and the angular velocity ω are the state variables of the system, which are also the outputs.

It can be useful, for example, for simulation or numerical control purposes, to discretize the model (1.1). This results in a textitdiscrete state space model in the following form:"

$$\begin{aligned} x(k+1) &= f_d(x(k),\, u(k)) \\ y(k+1) &= g_d(x(k),\, u(k)) \end{aligned} \tag{1.6}$$

where $x(t = k) \in \mathbf{R}^{n_x}$ is known as the *discrete state vector*, $u(k) \in \mathbf{R}^{n_u}$ represents the *discrete inputs vector*, which includes the control input vector $u_c(k)$ as well as the exogenous disturbances $d(k)$, *i.e.*, $u(k) = [u_c(k),\, d(k)]^T$, and $y(k) \in \mathbf{R}^{n_y}$ is the *discrete output vector*. The state variables which are the components of the state vector *i.e.*,

$$x(k) = [x_1(k), \ldots, x_n(k)] \tag{1.7}$$

has in the continuous case, it is possible to determine the future evolution of the system knowing the initial conditions of the state variables and the applied inputs.

The non linear vector function $f(x(k), u(k))$ is typically referred to as the evolution function while the non linear function $g(x(k), u(k))$ is usually called the output equation, we have

$$\begin{aligned} f(x(k), u(k)) &= [f_1(x_1(k), u(k)), \ \ldots, \ f_{n_x}(k), u(k))]^T \\ g(x(k), u(k)) &= [g_1(x(k), u(k)), \ \ldots, \ g_{n_y}(x(k), u(k))]^T \end{aligned} \tag{1.8}$$

If they explicitly depend on time, the system is said to be *time-dependent*. On the contrary, the system is said to be *time-independent*. In what follows, only time-independent systems will be considered.

A special case of great practical importance is that of linear systems. A system is considered linear if it satisfies the principle of superposition[2] In this case, whe have

$$\begin{cases} f(x(k), u(k)) = Fx(k) + Gu(k) \\ h(x(t), u(t)) = Cx(k) + Du(k) \end{cases} \tag{1.9}$$

where $F \in \mathbf{R}^{n_x \times n_x}$ is the *state matrix* also known as the *evolution matrix*, $G \in \mathbf{R}^{n_x \times n_u}$ is the *input matrix*, the matrix $C \in \mathbf{R}^{n_y \times n_u}$ and $D\mathbf{R}^{n_y \times n_u}$ is the *feedthrough, or feedforward matrix*. Note that the output equation remains unchanged by discretization.

- **Example I.2, Discretisation of the Robot Arm (1.5).**

We can discretize the system (1.5) using the Euler approximation, *i.e.*, we have:

$$\left[\frac{\dot{x}}{kT}\right]_{t=kT} \approx \frac{x((k+1)T) - x(kT)}{T} \tag{1.10}$$

where T is the *sampling time* and k is an integer. In order to simplify the notations, we will write for example $x(k)$ instead of $x(kT)$. The discrete state representation resulting from the Euler approximation is then written:

$$\begin{cases} \theta(k+1) = \theta(k) + T\omega(k) \\ \omega(k+1) = \dfrac{T}{J}u(k) + \left(1 - \dfrac{bT}{J}\right)\omega(k) - \dfrac{mglT}{J}\sin\theta(k) \end{cases} \tag{1.11}$$

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[2] Consider k input vectors $u_1(k)$, $u_2(k)$,…,$u_k(k)$ and let $y_i(k)$ be the system response to the input $u_i(k)$ $(i = 1, \ldots, k)$. The system is said to satisfy the principle of superposition.

1.2.2 Linearization of the State Space Model

An industrial system is very often intended to operate around an operating point, despite the various disturbances tending to deviate from it. Under these conditions, the use of a nonlinear state model for control or preventive maintenance purposes is not justified. We can be satisfied with a local linear state model, valid only in the vicinity of the desired operating point for the system.

