A Concise Course of Mathematics with Applications

A Concise Course of Mathematics with Applications:

A Conceptual Study

By

Nicolas Laos

Cambridge Scholars Publishing

A Concise Course of Mathematics with Applications: A Conceptual Study

By Nicolas Laos

This book first published 2024

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN: 978-1-0364-0978-4
ISBN (Ebook): 978-1-0364-0979-1

To my teachers and to my students.
—Nicolas Laos

CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

PREFACE

The word “mathematics” comes from the Greek word “manthānein,” which means “to learn.” Mathematics is mainly about forming ways to see problems in order to solve them by combining logical rigor, imagination, and intuition. Furthermore, mathematics is a peculiar sense that enables us to perceive realities that would otherwise be inaccessible to us. In fact, mathematics is our sense for patterns, relations, and logical connections. Mathematics, in its essence, is not so much about calculating as about understanding, and, thus, it is a way of knowing, searching for truth, thinking, and developing technology.

In general, “truth”—the pursuit of which remains at the heart of scientific endeavor—can be defined as a set of relations that determine if, and the extent to which, the representation of reality within consciousness (that is, the knowledge of reality) is in concordance with the presence of reality itself (that is, with the nature of reality).

The development of mathematical intuition depends on learning the basic concepts (thus, creating a powerful intellectual toolbox), using our intellectual toolbox in order to solve problems, and thinking creatively (rather than simply memorizing mathematical tools).

This book is mainly aimed at the following four categories of readers:

i. *Mathematics students:* Those who study mathematics can profitably use this book as a self-contained, conceptual and methodic guide and compendium of pure and applied mathematics and as a supplement to their standard textbooks in the courses of algebra, linear algebra, geometry (including classical Euclidean geometry, analytic geometry, non-Euclidean geometries, and metric geometry), infinitesimal calculus (single-variable, multivariable, and vector calculus), differential equations, and real analysis.
ii. *Natural-science and social-science students:* This book enables one to understand the significance of mathematical modeling (including analytic and statistical methods) both in the context of the natural sciences and in the context of the social sciences. Therefore, this book can be useful for both natural-science and social-science students, helping them to better understand the

importance of mathematics in their discipline and the mathematics courses included in their curriculum.

iii. *Philosophy students:* This book contains a systematic study of mathematical philosophy, philosophy of science, and the methodology of mathematics.

iv. *Any person who would like to enhance his/her ability to understand science in general, to get a better understanding of mathematics, and to fill cognitive gaps that he/she may have in mathematics and philosophy of science.*

Regarding my competence in mathematics, I would like to acknowledge the importance of the mathematical education and scientific guidance that I received from the following professors during my studies at the University of La Verne: the renowned research mathematician Professor Themistocles M. Rassias (Ph.D./University of California, Berkeley, former Chairman of the Department of Mathematics at the University of La Verne's Athens Campus and Professor at the National Technical University of Athens) taught me Calculus I, II & III, Advanced Calculus, Linear Algebra, Differential Equations, and Number Theory, and he supervised my research work in the foundations of mathematical analysis and differential geometry (a part of the research work and the dissertation that I completed at the University of La Verne under the supervision of Professor Themistocles M. Rassias was published in 1998 as the volume no. 24 of the scientifically advanced Series in Pure Mathematics of the World Scientific Publishing Company); the highly experienced applied mathematician Professor Christos Koutsogeorgis (Ph.D./City University of New York) taught me Discrete Mathematics, Abstract Algebra, and Probability Theory with mathematical statistics; and the distinguished IT Professor Chamberlain Foes (Ph.D./Portland State University) taught me PASCAL (programming language) and introduced me to mathematical informatics and management information systems.

Moreover, my cooperation with the prominent philosopher Dr. Giuliano Di Bernardo, who held the Chair of Philosophy of Science and Logic at the Faculty of Sociology of the University of Trento from 1979 until 2010, has helped me to explore several aspects of epistemology. Epistemology is the branch of philosophy that makes knowledge itself the subject matter of inquiry, and, therefore, every conscientious scholar has to be epistemologically sensitive and informed. Furthermore, epistemology is intimately related to ontology, also known as metaphysics, which investigates the nature of existence itself as well as the degree of existence of the phenomena that appear to us (and epistemology enables us to distinguish between theorems about models and theorems about reality; this distinction is very important in applied science, where models must be not only logically valid but also empirically validated).

Regarding my interdisciplinary studies and research work, I would like to acknowledge the contribution of the following professors to my education during my studies at the University of La Verne (1992–96): the historian Professor Vassilios Christides (Ph.D./Princeton University) taught me a comprehensive set of courses on the history of world civilization; the historian Professor Paul Angelides (Ph.D./Ohio State University) taught me the courses "U.S. Intellectual History" and "Development of American Democracy"; the political scientist Professor Blanca Ananiadis (Ph.D./University of Essex) taught me European politics and political institutions; and the sociologist Professor Gerasimos Makris (Ph.D./LSE) taught me Sociology. My studies in the history of civilization in general and in the history of science in particular have enabled me to articulate a typology of cultures, and, in this context, I have to mention that my approach to cultural issues, including science, is founded on certain aspects of classical philosophy and of what we call "modernity."

