Instability of the Earth's Lithosphere

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B.I. Birger

Cambridge
Scholars
Publishing

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This book first published 2024

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN: 978-1-0364-1008-7
ISBN (Ebook): 978-1-0364-1009-4

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INTRODUCTION

In order to mathematically describe the geophysical process associated with flows in the Earth's mantle, it is necessary to add a rheological equation to the basic equations of continuum mechanics that establishes a connection between stresses, strains and time. In other words, a rheological model of the mantle should be introduced that describes the flow of material at the high levels of temperatures and pressures characteristic of the mantle. Construction of a universal, i.e. suitable for studying geophysical processes with any characteristic times, rheological model of the mantle is one of the central problems of geophysics. Having such a model, it is possible to use estimates of rheological parameters obtained when considering fast processes, for example, the processes of attenuation of seismic waves, to study slow processes, for example, thermal convection in the mantle.

Laboratory experiments with rock samples show that at small deformations, transient creep occurs, in which the strain rate decreases and the effective viscosity increases with time. According to plate tectonics, deformations in the lithosphere are very small everywhere except in those regions where the boundaries between lithospheric plates are located. Therefore, in the lithosphere, or more precisely in lithospheric plates, transient creep occurs. Thus, the rheology of the lithosphere is fundamentally different from the rheology of the underlying mantle due to the difference in the levels of deformation.

This book is devoted to the study of lithospheric processes. When considering these processes, it is necessary to have an idea of the rheology of the entire mantle. Currents generated by the lithospheric process penetrate into the underlying mantle, where they are superimposed on the main convective current associated with large deformations described by nonlinear equations. Superimposed flows are described by a linear rheological equation, the form of which depends on the characteristics of the main and superimposed flows. If the creep of geomaterial were described by the rheological model of a viscous Newtonian fluid, then any flow in the lithosphere and mantle could be characterized by a viscosity coefficient depending on pressure and temperature, and, consequently, on depth. With such a simple rheology, estimates of viscosity at different depths obtained from studying post-glacial flows in the mantle could be used to study convection in the mantle and any other processes. In the case of non-Newtonian rheology, we can talk about effective viscosity, which depends not only on the depth, but also on the characteristics of the flow under consideration. In the hereditary (having a memory) rheological model used in this book, the effective viscosity depends on the characteristic duration or periodicity of the process under consideration. The effective viscosities characterizing lithospheric processes of various durations discussed in this book differ from each other by several orders of magnitude. But since these processes occur in the lithosphere, which is described by a single rheological model, it is possible to establish relationships connecting the effective viscosities of various processes.

In the 70s, nonlinear equations describing stationary convection in the mantle were solved within the framework of the Newtonian rheological model. It has been shown that steady-state convection forms an upper boundary layer, which can be divided into a mechanical boundary layer (the lithosphere) and a thin thermal boundary layer located beneath the lithosphere. At low Rayleigh numbers, stationary mantle convection occurs, but at high Rayleigh numbers, which characterize the mantle, the thermal boundary layer becomes unstable. This instability leads to "secondary" small-scale convection [Parsons, McKenzie, 1978] and, consequently, mantle convection becomes two-scale and non-stationary, i.e. time dependent. Secondary convection destroys the thermal boundary layer, but preserves the mechanical one. If the Rayleigh number were greater than its actual value for the mantle, unsteady mantle convection would be multi-scale and chaotic [Hansen et al., 1992; Larsen et al., 1995]. Numerical experiments in which nonlinear equations of unsteady convection were solved within the framework of a rheological model of a power-law fluid with a temperature-dependent rheological parameter showed that unsteady convection also forms a mechanical boundary layer [Christensen, 1984; Christensen and Yuen, 1989; Schubert et al., 2001], and secondary small-scale convection manifests itself in the form of the movement of thermals under the lithosphere [Davaille and Jaupart, 1993; Solomatov, 1995; Dumoulin et al., 1999]. Numerical experiments in which mantle convection is modeled within the framework of a rheological model of a power-law non-Newtonian fluid show that the strong dependence of effective viscosity on temperature excludes convective motion in the lithosphere [Christensen, 1985; Solomatov, 1995]. The cold and, therefore, very viscous lithosphere behaves like a stationary lid and does not sink into the mantle, i.e. there is no subduction required in plate tectonics. The upper cold boundary layer (lithosphere), formed by mantle convection, can only sink into the mantle if its effective viscosity is not too high. Since the effective viscosity in the case of transient creep is significantly lower than in the case of steady creep, it can be assumed that a lithospheric plate having transient creep can sink into the mantle.