1.2.2.1 Equilibrium point and equilibrium set

A physical system, represented by the equations $\dot{x} = f(x, u)$, is said to be in a stationary state if its state vector $x(t) = \mathbf{R}^{n_x \times n_u}$ does not change over time. Under these conditions, the time derivative of the state vector is zero ($\dot{x}(t) = 0$). We call equilibrium points the *stationary states of the system*, they are therefore solution of $f(x_0, u_0) = 0$ where the subscript zero is put to indicate that these are the quantities at equilibrium. We call the equilibrium set $\mathbf{E}$ of the system, the set of its equilibrium points, that is:

$$\mathbf{E} = \left\{(x_0, u_0) \in \mathbf{R}^{n_x \times n_u} \ : \ f(x_0, u_0) = 0\right\} \tag{1.12}$$

- **Example I.3, Points d'Equilibre the Robot Arm (1.5).** The equilibrium points of the Robot Arm (1.5) are has follows:

$$\begin{cases} \omega_0 = 0 \\ u_0 = mgl \sin(\theta_0) \end{cases} \tag{1.13}$$

1.2.2.2 Linearization of a State Space Model Around an Equilibrium Point

Consider a scalar continuous function $g(x) \in \mathbf{R} \to \mathbf{R}$. We wish to describe the behavior of this function in the vicinity of a point x_0 using a linear relation of the form $a\delta x + b$, where δx represents small variations of the variable x around x_0: x autour de x_0: $x = x_0 + \delta x$ and where a and b are constants to be determined. $x = x_0 + \delta x$ and where a and b sontare reals constants to be determined. We can see in the figure 1.3 that if the increase δx is sufficiently small then we can write that:

$$g(x_0 + \delta x) \approx g(x_0) + \left(\frac{dg(x_0)}{dx_0}\right) \delta x \tag{1.14}$$

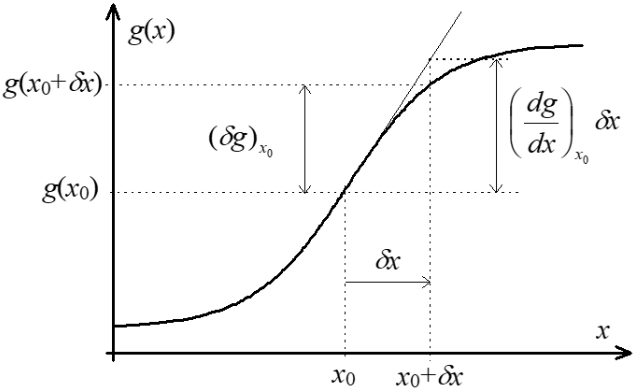

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Fig. 1.3 Variation of $g(x)$ for a variation δx of the variable around x_0

Note that the variation of the real variable of the function $g(x)$ in x_0, resulting from a variation δx of the variable around x_0, denoted $(\delta g(x_0) = g(x_0+\delta x) - g(x_0)$. Has it can be sees Fig. 1.14, has been approximated by the quantity $(dg(x_0)/dx_0)\delta x$, where $(dg(x_0)/dx_0)$ represents the slope of the line tangent to the function $g(x)$ at x_0. La relation (1.14) correspond to the first order Taylor serie expansion of the la function g. This approach can be generalized to the vector of multidimensional function: $g(x) = [g_1(x),\, g_2(x), \ldots, g_n(x)]^T$ with $x = [x_1,\, x_2, \ldots, x_n]^T$, we thus have:

$$g(x_0 + \delta x) = g(x_0) + \left(\frac{\partial g(x_0)}{\partial x_0}\right) \delta x \tag{1.15}$$

where $\left(\frac{\partial g(x_0)}{\partial x_0}\right)$ is the Jacobian of the vector function, $g(x_0)$ evaluated in $x_0 = [x_1^0,\, x_2^0,\, \ldots, x_n^0]^T$:

$$\left(\frac{\partial g(x_0)}{\partial x_0}\right) = \begin{bmatrix} \left(\frac{\partial g_1(x_0)}{\partial x_1^0}\right) & \cdots & \left(\frac{\partial g_1(x_0))}{\partial x_n^0}\right) \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial g_n(x_0)}{\partial x_1^0}\right) & \cdots & \left(\frac{\partial g_n(x_0)}{\partial x_n^0}\right) \end{bmatrix} \tag{1.16}$$