My gratitude extends to the following scholars: the political scientist Dr. Hazel Smith (Professor of International Security at Cranfield University, UK, and Fellow of the Royal Society of Arts, London) and the economist and epistemologist Dr. Michael Nicholson (Professor of International Relations at the University of Sussex), who supervised my research work in the epistemology and the mathematical modeling of International Relations and Political Economy during 1997–99 at the University of Kent's London Centre of International Relations; as well as my colleagues at the Faculty of Philosophy of the Theological Academy of Saint Andrew (Academia Teológica de San Andrés), Veracruz, Mexico, where I completed a series of Ph.D. courses (specifically, Methodology of Philosophical Investigation I & II, Theology and Philosophy I–IV, Selected Topics in Christian Philosophy I–IV, Seminar on Investigation in Christian Philosophy I–IV, and Interpretation of Philosophical Texts I & II), and the Dean of that Theological Academy, Metropolitan Dr. Daniel de Jesús Ruiz Flores of Mexico and All Latin America of the Ukrainian Orthodox Church (Iglesia Ortodoxa Ucraniana en México) helped me to explore and appreciate the interdisciplinary nature of the scholarly disciplines of theology and philosophy, and he signed my Doctoral Degree in Christian Philosophy.

In fact, I have systematically investigated theology and philosophy in order to investigate and analyze the meaning of reality, the dynamicity and the levels of the intentionality of human consciousness, and the general process of idealization. Moreover, my studies in theology and philosophy have helped me to study and understand the intellectual history of humanity and to study science in general and mathematics in particular within the context of the history of world civilization. The central theme of theology and philosophy is condensed in the meaning of the Greek word "logos," which means both language and thought, and refers to both the efficient cause and the final cause of the beings and the things that exist in the world. Indeed, ancient Greek and Roman scholars used the term "logos" in order to refer to the creative Nature, to the Norm of conduct, and to the Rule of discourse, and, gradually, the study of these three fundamental dimensions of reality was specialized in the context of particular scientific disciplines.

In the eleventh century C.E., the first "university" in the world was founded by an organized guild of students (*studiorum*) in Bologna. In fact, the founders of the University of Bologna created the word "uni-versity," and they invented an institution called "university" in order to give an adequate account of the "uni-verse," and, since the universe comes in many aspects, they thought that the study of each aspect of the universe requires the creation of a corresponding scholarly discipline. In fact, those students, acting as a mutual aid society, hired scholars to teach them liberal arts (grammar, logic, rhetoric, geometry, arithmetic, astronomy, and music), law, theology, and *ars dictaminis* (the composition of official letters and other epistolary documents). Thus, in the context of the "uni-versity," which reflects and gives an adequate account of the "uni-verse," each scholarly discipline informs and is informed by every other scholarly discipline, and this synthetic approach to knowledge underpins the classical ideal of education.

In this book, I study and delineate the following topics: Mathematical Philosophy; Mathematical Logic; the Structure of Number Sets and the Theory of Real Numbers, Arithmetic and Axiomatic Number Theory, and Algebra (including the study of Sequences and Series); Matrices and Applications in Input-Output Analysis and Linear Programming; Probability and Statistics; Classical Euclidean Geometry, Analytic Geometry, and Trigonometry; Vectors, Vector Spaces, Normed Vector Spaces, and Metric Spaces; basic principles of non-Euclidean Geometries and Metric Geometry; Infinitesimal Calculus and basic Topology (Functions, Limits, Continuity, Topological Structures, Homeomorphisms, Differentiation, and Integration, including Multivariable Calculus and Vector Calculus); Complex Numbers and Complex Analysis; basic principles of Ordinary Differential Equations; as well as mathematical methods and mathematical modeling in the natural sciences (including physics, engineering, biology, and neuroscience) and in the social sciences (including economics, management, strategic studies, and warfare problems). The option of, firstly, presenting algebra, geometry, and mathematical analysis in a new, creative, and synthetic way (emphasizing a methodical and thorough conceptual study of the subject-matter) and, secondly, combining different branches of pure and applied mathematics as well as philosophy into one self-contained course gave rise to a unique, innovative project.