The model of a power-law non-Newtonian fluid, usually used in modern theoretical geophysics to describe slow and, in particular, convective flows in the mantle, well describes the steady-state creep observed in experiments, at which the deformation rate is constant (stationary flow). A large number of studies, in particular classical works [Weertman, Weertman, 1975; Weertman, 1978] and more recent works [Karato, 2008; Karato, Spetzler, 1990; Karato, Wu, 1993; Zhang and Karato, 1995;], is devoted to the micromechanisms of creep in the mantle. It has been established that the movement of dislocations, as a rule, leads to a power-law fluid model (the exception is Harper-Dorn dislocation creep, which is described by a Newtonian fluid model), and the diffusion micromechanism leads to a Newtonian fluid model. However, there is no reason to believe that the rheological model of a power-law fluid, which adequately describes steady-state creep, is applicable when considering unsteady flows in the mantle, since this model, like the Newtonian fluid model, does not have memory, unlike real geomaterial. In addition, the power-law fluid model does not take into account transient creep, which is observed in laboratory experiments at small deformations not exceeding a few percent. **In the first chapter**, a nonlinear hereditary (having a memory) rheological model is proposed to describe the creep of geomaterial, which is consistent with the theory of a simple fluid with a decaying memory and with laboratory studies of creep. The proposed model is reduced to the power-law fluid model in the case of stationary flows and to the linear hereditary Andrade model in the case of flows which cause only small deformations. The first article also discusses the already mentioned superimposed flows. When solving linear differential equations with an initial condition corresponding to the initial disturbance in a fixed region, it is convenient to apply the Laplace transform in time and the Fourier transform in the horizontal spatial coordinate. It is shown in the first article that the instability of the lithosphere displays itself in the form of low-amplitude thermoconvective oscillations. The occurrence of thermoconvective waves in the lithosphere is due to the vertical temperature gradient present in it. Linear stability analysis shows that for the cratonic lithosphere, i.e., in its thickened section, which is located under the craton, the actual value of the Rayleigh number Ra is close to the minimum critical Rayleigh number Ra_m. Thus, the cratonic lithosphere is in a regime of threshold instability.

If $Ra \leq Ra_m$, the linearized equations of convective stability, the solution of which has the form of thermoconvective waves, completely determine the evolution of small initial disturbances arising in the lithosphere. The smaller Ra, compared to Ra_m, the more the thermoconvective waves attenuate. If $Ra > Ra_m$, the initial disturbance increases with time, and at sufficiently large times the use of linearized equations becomes illegal. To investigate an unstable system, it would be necessary to solve nonlinear equations of continuum mechanics which describe large deformations. However, since Ra only slightly exceeds Ra_m, the instability develops very slowly and the deformations remain small even over long times that have passed since the onset of the small initial disturbance. According to the concepts of plate tectonics, deformations and their rates in lithospheric plates are always small, and, therefore, linear equations completely describe dynamics of the lithosphere.

The second chapter shows that the universal, i.e. suitable for studying geophysical processes with any characteristic times, rheological model of a geomaterial can be represented as a series connection of rheological elements. With such a connection, the stresses in all elements are the same, and the deformation rate of the medium is the sum of the deformation rates of the rheological elements. The rheological chain consists of an elastic element, a brittle (pseudo-plastic) element, an element with diffusion creep, which is described by the Newtonian viscous fluid model, an element with high temperature dislocation creep, which is described by the nonlinear hereditary Andrade model, and, finally, an element with low temperature dislocation creep, which described by the linear hereditary Lomnitz model. It is the low-temperature creep that determines the attenuation of seismic waves everywhere in the mantle except the asthenosphere, where the attenuation of seismic waves is due to high-temperature creep.

In the third chapter, two stability problems are solved for the lithosphere, under which developed mantle convection occurs. When solving both problems, the Andrade rheological model is used, which describes the transient creep of the lithosphere. The solution to the first problem shows that the state in which the convective flow in the mantle forms a stationary upper boundary layer (lithosphere) is unstable and bifurcation occurs into a state in which the convective flow breaks the lithosphere into mobile plates. The solution to the second problem, set for a moving lithospheric plate, shows that thickened regions of the continental lithosphere are in a regime of threshold small-scale oscillatory instability.