Now let's use the result (1.15) to linearize the differential system (1.1) around the operating point, or equilibrium point $(x_0, u_0) = (x_1^0,\, \ldots, x_{n_x}^0,\, u_1^0,\, \ldots, u_{n_u}^0)$. For this purpose, we consider small variations δx of the state, δx of the state, around x_0: $x = x_0 + \delta x$ and small variations δu of the input, around u_0: $u = u_0 + \delta u$. We therefore have:

$$\begin{cases} \dot{x} = \dot{x}_0 + \delta\dot{x} = f(x_0 + \delta x, u_0 + \delta u) \\ y = h(x_0 + \delta x, u_0 + \delta u) \end{cases} \tag{1.17}$$

but x_0 is a constant vector, it depends on the equilibrium point around which the is realized we therefore have $\dot{x}_0 = 0$. By carrying out a first-order Taylor series expansion of the functions f and h, we have:

$$\begin{cases} \delta\dot{x} = f(x_0, u_0) + \left(\dfrac{\partial f(x_0)}{\partial x}\right)\delta x + \left(\dfrac{\partial f(u_0)}{\partial u_0}\right)\delta u \\ \\ y = h(x_0, u_0) + \left(\dfrac{\partial h(x_0)}{\partial x_0}\right)\delta x + \left(\dfrac{\partial h(u_0)}{\partial u_0}\right)\delta u \\ \\ \delta x = x - x_0, \quad \delta u = u - u_0 \end{cases} \tag{1.18}$$

where the subscript zero indicates that the partial derivatives must be evaluated at equilibrium point (x_0, u_0). According to the definition of an equilibrium point we have: $f(x_0, u_0) = 0$, on the other hand $h(x_0, u_0)$ represents the output vector at equilibrium, we we will denote by y_0. The linearized model is then written:

$$\begin{cases} \delta\dot{x} = f(x_0, u_0) + \left(\dfrac{\partial f(x_0)}{\partial x}\right)\delta x + \left(\dfrac{\partial f(u_0)}{\partial u_0}\right)\delta u \\ \\ y = h(x_0, u_0) + \left(\dfrac{\partial h(x_0)}{\partial x_0}\right)\delta x + \left(\dfrac{\partial h(u_0)}{\partial u_0}\right)\delta u \\ \\ \delta x = x - x_0, \quad \delta u = u - u_0 \quad \delta y = y - y_0 \end{cases} \tag{1.19}$$

Let us pose:

$$A(x_0,u_0) = \begin{bmatrix} \left(\frac{\partial f_1(x_0)}{\partial x_1^0}\right) & \cdots & \left(\frac{\partial f_1(x_0)}{\partial x_{n_u}^0}\right) \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial f_n(x_0)}{\partial x_1^0}\right) & \cdots & \left(\frac{\partial f_n(x_0)}{\partial x_{n_x}^0}\right) \end{bmatrix}, B(x_0,u_0) = \begin{bmatrix} \left(\frac{\partial f_1(u_0)}{\partial u_1^0}\right) & \cdots & \left(\frac{\partial f_1(u_0)}{\partial u_{n_u}^0}\right) \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial f_n}{\partial u_1}\right) & \cdots & \left(\frac{\partial f_n(x_0)}{\partial u_{n_u}^0}\right) \end{bmatrix}$$

$$C(x_0,u_0) = \begin{bmatrix} \left(\frac{\partial h_1(x_0)}{\partial x_1^0}\right) & \cdots & \left(\frac{\partial h_1(x_0)}{\partial x_{n_x}^0}\right) \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial h_p(x_0)}{\partial x_1^0}\right) & \cdots & \left(\frac{\partial h_p(x_0)}{\partial x_{n_x}^0}\right) \end{bmatrix}, D(x_0,u_0) = \begin{bmatrix} \left(\frac{\partial h_1(u_0)}{\partial u_1^0}\right) & \cdots & \left(\frac{\partial h_1(u_0)}{\partial u_{n_u}^0}\right) \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial h_p(x_0)}{\partial u_1^0}\right) & \cdots & \left(\frac{\partial h_p}{\partial u_{n_u}^0)}\right) \end{bmatrix} \tag{1.20}$$