The present book was originally presented as a series of lectures that I delivered during the academic year 2022–23 in the context of a Laboratory of Interdisciplinary Mathematics and Epistemology (for both scholars and professional technocrats) that I organized inspired by, and in honor of, my philosophy mentor Professor Giuliano Di Bernardo and with the support of an international private Masonic Lodge of literati that I have created and manage on the basis of the teachings and honors that I have received from Professor Giuliano Di Bernardo in the context of the Dignity Order (which is a private exclusive membership association for the defense of the dignity of humanity, and it was founded by Professor Giuliano Di Bernardo in 2012 under Austrian Law). As an independent scholar and consultant, with some informal international scholarly affiliations, I have the opportunity to consider and study several mathematical and methodological-epistemological problems as well as other analytical issues in the context of several projects in the fields of physics, engineering, biology, economics, management, social policy, and strategic studies. Furthermore, my inspiration for writing this book was enhanced by the legacy of the Royal Society of Arts (London), which approved my Fellowship in December 2023 (my Fellowship No. being 8289155), as well as by my experience as an instructor at the University of Indianapolis (Athens Campus, Greece, 2012–13), where I taught epistemological and methodological issues to students of International Relations, and as an analyst in financial-services, construction, IT, and shipping companies.

A Few Preliminary Thoughts

It is due to the intentionality, or the referentiality, of consciousness, or, in other words, due to the fact that consciousness is the consciousness of its contents, that the contents of consciousness become experiences for it. In fact, as the Austrian-German philosopher and mathematician Edmund Husserl (1859–1938) has taught, consciousness not only treats the presence of experiences within itself in a critical way, but also causes their presence, as it is implied by the term “intentionality.” Intentionality is not only the ability to refer to something, but also the ability to cause something. Given that, as the French philosopher Henri Bergson (1859–1941) has taught, intentionality consists of both the ability to refer and the ability to cause, we realize that the term "intentionality" expresses the dynamism of consciousness; and the dynamism of consciousness manifests itself in the manner in which consciousness intervenes in the reality of the world and restructures it.

Furthermore, regarding the creativity of human consciousness, it should be mentioned that the American neuroscientist Benjamin Libet and his collaborators clarified an aspect of free will through their discovery that humans consciously decide to act before they even think about making the decision to act. In his book *Mind Time*, Libet maintains that free will is not only an expression of the brain's conscious activity, but it begins earlier in the unconscious mind, and it has a power of *veto* over whether or not the action takes place.

The evolutionary history of humanity is defined by the increase in brain size, as the latter made possible the rise of consciousness, both in its primitive form and in its higher order. Higher order consciousness, in turn, made possible the birth of language and intentionality or purposefulness. Consciousness is essentially linked to intentionality, through which human beings can access external reality and enter into relationships with each other. Undoubtedly, there are conscious states that are not intentional, and there are intentional states that are not part of our consciousness. Nevertheless, the connection between consciousness and intentionality plays a crucial role in understanding human beings and history.

Intentionality can operate according to a hierarchy of relations ranging from a minimum to a maximum. The levels of this hierarchy of relations are called "orders of intentionality," in the terminology of the prominent British cognitive anthropologist Robin Dunbar (University of Oxford). In his book *The Human Story*, Dunbar has analyzed the development of the different orders of intentionality. Specifically, bacteria and certain insects have zeroth-order intentionality, while brain-equipped organisms are conscious of their mental states. For instance, brain-equipped organisms know when they are in danger or hungry. Therefore, brain-equipped organisms have first-order intentionality. First-order intentionality means that a being is self-aware, consciously referring to itself. However, there are also types of higher-order intentionality. Intentionality can be directed towards the beliefs of other people—we say that it is second-order intentionality. In other words, in the terminology of Robin Dunbar, we can distinguish the orders of intentionality as follows: most vertebrates can recall their mental states, at least in an elementary way, that is, by knowing that they know. Organisms that know that they know have first-order intentionality. Organisms that, moreover, know that someone else knows something have second-order intentionality. Organisms that, in addition, know that someone else knows that someone else knows something have third-order intentionality. As the number of subjects in the intentionality sequence increases, so does the number of hierarchical orders. This sequence can reflexively be extended indefinitely, but, in the context of their everyday life, most people rarely reach intentionality of an order higher than fourth, and they can very hardly rise to the fifth order—that is, to the following type of reasoning: "Theodore knows that Christina believes that George thinks that Nicolas supposes that Natasha intends to do something." Fourth-order intentionality is required, at a minimum, for the development of literature that goes beyond mere narrative, because, for example, an author wants his/her readers to believe that literary hero A thinks that literary hero B intends to do something. The same level of minimum skills is required for the development of science, since doing a scientific task requires asking whether the world could exist otherwise and going beyond the level of sensory experience, and then asking someone else to do the same.

As Robin Dunbar argues in his book *How Religion Evolved and Why It Endures*, the invention of religion by the species *Homo* is one of the earliest and most impressive manifestations of humanity's ascent to very high levels of intentionality, and, indeed, religion represents an extremely advanced and complex expression of humanity's creative capacity. In fact, the ability to conceive religion is an exclusive privilege of the human species. No other biological species living on Earth can formulate anything even remotely resembling religion. Since humans are a product of evolution, we must carefully investigate the factors that may have favored the emergence of our religious impulse.