In the fourth chapter, the results of the study of thermoconvective oscillations obtained in the first and third articles are refined by taking into account the dependence of the Andrade rheological parameter on temperature. This chapter examines the continental craton located between orogenic belts. Disturbances in the relief of the earth's surface caused by thrusts in orogenic belts excite amplitude-modulated thermoconvective waves (wave packets) in the lithosphere. Packets of thermoconvective waves [Birger, 2000] propagate from the boundaries of the craton to its center and form a zone of thermoconvective oscillations (standing waves) in the lithosphere under the central region of the craton. Sedimentary basins are formed above the oscillation zone, which is a system of convective cells in the lithosphere with periodically changing current directions. When solving the problem of thermoconvective waves, the change in the rheological parameter of the lithosphere with depth, associated with the strong dependence of rheological properties on temperature, is taken into account, and the upper boundary of the lithosphere (the earth's surface) is considered as a deformable surface on which processes of sedimentation and erosion occur. The mobility of the upper boundary of the lithosphere, in the upper layers of which the rheological parameter greatly increases, is significantly higher than for a rheologically homogeneous model of the lithosphere.

The characteristic duration of restoration of isostatic equilibrium after an initial disturbance of the relief of the earth's surface is several thousand years, and therefore the distribution of rheological properties along the depth of the lithosphere and crust differs from the distribution that corresponds to slower geological processes. It is shown that when considering the process of restoration of isostasy, it is possible to model the upper crust as a thin elastic plate, and the underlying lower crust and lithosphere as a half-space with transient creep. For such a system, **in the fifth chapter**, using the Fourier and Laplace transformations, solutions to the equations of continuum mechanics are obtained in the form of transverse waves, which, strongly attenuated, propagate from the region of initial disturbance along the earth's surface and cause its vertical displacements. Such solutions, called inertialess Rayleigh waves, depend on the nature of the initial disturbance. In the case of a point initial disturbance, an analytical expression for these waves is found, which gives an explicit dependence of the vertical displacement of the earth's surface on the horizontal coordinate and time. Inertialess Rayleigh waves can be considered as a mechanism of modern vertical movements of the earth's crust.

The sixth chapter examines the influence of the rheology of the lithosphere, which has elasticity, brittleness (pseudo-plasticity) and creep, on folding in the earth's crust. Folding is created by horizontal compression that occurs when lithospheric plates collide. The effective viscosity, which characterizes transient creep, is lower than the effective viscosity during steady-state creep, and depends on the characteristic time of the process under consideration. The transient creep leads to such a distribution of rheological properties of the horizontally compressed lithosphere, in which the upper crust is brittle, and transient creep dominates in the lower crust and in the mantle lithosphere. It is shown that the flows, which arise in the lithosphere due to instability during horizontal compression and cause folding, are concentrated in the upper brittle crust, create a short-wave relief of the earth's surface, penetrate to a small depth into the lower crust but do not penetrate the mantle and, therefore, do not bend the Moho boundary.

The seventh chapter examines the role of the rheology of the lithosphere, which has elasticity, brittleness (pseudo-plasticity) and creep, in the process of accumulation of shear elastic deformations and stresses in the vicinity of locked faults of the earth's crust, i.e. in the earthquake preparation process. The typical duration of this process is several decades. At such times, the thin upper crust layer behaves as brittle, the underlying layer behaves as elastic (it is in this layer that stress accumulates, leading to an earthquake), and transient creep dominates in the lower crust and mantle lithosphere. The transient creep leads to a nonlinear time dependence of the deformations that occur in the vicinity of a locked fault in the elastic crust. Deformation of the basalt layer of the upper crust, where the iron content is quite high, causes a disturbance in the magnetic field by the mechanism of piezomagnetism. If creep were steady-state, the strain rate would be constant. In this case, the piezomagnetic mechanism leads to a linear dependence of the magnetic field disturbance on time, which does not correspond to observational data. The transient creep of the layer underlying the elastic crust leads to the fact that the growth rate of the magnetic field gradually decreases, reaching a constant value closer to the end of earthquake preparation process. The weaker the change in the rate of deformation and the rate of increase of the magnetic field, the closer the rupture of the locked fault. Thus, the nonlinear time dependence of the magnetic field caused by transient creep can be considered as a magnetic precursor of an earthquake.