$A(x_0, u_0) \in \mathbf{R}^{n_x \times n_x}$ is the Jacobian matrix of the vector function $f(x_0, u_0)$ evaluated in x_0. The matrix $B(x_0, u_0) \in \mathbf{R}^{n_x \times n_y}$ is the Jacobian matrix of the vector function $f(x_0, u_0)$ evaluated in en u_0. The matrix $C(x_0, u_0) \in \mathbf{R}^{n_y \times n_u}$ is the Jacobian matrix of the vector function $f(x_0, u_0)$ evaluated in x_0. The matrix $D(x_0,u_0) \in \mathbf{R}^{n_x \times n_u}$ is the Jacobian matrix of the vector function $f(x_0,u_0)$ evaluated in u_0.

The linearized system around an equilibrium point (x_0,u_0) is written, in matrix formhas follows:

$$\begin{cases} \delta\dot{x}(t) = A(x_0,u_0)\delta x(t) + B(x_0,u_0)\delta u(t) \\ \\ \delta y(t) = C(x_0,u_0)\delta x(t) + D(x_0,u_0)\delta u(t) \end{cases} \tag{1.21}$$

we therefore obtain in this way a linear description of the nonlinear dynamic system in the vicinity of an equilibrium point. It should be noted that the linearized model thus obtained depends on the equilibrium point around which the linearization was carried out.

- **Example I.4, Linearizarion of the model (1.5) around the equilibrium point $\theta_0 = \pi/4$.**

$$\begin{cases} f(x,u) = \begin{bmatrix} \omega \\ \frac{1}{J}u(t) - \frac{b}{J}\omega(t) - \frac{mgl}{J}\sin\theta(t) \end{bmatrix} \\ \\ x = [\theta \quad \omega]^T \end{cases} \tag{1.22}$$

where

$$A(\theta_0) = \left(\frac{\partial f(x_0)}{\partial x_0}\right) = \begin{bmatrix} 0 & 1 \\ -\frac{mgl}{J}\cos\theta_0 & -\frac{b}{J} \end{bmatrix}, \; B(\theta_0) = \left(\frac{\partial f(x_0)}{\partial u_0}\right) = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} \tag{1.23}$$

the linear state representation is then as follows:

$$\delta\dot{x} = \begin{bmatrix} \delta\dot{\theta} \\ \delta\dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{mgl\sqrt{2}}{2J} & -\frac{b}{J} \end{bmatrix} \begin{bmatrix} \delta\theta \\ \delta\omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} \delta u \tag{1.24}$$

1.2.3 Transfer model or external representation

A representation widely used in the study of linear systems is the representation by transfer function and its generalization to the multivariable case by transfer matrix. In the SISO[3] case, the transfer function represents the Laplace transform[4]

The generalization of this definition to the MIMO[5] conduct to the notion of transfer matrix. The transfer matrix $G(s)$ of a MIMO system is obtained by

---

[3] Single Input Single Output

[4] The Laplace transform of a continuous signal $x(t)$ is written has $X(s) = \mathcal{L}(x(t) = \int_0^\infty x(t)e^{-st}dt$, where s is the Laplace complex variable: $s = \alpha + j\beta$.

[5] Multiple Inputs Multiple Outputs

taking the Laplace transform of the linear state representation of the system considered. Knowing that for zero initial conditions: $\mathcal{L}(\dot{x}(t)) = sX(s)$, thus we have:

$$\mathcal{L}\left[\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}\right] = \begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases} \tag{1.25}$$

by eliminating the state between these two equations we obtain a relationship between the outputs of the system and its inputs, hence the name external representation:

$$Y(s) = G(s)U(s), \quad \text{avec}: \quad G(s) = C(sI - A)^{-1}B + D \tag{1.26}$$

where I is the identity matrix. The transfer matrix $G(s)$ represents the Laplace transform of the impulse responses of the system with respect to each of the inputs.