In order to explain religion as a social activity and as a social institution, we need at least fourth-order (perhaps even fifth-order) intentionality, so that we can handle syllogisms of the following type: "John supposes (1) that Mary believes (2) that John believes (3) that there is a divine being intending (4) to influence people's future (because this divine being understands people's desires (5))." Until people can interact and form a community on the basis of fourth-order (or even fifth-order) intentionality, we cannot yet speak of a fully developed religion, but only of religious beliefs. The existence of a common belief—that is, the fact that there are things that mean the same to everyone—is the keystone of religion. Hence, a true communion of words, a sharing of words as a basic characteristic of any genuine dialogue, is a major underpinning of religion.

Based on the point that, in order to *understand* religion, one needs a well-formed language and at least fourth-order intentionality (while the *creation* of a religion requires at least fifth-order intentionality), we can determine when religion made its first appearance in the evolutionary history of hominids. Specifically, in view of the foregoing, we can argue that the first appearance of religion in the evolutionary history of hominids coincided with the time of the first appearance of language. Fifth-order intentionality, associated with *Homo sapiens*, manifested itself much later, when fifth-order intentionality in conjunction with a well-formed language equipped with advanced grammar and advanced syntax expressed religion as both a social institution and a metaphysical system.

Darwin's theory of evolution favors everything that can help the species to survive. As Giuliano Di Bernardo argues in his book *The Epistemological Foundation of Sociology* (Amazon, 2021), the evolutionary advantages that the human species has derived from religion are social cohesion, social control, creative imagination (especially regarding the conception of a better world), and creative management of existential anxiety. To achieve these goals, religion uses powerful means, such as belief in immortality, metaphysics, mysticism, and rituals.

However, in the context of the ancient Greek civilization, in the Aegean, certain Greek intellectuals became aware of and highlighted the fact that the human mind can discern and differentiate itself from the surrounding body of nature and can discern similarities in a multiplicity of events, abstract these from their settings, generalize them, and deduce therefrom other relationships consistent with further experience. "Abstraction" means getting rid of what we consider unnecessary details (so that, after getting rid of unnecessary details, things that were different because of unnecessary details become identical), and, therefore, we have a non-trivial concept of "identity," on the basis of which we study the "sameness" of certain things, or we look at certain things as if they were the same. "Composition" means that we combine certain abstract objects into bigger abstract objects, so that, when we have to deal with complex problems, we need to be able to divide ("analyze") the bigger problem into smaller problems, solve them separately, and then combine the solutions together. These concepts underpin "operational structuralism," which, in turn, underpins the development of modern mathematics (by the term "operation," we mean a rule according to which we can combine any two elements of a given system). The origin of "operational structuralism" can be traced back to ancient Greek philosophy. In the context of modern science and philosophy, the scholar that put operational structuralism within a rigorous mathematical-logical setting was René Descartes, the acknowledged founder of modern analytic geometry and of modern philosophy.

Philosophy and the scientific method were invented by the ancient Greek civilization. Initially, philosophy was developed within the context of the ancient Greek mystery cult of Orphism, but, gradually, it achieved its structural autonomy from religion; and a philosophical approach to religion (that is, a reflection on religion, which is something different from religion) gave rise to theology. Hence, with the invention of philosophy, the ancient Greek civilization created a method that enables humanity to rise to the highest levels of intentionality without having to resort to religion, as well as to secure for the human beings the evolutionary advantages offered by religion without being dependent on religion. In the context of philosophy, we study truth itself, what we can know, what makes an argument rational, valid, or fallacious, the reality of being, the relationship between consciousness and the world, moral criteria, and the interplay between different scholarly disciplines in the most abstract and most rigorous way possible. Thus, the invention of philosophy by the ancient Greek civilization made the ancient Greek civilization capable of becoming the inventor of science, too. For instance, the mathematical and philosophical problems suggested and studied by Aristotle, Plato, Zeno, and Pythagoras inspired and guided the mathematical works of Eudoxus, Archimedes, Apollonius of Perga, and Nicolas d'Oresme, who were leading pioneers of infinitesimal calculus, and, in turn, the latters' achievements inspired and guided the mathematical works of Torricelli, Cavalieri, Galileo, Kepler, Valerio, and Stevin, who made decisive contributions to the development of infinitesimal calculus and its applications, and, in turn, the latters' works inspired and guided Barrow and Fermat, who developed infinitesimal calculus even further and set the stage for the systematic and rigorous formulation of infinitesimal calculus by Newton and Leibniz. For a systematic study of the importance of ancient Greek thought for the development of science and philosophy, I strongly recommend the books written and edited by the British classical scholar, educationalist, and academic administrator Sir Richard Livingstone (1880–1960).

Based on the principles of abstraction and syllogism, mathematicians study the quantitative and the qualitative relations and the forms of a space (structured set), identify various connections in the processes that take place in reality, and they formulate them in the form of logical sentences written in symbols. The heuristic role of mathematics, that is, the articulation of new results, which then acquire empirical significance and confirmation or a new interpretation, is based on the correct representation of reality by mathematical models.