Within the framework of the rheological model, which takes into account transient creep at small deformations, elasticity, and brittleness of the Earth's medium, the following geodynamic processes are studied in the last articles of the book: **In chapter 8**, propagation of diffusive stress waves which can be considered as a mechanism of resent movements of the Earth's crust. Propagating along the locked transform fault from a segment where the fault was ruptured that was accompanied by an earthquake, such a wave causes rupturing at the distant section of the locked fault and hence a new earthquake. **In chapter 9**, creation of earthquake foci in weakened zones of the lithosphere plunging into the mantle. In these zones, the effective viscosity, determined by transient creep, is very low, and in a short time after the application of the shear stress, significant displacements of the zone sides occur that causes seismic waves of large amplitude. **In chapter 10**, attenuation of a shear harmonic seismic wave that propagates through the fractured Earth's crust toward the Earth's surface. This attenuation is due to internal friction, i.e. friction between the sides of microcracks. Internal friction in the crust leads to distortion of the harmonic wave profile that can lead to the transformation of a harmonic wave to a shock wave. **In chapter 11**, stability analysis of heavy inclusions in the Earth's crust associated with chemical inhomogeneity or phase transitions.

References

Birger, B.I. Excitation of thermoconvective waves in the continental lithosphere //. Geophys. J. Inter. 2000. Vol.140. P. 24-36.

Christensen U. Convection with pressure- and temperature-dependent non-Newtonian rheology // Geophys. J. Roy. Astron. Soc. 1984. Vol.77. P. 343-384.

Christensen U. Heat transport by variable viscosity convection: pressure influence, non-Newtonian rheology and decaying heat sources // Phys. Earth Planet. Inter. 1985. Vol. 37. P. 183-205

Christensen U.R., Yuen D.A. Time dependent convection with non-Newtonian rheology // J. Geophys. Res. 1989. Vol. 94. P. 814-820.

Davaille, A., Jaupart, C. 1993. Transient high-Rayleigh-number thermal convection with large viscosity variations. J. Fluid Mech. Vol. 253, P. 141-166.

Dumoulin C., Doin M.-P., Fleitout L. Heat transport in stagnant lid convection with temperature- and pressure-dependent Newtonian or non-Newtonian rheology // J. Geophys. Res. 1999. Vol. 104. P. 12759-12777.

Hansen, U., Yuen D. A., Malevsky A. V. 1992. Comparison of steady-state and strongly chaotic thermal convection at high Rayleigh number. Phys. Rev. A, Vol. 46, P. 4742-4754.

Karato S. Deformation of Earth Materials – An Introduction to the Rheology of Solid Earth. New Jork. Cambridge university press. 2008. 463 p.

Karato S., Spetzler H. A. Defect microdynamics in minerals and solid-state mechanisms of seismic wave attenuation and velocity dispersion in the mantle // Reviews of Geophysics. 1990. Vol. 28. P. 399-421.

Karato S., Wu P. Rheology of the upper mantle: a synthesis // Science. 1993. Vol. 260. P. 771-778.

Larsen, T. V., Yuen D. A., Malevsky A. V., Smedsmo A. J. 1995. Dynamics of thermal convection with Newtonian temperature-dependent viscosity at high Rayleigh number. Phys. Earth Planet. Inter. Vol. 89, P. 9-33.

Parsons B., McKenzie D. P. Mantle convection and the thermal structure of the plates // J. Geophys. Res. 1978. Vol. 83. P. 4485-4496.

Schubert, G., Turcotte, D. L., Olson P. 2001. Mantle Convection in the Earth and Planets. Cambridge Univ. Press, UK. 915 p.

Solomatov V. Scaling of temperature- and stress-dependent viscosity convection // Phys. Fluids. 1995. Vol. 7. P. 266-274.

Weertman J. Creep laws for the mantle of the Earth // Phil. Trans. Roy. Soc. London. 1978. Vol. A288. P. 9-26.

Weertman J., Weertman J. R. High temperature creep of rock and mantle viscosity // Annual review of earth and planetary sciences. 1975. Vol. 3. P. 293-315.

Zhang S., Karato S. Lattice preferred orientation of olivine aggregates deformed in simple shear // Nature. 1995. Vol. 375. P. 774-777.