- **Example I.5, transfer function between the input δu and the angular position $\delta\theta$ of the system (1.24) is written**

$$G(s) = \begin{bmatrix} 1 \; 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{mgl\sqrt{2}}{2J} & s + \frac{b}{J} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} \tag{1.27}$$

with

$$\begin{bmatrix} s & -1 \\ \frac{mgl\sqrt{2}}{2J} & s + \frac{b}{J} \end{bmatrix}^{-1} = \frac{1}{s(s + \frac{b}{J}) + \frac{mgl\sqrt{2}}{2J}} \begin{bmatrix} s + \frac{b}{J} & 1 \\ -\frac{mgl\sqrt{2}}{2J} & s \end{bmatrix} \tag{1.28}$$

thus, we have

$$G(s) = \frac{1/J}{s^2 + \frac{b}{J}s + \frac{mgl\sqrt{2}}{2J}} \tag{1.29}$$

In the case of discrete linear systems, we can define the discrete transfer matrix $G(z)$ in the same way by taking the transform[6] the discrete transfer matrix $G(z)$ is obtenaid by taking the discrete transforms of the state space model for zero initial conditions, we have:

$$\mathcal{Z}\left[\begin{cases} x(k+1) = Fx(k) + Gu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}\right] = \begin{cases} zX(z) = FX(z) + GU(z) \\ Y(z) = CX(z) + DU(z) \end{cases} \tag{1.30}$$

by eliminating the state between these two equations we obtain the following discrte transfer matrix:

$$Y(z) = G(z)U(z), \quad \text{with:}, \; G(z) = C(zI - F)^{-1}G + D \tag{1.31}$$

[6] The z transform of a signal $x(t)$ with a sampling rate of T is written has $X(z) = \mathcal{Z}(x(t)) = \sum_{k=0}^{\infty} x(kT)z^{-k}$

We will have the opportunity in Chapter 2 to return in more detail to the state and transfer models.

1.3 Modeling Principle of a Physical System

The search for a mathematical model of a given system can be approached by assuming that it is possible to obtain a complete description of the system using a set of equations resulting from the application of the laws of physics. The model thus obtained is called a *physical model*.

1.3.1 Schematic diagram.

The schematic diagram is a simplified representation of the physical system as it appears (or will appear) in the reality. It makes it possible to understand the operation of the system and the way in which the main function is ensured. It make it possible to have a global view of the system and to identify the main components as well as their interconnections. It gives an outline of technical solutions used during the design phase of the system.

As an example, figure 1.4 shows the schematic diagram of a very simple system composed of a tank and an electro-valve.

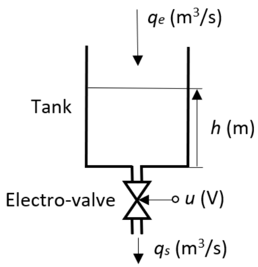

Fig. 1.4 Schematic diagram of a tank equipped with an electro-valve

In this installation, the fluid is introduced in the tank with an unknown flow q_e. the volume of fluid in the tank is evacuated, according to the control input u, with a flow q_s. The variable h represents the level of the fluid in the tank. Note that q_e is a disturbance variable (not controlled) whereas q_s is the controlled variable via the input voltage applied to the electro-valve.

A mathematical model of the system can then be obtained by formalizing the various components of the system through the application of the physical laws to each component of the schematic diagram.

For instance, considering the schematic diagram given in figure 1.4, the mathematical model associated to the bloc "tank" is given by applying the principle of mass conservation, *i.e.*

$$S\frac{dh}{dt} = q_e - q_s \tag{1.32}$$

Where S (m^2) is the section of the tank and h the level of fluid. Now considering the bloc "electro-valve" it can be seen that the output flow depends on h and u. By definition, the flow is the product of a section by the velocity of fluid through the section. Let s be the section exerted by the electro-valve. Assuming that s is proportional to the voltage input u we have $s = ku$, where k is a constant depending on the electro-valve. Thus the output flow is given by $q_s = kuv$, where v is the velocity of fluid through the sections s. According to the Bernoulli[7]law this velocity is given by $v = \sqrt{2gh}$, where g is the acceleration due to gravity. Finally the output flow is given by

$$q_s = k\sqrt{2gh}u \tag{1.33}$$

By reporting relation (1.33) into (1.32), we obtain the global model of the system

$$\dot{h} = \frac{1}{S}\left(q_e - k\sqrt{2gh}u\right) \tag{1.34}$$

where $\dot{h} = dh/dt$. This is a first order time-invariant nonlinear differential equation. The expression "time-invariant" comes from the fact that the parameters model are not time dependent.