A "model" is intended as a carefully and methodically simplified analogue of real-world phenomena and situations, and its deductive structure helps scientists to explore the consequences of alternative assumptions. Given that scientific modeling aims to explain how things are and why things are the way they are, as well as to analyze and evaluate alternative assumptions, it contrasts, for instance, with the use of basic statistical methods solely to summarize empirical data. Furthermore, one can experiment with the model (by changing the assumptions) when it would be epistemologically, technically, and/or morally impossible and/or too risky to experiment with the real world.

In general, we should be aware that, usually, the object of scientific investigation is not an object of the real world but an ideal image of it. For instance, physics introduces many idealized objects to use in the idealization of physics problems; such as the following: (i) "Particle": this term refers to a fundamental and universal physical object, and, when physicists use this concept, they ignore the geometric dimensions of an object in comparison with the characteristic distances of the corresponding problem. (ii) "Rigid body": in this idealized object, all possible strains are ignored. (iii) "Elastic body": in this idealized object, the remnant strain is ignored. (iv) "Inelastic body": this idealized object is incapable of sustaining deformation without permanent change in size or shape, and, in this case, elastic deformation is ignored. Moreover, physics introduces idealized physical processes, too; such as the isochoric, isobaric, isothermal, and adiabatic processes. Similarly, in microeconomics and econometrics, the "model" of a real economic phenomenon reflects the essentials, allows only for the most important interrelations and interactions, and considers idealized actors rather than real actors. In particular, the mainstream of economic theory does not deal with real businesses or interests, or real markets, but it deals with theoretically representative firms, abstract markets, and generalities like the interest rate and the flow of money, and, therefore, it predicts general (rather than individual) behavior, and, more specifically, it predicts what the consequences of different kinds of behavior will be *under certain hypotheses*.

Scientific explanation is based on the fact that real objects and phenomena themselves are so complicated and interrelated that their study and quantitative investigation with due account for all aspects, interrelations, and interactions would lead to insurmountable mathematical difficulties. Therefore, a reasonable level of idealization of concrete problems characterizes every meaningful task in the context of applied science. If applied scientists did not idealize their problems, then they could not solve a single concrete problem in full. The simplifying assumptions vary from problem to problem, but a common feature of every scientific idealization consists of the methodical identification of non-essential, secondary interrelations and interactions and the decision to ignore them. Hence, a question of criteria arises. When, in what conditions, can an interrelation or interaction be characterized as non-essential and ignored, and when not? The answer depends on the method used in analyzing the solution to a problem and on the estimate method. However, two ways of idealization are most commonly used in applied science: the introduction of idealized objects (or idealized actors) and the decision to ignore non-essential interactions and processes (once we have clearly identified them as such).

In the context of my work in mathematical modeling, I have been using two categories of mathematical models: one category of mathematical models depends on the Italian physicist and engineer Galileo's method (consisting of intuition or resolution, demonstration, and experiment), and the other category of mathematical models depends on General System Theory (originally due to the Austrian biologist Ludwig von Bertalanffy). Applied firstly to celestial mechanics, Galileo's method is characterized by a mechanistic conception, according to which formal rules ("reasons") cause behavior (in an automatic way), and it is ideally suited for the study of classical physics. In fact, the estimation of a physical phenomenon consists of finding the fundamental law governing the phenomenon and, subsequently, numerically calculating the order of magnitude of the respective physical quantity. However, applied firstly in biology and, subsequently, in certain aspects of modern physics and ecological studies, as well as in behavioral and social sciences (where formal rules ("reasons") do not necessarily cause behavior), the "working attitude" of General System Theory is that of the "open system," delineated by Ludwig von Bertalanffy in his book *General System Theory* (originally published in 1968).

The closed system, reflecting the model of thought of classical physics, is axiomatic in a way that the object of scientific research is separated from the outer environment, and the outcome results from the initial conditions. From this perspective, scientific research is concerned with the analysis of the characteristics and the quantities of the elemental components, which are held in isolation for the purpose of study. Moreover, it is based on an additive methodology that underpins the deduction of the meaning of the whole from a specific *corpus* of knowledge of the character of its elementary parts. Thus, it is characterized by "reductionism." Thinking according to this model (the "machine model") has introduced both useful and misleading insights in the study of human systems. For instance, the "machine model" provides a rigorous reference system (specifically, a platform equipped with a ruler and a clock, enabling us to determine the position of the bodies under consideration and the course of time) as well as powerful analytical methods, but the actions of living things, in general, do not fit the conceptual models of classical physics.