CHAPTER 1

RHEOLOGY OF THE EARTH AND A THERMOCONVECTIVE MECHANISM FOR SEDIMENTARY BASIN FORMATION

Summary

A power-law non-Newtonian fluid is usually assumed to model slow flows in the mantle and, in particular, convective flows. However, the power-law fluid has no memory, in contrast to a real material. A new non-linear integral (having a memory) model is proposed to describe the rheology of rocks. The model is consistent with the theory of simple fluids with fading memory and with laboratory studies of rock creep. The proposed model reduces to the power-law fluid model for stationary flows and to the Andrade model for flows associated with small strains. Stationary convection beneath continents has been studied by Fleitout & Yuen (1984), who used the power-law fluid model and obtained the cold immobile boundary layer (continental lithosphere). In a stability analysis of this layer, the Andrade model must be used. The analysis shows that the lithosphere is overstable (the period of oscillation is about 200 Ma). In the present study, it is suggested that these thermoconvective oscillations of the lithosphere are a mechanism for sedimentary basin formation. The vertical crustal movement in sedimentary basins can be considered as a slow subsidence on which small-amplitude oscillations are superimposed. The longest period of oscillatory crustal movement is of the same order of magnitude as the period of convective oscillation of the lithosphere found in the stability analysis. Taking into account the difference between depositional and erosional transport rates we can explain the permanent subsidence as well as the oscillations.

Introduction

The long history of deposition in sedimentary basins is inevitably interrupted by episodes of erosion that indicate uplift. The vertical crustal movement in a sedimentary basin can be roughly represented as an oscillation (period of the order of 200 Ma) superimposed on a permanent slow subsidence. Although many geologists consider vertical oscillatory crustal movements to be a basic type of tectonic movement (see e.g. Beloussov et al. 1974; van Bemmelen 1976) there is no generally accepted explanation for them.

Several mechanisms have been proposed to explain the vertical crustal movements that form sedimentary basins (see e.g. Quinlan 1987). It is usually suggested that the driving force for subsidence is the isostatically uncompensated ancient mass excess (e.g. Lambeck, 1983), in which case subsidence rates should rapidly decrease with time. However, cratonic basins undergo discrete periods of increased subsidence rates (reactivations). DeRito, Cozzarelli & Hodge (1983) and Cloetingh, McQueen & Lambeck (1985) explain the periods of reactivation by the application of compressional stresses induced by the collision of lithospheric plates. After the collision, a global reorientation of mantle flows and lithospheric plate movements begin. As a result of this reorientation, the plates diverge and the compressional stresses are removed. The typical duration of this tectonic cycle (of the order of 200 Ma) corresponds to the typical period of reactivation in sedimentary basins. Thus, the compressional stress mechanism for sedimentary basin formation is very attractive and is now widely used in geological studies (Cloetingh 1986).

The time dependence of the vertical displacement of the surface of the lithosphere after the compressive stress application is represented as

$$w(t) = (w_0 - w_{eq}) \exp(-t/\tau) + w_{eq} ,$$

where w_{eq} is the deflection in the equlibrium state which is settled in the presence of horizontal compression, w_0 is the initial deflection at $t = 0$, and t is the relaxation time. Thus, the reactivation period is determined by the relaxation time t rather than by the duration of the compressive stress application. Estimates show that the relaxation time is small in comparison with the typical reactivation period (about 100 Ma) implied by geological data.

To avoid difficulties related to small relaxation times, DeRito et al. (1983) have assumed that pulsing, rather than constant, stresses are applied during reactivation periods (the duration of a pulse is much shorter than the reactivation period). Furthermore, the mechanism proposed by DeRito et al., implies that the basin history has no periods of uplift accompanied by erosion, and explains slow and fast subsidences.

Birger (1995) relates the oscillatory crustal movements to thermoconvective oscillations of the upper thermal boundary layer (lithosphere) formed by convection in the Earth's mantle. The period of convective oscillations is of the same order of magnitude as the typical reactivation period.

The theory of thermoconvective oscillations, developed by the author, is based on the study of mantle rock rheology. In the present study, a new non-linear integral (having a memory) rheological model, consistent with the theory of simple fluids with fading memory, is suggested for mantle rocks instead of the previous model (Birger 1991, 1995). Permanent subsidence as well as oscillatory movements in sedimentary basins are explained by thermoconvective oscillations of the lithosphere, in contrast to the previous study (Birger, 1995) where only oscillatory crustal movements were considered.