This kind of system is said to be a dynamical system because the evolution of h over a time interval depends on the initial level as well as the evolution of the input q_e and u on $[t_0,\ t]$. In the modeling terminology, h is called the state variable. It represents the minimal information required to compute the evolution of the system for any given input variables. The representation (1.34) is known as being a state space model of the system.

[7] For all points along a streamline, we have: $\frac{p}{\rho} + \frac{1}{2}v^2 + E_p =$ constant, where p is the pressure, v the flow velocity and E_p the potential energy.

1.4 Examples of modeling

In this section three examples of modeling (physical model) are presented: the *electrical actuator* (EA), the *device for heating a fluid* (DHF) and the *magnetic levitation of a mass* (MSM).

1.4.1 Modeling of an Electrical Actuator

We consider the direct current motor with independent magnetic excitation shown in the Figure 1.5, as well has the nomenclature presented Table 1.1.

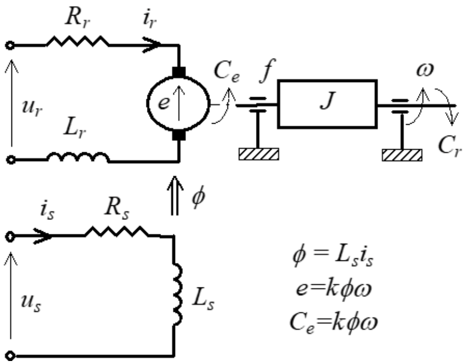

Fig. 1.5 Schematic diagram of the electrical actuator whit independent magnetic excitation.

As shown in the schematic diagram (see Figure 1.5), this machine has two ferromagnetic parts on which windings are wound. One of these parts is fixed and is called the inductor or stator, intended to produce a magnetic flux ϕ. The other is mobile and is called the armature or rotor, intended for the production of an electromagnetic torque C_e. This machine rotates a mechanical load whose overall moment of inertia seen from the motor shaft is denoted J. The driven load exerts a resisting torque C_r.

The assumptions are as follows: the machine is unsaturated (the magnetic flux is proportional to the current), the armature magnetic reaction is negligible (the armature magnetic flux does not alter the inductor magnetic flux), and dry friction is negligible (there is rotation of the rotor for an armature current, however small it may be).

Table 1.1 Nomenclature of the electric actuator.

Notation	Name and units
R_s	Stator winding resistance (Ω)
L_s	Stator winding inductance (H)
i_s	Stator current (A)
u_s	Stator voltage (V)
Φ	Magnetic excitation flux (Wb)
R_r	Rotor winding resistance (Ω)
L_r	Rotor winding inductance (H)
i_r	Rotor current (A)
u_r	Rotor voltage (V)
e	Electromotive forces (V)
k	Motor constant
f	Viscous coefficient ($Nm/rd/s$)
J	Total moment of inertia ($Kg.m^2$)
ω	Angular speed (rd/s)
C_e	Electromagnetic torque (Nm)
C_r	Load torque (Nm)

- **Modeling of the Stator Circuit**

The application of the mesh law, Ohm's law and Faraday's law makes it possible to write:

$$u_s = R_s i_s + L_s \frac{di_s}{dt} \tag{1.35}$$

- **Modeling of the Rotor Circuit**

We obtain in the same way:

$$u_r = R_r i_r + L_r \frac{di_r}{dt} + e \tag{1.36}$$

where e represents the back electromotive force (cemf) due to the rotation of the rotor windings in the inducing magnetic field. This cemf is proportional to the magnetic excitation flux ϕ and to the variation of magnetic flux through the rotor windings. Therefore, at the angular speed ω, we have:

$$e = k\phi\omega \tag{1.37}$$

la constante de proportionnalité k dépend du moteur utilisé. La machine étant non saturée, le flux d'excitation magnétique est proportionnel au courant $\phi = L_s i_s$.

- **Modeling of the Mechanical Part**