On the other hand, General System Theory endorses and absorbs the clarity of thought and the rigor that characterize the "machine model," but it is based on the empirically verified fact that living beings and their organizations are not collections of isolated and uniform units, the sum of which accounts for a total phenomenon. Even though there is a structural continuity between inorganic matter and organic matter, life—by transforming inorganic matter into organic matter—implies an important differentiation in matter. Some characteristic differences between inorganic matter and organic matter are the following: Firstly, inorganic matter is governed by inertia, whereas organically structured living beings sense things, react to external stimuli, and move on their own. Secondly, inorganic matter reacts according to the laws of mechanics, but the reactions of organically structured living beings manifest peculiar qualitative features that are not strictly analogous to the stimuli that cause reaction, and they depend on organic relations that govern each living being according to its structural program. Thirdly, according to the Standard Model of particle physics, the minimal constituent matter elements of inorganic bodies are uniform—that is, subatomic particles are identical (so that no exchange of two identical particles, such as electrons, can lead to a new microscopic state)—but the minimal constituent matter elements of organic matter (such as DNA) are subject to differentiations, which underpin the actualization and the manifestation of the structural program of an organic being. Fourthly, inorganic bodies are connected with each other under specific conditions in order to form chemical compounds, which are always characterized by the same quantitative data (e.g., Antoine Laurent Lavoisier's "law of conservation of matter," Louis Proust's "law of constant composition," and John Dalton's "law of multiple proportions"), but organically structured living beings exchange some of their constituent elements with some of their environment's constituent elements in the context of a dynamical process that is called assimilation. Fifthly, inorganic bodies exist in definite and fixed quantities, but organically structured living beings (specifically, "parents") create new living beings (specifically, "offspring") similar to them in the context of the reproductive process. Sixthly, with few exceptions (such as radioactive nuclides, or nuclear species which are unstable structures that decay to form other nuclides by emitting particles and electromagnetic radiation), inorganic bodies are incapable of self-transformation, but organically structured living beings follow life cycles (namely, developmental stages that occur during an organism's lifetime).

The phenomena of the living world must be modeled as open systems, in which the "components" are sets of organized actions that are maintained by exchanges in the environment, and the issue of teleology (normative action) must be explicitly addressed in the models of human systems. Therefore, the postulates that refer to the dynamism of an open system and the rules that relate means to ends must explicitly find their place in any meaningful study of the social sciences. For this reason, deontic logic, approximation theory, stochastic processes, and a dynamical approach to structural analysis play an important role in the modeling of human systems.

The Meaning of a "Conceptual Study"

Mathematics plays a very important role in the world (and this is easily and thoroughly understood in applied mathematics); and, therefore, the student of mathematics must be properly instructed to understand the true nature of mathematics. The understanding of the true nature of mathematics is a key underpinning of the progress of civilization. As Euclid has taught, the true nature of mathematics is inextricably linked to deductive reasoning: from various hypotheses (often related to the perception of the world), we logically proceed to proofs.

However, hypothetico-deductive systems, especially when we rise to very high levels of abstraction, give rise to paradoxes, that is, contradictions of understanding, contradictions of logic, contradictions of semantics, and contradictions of thinking. Paradoxes have played an important role in the development of mathematics and logic. Some well-known mathematical paradoxes are the following: Zeno's paradox, Eubulides's heap ("sorites") paradox, Epimenides's "liar paradox," Hilbert's "Grand Hotel" paradox, Russell's paradox, etc.

In the context of hypothetico-deductive systems, we have to accept axiomatic truths, and, simultaneously, we have to be ready to concede that various problems stem from these axiomatic truths. On the one hand, we have to accept the existence of mathematical truths, and, on the other hand, we have to concede that mathematical truths give rise to paradoxes and comprehension problems. A way out of this uncomfortable situation was offered by the great philosopher and logician Ludwig Wittgenstein, who explained this uncomfortable situation, especially regarding the capacity of our evidentiary tools. Specifically, Wittgenstein maintains that the limits of our language together with our perceptual skills determine the limits of our thinking, since they construct the image (intellectual representation) of the world that we can perceive. As we rise to higher and higher levels of abstraction, we must be prepared to confront paradoxes. Moreover, another great philosopher and logician, Kurt Gödel, proved that, in the world, there exist propositions whose truth is valid but unprovable by (i.e., within the context of) the formal mathematical framework that we have established. In other words, as I shall explain in Chapter 1, Gödel proved that, in a formal mathematical framework, there exist mathematical propositions that are necessarily true and simultaneously unprovable by means of the tools provided by the given formal mathematical framework (and, therefore, they urge us to expand our conceptual and mathematical toolbox). Gödel has mathematically proved that a completely formalized system of arithmetic (like a machine) is either inconsistent (leading to a contradiction) or incomplete (lacking in its axiomatic foundations). Absolute, mechanistic rigor is impossible.

Conceptual knowledge (which is a formally constructed and linguistically expressed kind of knowledge), far from contradicting or excluding intuitive knowledge (which is a way of knowing that is more direct, immediate, and expressing a felt sense of things), is in a relationship of mutual complementarity with intuition, specifically, rational intuition (to which I shall refer in the Introduction). Conceptual knowledge is a necessary underpinning of rational intuition, and rational intuition, in turn, provides the mental readiness of the knowing subject to recognize and accept a truth that lies before him/her. This creative synthesis between conceptual knowledge and rational intuition underpins the ancient Greek notion of "epopteia," which means having seen an object in a comprehensive way ("global vision"). Moreover, as the philosopher Michael Dummett has pointedly argued, "intuition is not a special source of ineffable insight: it is the womb of articulated understanding" (Dummett, *Truth and Other Enigmas*, p. 214).