Simple fluids with fading memory

A variety of creep micromechanisms are discussed in geophysics (see, for example, Poirier 1985; Turcotte & Schubert 1982), but, from the point of view of continuum mechanics, only two rheological modes are used. One of these models is a Newtonian fluid (a micromechanism based on the diffusion of vacancies in a crystal lattice leads to this model). The other is a power-law non-Newtonian fluid (micromechanisms based on the motion of dislocations lead to this model). In contrast to real materials, neither model has a memory or, more precisely, they have an infinitely short memory. However, models with memories are widely used in continuum mechanics. Such models are necessary to describe the behavior of materials under time-dependent stresses.

So far, only creep tests have been used in geophysics. (Creep is defined here as a flow of material under constant stress applied at a certain time.) The power-law model provides an adequate explanation for the creep tests; however, this does not imply that the power-law model is relevant to the case of time-dependent stresses. In other words, although the power- law fluid is suitable for describing creep, it is not an adequate rheological model of geomaterial.

The memory of materials is not present in the creep micromechanisms because only cases of constant stresses are considered. The rate of strain in dislocation creep is proportional to stress and inversely proportional to the squared mean distance between dislocations. Assuming this distance to be inversely proportional to stress, we arrive at a cubic dependence of strain rate on stress, i.e. at a power-law (cubic) fluid. In the case of time-dependent stress, the mean distance between dislocations is controlled by the stress history rather than by the current stress value. Thus, the dislocation micromechanism leads to rheological models with memory.

The theory of simple fluids with fading memory is a fundamental rheological theory in modern continuum mechanics (see, for example, Truesdell 1972; Astarita & Marrucci 1974). The following are basic principles in this theory: (1) determinism (the current stress is controlled by the strain history and is independent of future strains); (2) locality (the stress at a given point in space is determined by the strain history in an infinitely small vicinity of the material point coincident with the given point at the instant of observation); (3) non-existence of a natural state (this principle distinguishes fluids from solids and implies that strain is measured from the current state); (4) fading memory (the current stress depends on the recent history of strain more strongly than on its more distant past).

A simple fluid with fading memory is described by the equation

$$\sigma_{ij}(x_i, t) = F\big[\varepsilon_{ij}(t-s)\big], \qquad 0 \le s \le \infty, \tag{1}$$

where σ_{ij} is the deviatoric stress tensor, ε_{ij} is the strain tensor, x_i are the Eulerian spatial coordinates, $\tau = t - s$ is time, t is the observation time, s is the time-lag, and F is the tensor functional of the strain prehistory tensor, where the strain prehistory is the strain at times preceding the observation time. The strain in (1) is defined such that

$$\varepsilon_{ij}(t) = 0, \tag{2}$$

that is, the strain at the observation time (zero time-lag) vanishes.

Coleman & Noll (1960) showed that, for a very slow flow of a fluid with the rheology of eq. (1), the Newtonian fluid rheological equation

$$\sigma_{ij} = 2\eta\dot{\varepsilon}_{ij} \tag{3}$$

is valid. In eq. (3), $\dot{\varepsilon}_{ij}$ is the strain rate tensor (a dot denotes ij the time derivative) and η is the Newtonian viscosity. When strains are small, eq. (1), as Coleman & Noll (1961) have shown, reduces to Boltzmann's rheological relationship

$$\sigma_{ij} = 2\int_0^\infty R(s)\varepsilon_{ij}(t-s)ds, \tag{4}$$

where *R(s)* is the relaxation kernel.

Integrating the right-hand side of eq. (4) by parts, we obtain the alternative form

$$\varepsilon_{ij} = 2\int_0^\infty \Pi(s)\dot{\varepsilon}_{ij}(t-s)ds, \tag{5}$$

where $\prod(s)$ is the memory kernel related to the relaxation kernel by

$$R(s) = -\frac{d\Pi}{ds}, \qquad \Pi(\infty) = 0. \tag{6}$$

An experimental determination of the functional *F* in eq. (1) is impracticable, because it would require an inconceivable number of tests. Another approach to the rheometry of simple fluids focuses on the analysis of specific (rheometric) flows. A rheometric flow of a given type is defined as an infinite family of flows completely described by one or several parameters. A periodic flow characterized by two parameters, frequency and amplitude, is rheometric, and so is a linear Couette flow (simple shear flow) depending on a single parameter, the rate of shear strain. Experimental tests can yield the rheological relationship for rheometric flows. However, material can behave in an unpredictable manner in flows of other types. A third approach reduces eq. (1) to a more specific form of non-linear integral equation. This is the approach that will be used in the present study.