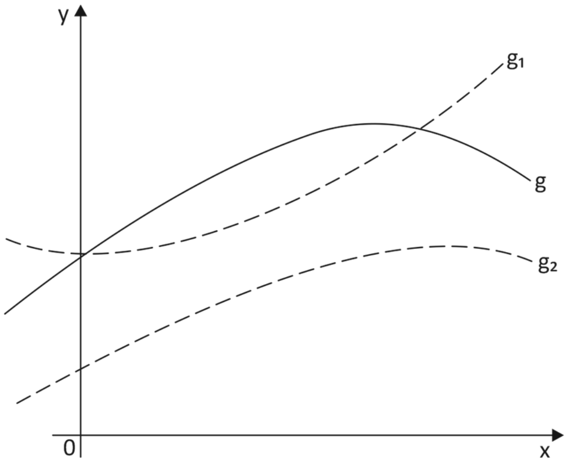
The value and the utility of mathematics do not derive from the "beauty" of mathematical formalism or from the complexity of mathematical abstractions, but from the fact that mathematics helps us to articulate representations of reality, which are useful in order to understand and/or restructure reality according to the intentionality of consciousness. I endorse the argument of the great French mathematician René Thom (1923–2002) that "what justifies the 'essential' character of a mathematical theory is its ability to provide us with a representation of reality"; and the possibility of abstracting mathematical entities from concrete situations derives from the fact that mathematics provides us with a model of the "real" (Thom, *Mathématiques essentielles*, pp. 2–3). Moreover, René Thom has brilliantly explained the term "real," which is the object of mathematical modeling, by arguing that, by the term "real," he means "both aspects of the reality of the external world—whether it is given to us by the immediate perception of the world around us, or by a mediated construction such as scientific vision" (*ibid*).

Thus, Thom's understanding of mathematics is not focused on formalism, but on a broad perspective of motion, form, and change of form, where "form" is interpreted according to Aristotle's hylozoism. According to formalism, mathematics is a game of symbols, bringing with it no more commitment to an ontology of objects or properties than chess or ludo, whereas, from the perspective of Aristotle's philosophy, mathematics is a body of propositions representing an abstract sector of reality. According to Aristotle's hylozoism, every being is composed, in an indissociable way, of matter and form, and matter is a substratum awaiting and needing to receive a form in order to become a substance, the substance of being. When Aristotle says that a being exists with regard to its substance, he refers to the "material" of which a being is composed, namely, to the "material cause" of a being. The "substantive" mode of being is complemented by form (i.e., by the "formal" mode of being), which is due to species. In his *Metaphysics*, Aristotle replaced the Platonic term "idea" with the concept of species. Form is a mode of being that is assumed by substance, and, due to its form, a being is even more sharply differentiated from every other being.

According to Thom, in the context of Aristotle's hylozoism, the notion of a bounded open set can exist as the substratum of being, whereas the notion of an unbounded open set cannot (Thom, "Les intuitions topologiques primordiales de l'aristotélisme," p. 396). Furthermore, following Aristotle's hylozoism, Thom maintains that, in mathematical modeling, the ideal of quantitative accuracy in description must always be pursued in conjunction with the ideal of qualitative accuracy in explanation. The ideal of qualitative accuracy in explanation refers to the elucidation of structure, that is, of the coherent link between the substance and the form of the phenomenon under study. In particular, Thom has considered the following case: Let us suppose that the experimental study of a phenomenon Φ gives an empirical graph g whose equation is $y = g(x)$, and that a researcher attempting to explain Φ has available two theories, say θ_1 and θ_2. In Figure 0-1, we see the empirical graph $y = g(x)$ of the phenomenon Φ, the graph $y = g_1(x)$ of theory θ_1, and the graph $y = g_2(x)$ of theory θ_2. Neither the graph $y = g_1(x)$ nor the graph $y = g_2(x)$ fits the graph $y = g(x)$ well. As shown in Figure 0-1, the graph $y =$

$g_1(x)$ fits better *quantitatively*, in the sense that, over the interval considered, $\int|g - g_1|dx$ is smaller than $\int|g - g_2|dx$. On the other hand, Figure 0-1 clearly shows that the graph $y = g_2(x)$ fits better *qualitatively*, in the sense that it has the same shape and appearance as $y = g(x)$ (e.g., more specifically, in terms of monotonicity and curvature). Hence, René Thom argues that, in this situation, the researcher should retain θ_2 rather than θ_1 "even at the expense of a greater quantitative error," because " θ_2, which gives rise to a graph of the same appearance as the experimental result, must be a better clue to the underlying mechanisms of Φ than the quantitatively more exact θ_1 " (Thom, *Structural Stability and Morphogenesis*, p. 4).

Figure 0-1: Quantitative and qualitative aspects of modeling.



In view of the foregoing, there is a strong interplay between philosophy, logic, and mathematics; and mathematical education must include a deep understanding of mathematical concepts, the methodology of mathematics, and epistemology in general. The importance of the interaction between mathematical education and philosophical education becomes even clearer in the context of interdisciplinary mathematics.

My present work on pure and applied mathematics and epistemology expresses my efforts to educate various groups of people in mathematical thinking and epistemology, starting from the basics. Mastering the basics enables one to understand the progress of science and technology and to think creatively. Moreover, this book aims to equip every aspiring person with a self-contained reference work for self-study in the fields of mathematics and epistemology.

The care this book has received from Cambridge Scholars Publishing needs special mention here. For any remaining typing errors, I am wholly responsible, and I would deeply appreciate if they are brought to my notice by the readers.

—Nicolas Laos
May 2024

INTRODUCTION: MATHEMATICAL PHILOSOPHY

Every scientific activity is based on consciousness, thinking, perception, memory, judgment, imagination, volition, emotion, attention, as well as intuition.

Consciousness can be construed as an existential state that allows one to develop the functions that are necessary in order to know both one's existential environment as well as the events that take place around oneself and within oneself. Thinking is based on symbols, which represent various objects and events, and it is a complex mental faculty characterized by the creation and the manipulation of symbols, their meanings, and their mutual relations. Perception is a process whereby a living organism organizes and interprets sensory-sensuous data by relating them to the results of previous experiences. In other words, perception is not static, but a developing attribute of living organisms; it is active in the sense that it affects the raw material of scattered and crude sensory-sensuous data in order to organize and interpret them; and it is completed with the reconstruction of the present (present sensory-sensuous data) by means of the past (data originating from previous experiences). Therefore, perception is intimately related to memory and judgment. Judgment is one's ability to compare and contrast ideas or events, to perceive their relations with other ideas or events, and to extract correct conclusions through comparison and contrast. Memory is one's ability to preserve the past within oneself—or, equivalently, the function whereby one retains and accordingly mobilizes preexisting impressions. Imagination is a mental faculty that enables one to form mental images, representations, that do not (directly) derive from the senses. Imagination is not subject to the principle of reality, as the latter is formed by the established institutions. Imagination develops because consciousness cannot conceive the absolute being in an objective way. Volition, or will, is one's ability to make decisions and implement them kinetically. Emotion or affect is the mental faculty that determines one's mood. Attention is a mental faculty that focuses conscious functions on particular stimuli in a selective way, and it operates as a link between perception and consciousness. Intuition means that consciousness conceives a truth (that is, it formulates a judgment about the reality of an object) according to a process of conscious processing that starts from a minimum empirical or logical datum and rises to a whole abstract system with which consciousness realizes that it is connected or to which consciousness realizes that it belongs (rational intuition, in particular, is intimately related to a type of subconscious thinking). In his *Republic*, Plato tries to define intuition as a fundamental capacity of human reason to comprehend the nature of the object of consciousness, and, in his works *Meno* and *Phaedo*, Plato understands intuition as the awareness of knowledge that previously existed in a dormant form within the mind. Moreover, David Hume, in his book entitled *A Treatise of Human Nature*, explains intuition as the power of the mind to recognize relationships (relations of time, place, and causation) without requiring further examination.

In general, philosophers are preoccupied with methodic and systematic investigations of the problems that originate from the reference of consciousness to the world and to itself. In other words, philosophers are preoccupied with the problems that originate from humanity's attempt to articulate a qualitative interpretation of the integration of the consciousness of existence into the reality of the world. The aforementioned problems pertain to the world itself, to consciousness, and to the relation between consciousness and the world.

It goes without saying that scientists are also preoccupied with similar problems. However, there are two important differences between philosophy and science. Firstly, from the perspective of science, it suffices to find and formulate relations and laws (generalizations) that, under certain conditions and to some extent, can interpret the objects of scientific research. Philosophy, on the other hand, moves beyond these findings and formulations in order to evaluate the objects of philosophical research and, ultimately, to articulate a *general method* and a *general criterion* for the explanation of every object of philosophical research. Whereas sciences consist of images and explanations of these images, philosophies are formulated by referring to wholes and by inducing wholes from parts. Hence, for instance, a philosopher will ask what is "scientific" about science, or what is the true meaning of science? Therefore, philosophy and science differ from each other with regard to the level of generality that characterizes their endeavors, and philosophy is a reflection on science. Secondly, as the French philosopher Pierre Hadot pointed out in his book *Philosophy As a Way of Life*, unlike the various scientific disciplines, philosophy is not merely a science, but it is a "way of life." More specifically, philosophy implies a conscious being's free and deliberate decision to seek truth for the sake of knowledge itself, since a philosopher is aware that knowledge is inextricably linked to the existential freedom