Creep of geomaterials

Fig. 1 demonstrates a typical creep curve derived from laboratory tests. A constant stress is applied to the specimen at time $\tau = 0$. The curve can be divided into two stages. The first stage represents transient creep (the strain rate decreases with time). The second stage shows steady-state creep (the strain rate is constant). The transition to steady-state creep occurs at creep strains of several per cent, regardless of the stress at which the experiment is carried out. Furthermore, for transient creep it is established that the strain depends linearly on applied stress. For mantle rocks at high temperatures, transient creep is well described by the Andrade law:

$$2\varepsilon_{ij} = \sigma_{ij} f(\tau), \qquad f(\tau) = \tau^m / A\,, \tag{7}$$

where $f(\tau)$ is the creep function (an analytical form of the transient creep curve), A is a temperature- and pressure-dependent rheological parameter, and m is a dimensionless exponent ($0 < m < 1$). The typical value of m for mantle rocks is 1/3.

Strain in (7) is measured from the state at time $\tau = 0$ when a constant stress is applied, rather than from the state at the time of observation. The strain vanishes at $\tau \leq 0$ and is nonzero at $\tau = t$, thus violating condition (2) for simple fluids. The notation e_{ij} denotes strain measured from the state at $\tau = 0$. The second invariant of the strain tensor $e = (e_{kl}e_{kl} / 2)^{1/2}$ is shown in Fig. 1 as a function of time. In the case of simple extension usually used in tests, one has $e = \sqrt{2}\, e_{11}/2$, where e_{11} is the only non-zero strain component. On the introduction of the time-lag $s = t - \tau$ and with strain measured from the state at observation time $\tau = t$, the Andrade law (7) takes the form

$$2\varepsilon_{ij} = \sigma_{ij} \frac{[t^m - (t-s)^m]}{A}, \quad 0 \leq s \leq t, \tag{8}$$
$$\varepsilon_{ij} = \frac{\sigma_{ij} t^m}{A}. \qquad s \geq t.$$

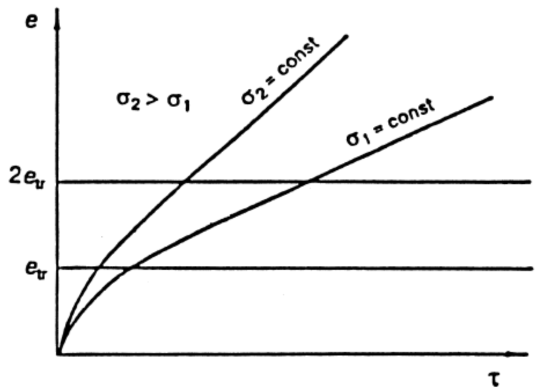

Figure 1. Experimental creep curves.

The constant strain rate at the stage of steady-state creep depends non-linearly on the constant applied stress

$$\dot{\varepsilon}_{ij} = \mathrm{B}\sigma^{n-1}\sigma_{ij}, \qquad \sigma = (\sigma_{kl}\sigma_{kl} \,/\, 2)^{1/2}\,. \tag{9}$$

Inversion of this relationship yields

$$\sigma_{ij} = B^{-1/n}\dot{\varepsilon}_{ij}{}^{1(1-n)/n}\dot{\varepsilon}_{ij}, \qquad \dot{\varepsilon}_{ij} = (\dot{\varepsilon}_{kl}\dot{\varepsilon}_{kl} \,/\, 2)^{1/2}\,, \tag{10}$$

where σ and $\dot{\varepsilon}$ are the second invariants of the deviatoric stress tensor and strain rate tensor, respectively, and B is a temperature- and pressure-dependent rheological parameter. A typical value of the exponent is $n = 3$.

Eq. (9) is identical to the rheological equation of a power-law non-Newtonian fluid. However, the power-law model requires eq.(9) to be valid notonly for constant stress (steady-state creep), but also for any time variation of stress. As indicated above, the power-law model, which is suitable in the case of creep, is not adequate in the general case of variable stress. In the next section, a rheological model, consistent with the theory of simple fluids with fading memory and satisfying creep laws (7) and (9), will be constructed.

Rheological model of the mantle

Boltzmann's linear integral relations (4) and (5) reduce to the Andrade law if the memory and relaxation kernels take the form

$$\Pi(s) = As^{-m}/m\Gamma(m)\Gamma(1-m), \tag{11}$$

$$R(s) = -\frac{d\Pi}{ds} = As^{-m-1}/\Gamma(m)\Gamma(1-m), \tag{12}$$

where Γ is the gamma function. To verify this it is sufficient to substitute (8) into the right-hand side of (5) and to observe that eq. (5) results in a stress that is independent of time only when $\Pi(s)$ has the form (12). It is easy to integrate the right-hand side of (5) using Laplace transforms. The integral model governed by eqs (4), (5), (11), and (12) will be called the Andrade rheological model in this paper. It generalizes the Andrade law for transient creep and describes the mantle rheology at small strains but, being linear, is not relevant to large strains.

The simplest non-linear integral model consistent with the theory of a fluid with fading memory was suggested by Tanner & Simmons (see Astarita & Marrucci 1974):

$$\sigma_{ij} = 2\int_0^{\infty} R(s)g(\varepsilon)\varepsilon_{ij}(t-s)ds, \qquad (13)$$

where

$$g(\varepsilon) \equiv 1, \quad if\ \varepsilon \leq \varepsilon_{tr}$$

$$g(\varepsilon) \equiv 0, \quad if\ \varepsilon > \varepsilon_{tr}\ .$$

Here, $e = (e_{kl}e_{kl}\ /\ 2)^{1/2}$ is the second invariant of strain and ε_{tr} is its transition value ($\varepsilon_{tr} = e_{tr}$) observed in creep tests (see Fig. 1).

Consider the steady-state creep within the framework of RMM. Substitute $\varepsilon_{ij} = \dot{\varepsilon}_{ij}s$ into (13), where $\dot{\varepsilon}_{ij}$is independent of *s* and *t*. The result is

$$\sigma_{ij} = 2\dot{\varepsilon}_{ij}\int_0^{s_m} sR(s)ds, \qquad (14)$$

where the memory depth s_M (strain at $s > s_M$ does not affect the stress) is defined as

$$s_m{=}\varepsilon_{tr}/\dot{\varepsilon}, \qquad (15)$$

where $\dot{\varepsilon}$ is the constant second invariant of the strain rate $\dot{\varepsilon}_{ij}$. It follows from (12), (14), and (15) that

$$\sigma_{ij} = 2A(\varepsilon_{tr}/\dot{\varepsilon})^{1-m}\dot{\varepsilon}_{ij}/(1-m\Gamma(m)\Gamma(1-m). \qquad (16)$$

Thus, RMM leads to eq. (10) for steady-state creep, with the parameters n and B related to the parameters of RMM by

$$n = \frac{1}{m}\,, \qquad \frac{1}{B} = \left(\frac{2A}{(1-m\Gamma(m)\Gamma(1-m)}\right)^{1/m} \varepsilon_{tr}^{(1-m)/m}\ . \qquad (17)$$

The numerical factor in (16) and (17) is (1− *m*) Γ(*m*) Γ(1−*m*) ≈ 3 for the value *m* = 1/3 typical of mantle rocks.

Eq. (14) is valid if a flow with constant strain rate (steady-state creep) exists over a time interval exceeding the memory depthss_M. Therefore, RMM implies the steady-state creep law (10) when $\varepsilon \geq 2\varepsilon_{tr}$ (under this condition the medium completely 'forgets' the initial stage of transient creep). When $\varepsilon \leq \varepsilon_{tr}$, this model leads to the Andrade law for transient creep. The range $\varepsilon_{tr} < e < 2\varepsilon_{tr}$ is not considered here. This is an intermediate range (see Fig. 1) where the experimental creep law is not yet clear.

In RMM, eqs (14) and (16) describe a stationary flow (s is not bounded) as well as the steady-state creep in a sufficiently long time interval ($s \geq s_M$). In the time interval $s \geq s_M$, the steady-state creep flow behaves like a stationary flow. The initial stage of creep (transient creep) obeys the Andrade law.

Periodic flows

Consider a periodic flow of small amplitude where the strain rate takes the form

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}(x_i)\exp[i\omega(t-s)], \quad (18)$$

where $\dot{\varepsilon}_{ij}(x_i)$is a space-dependent amplitude and ω is the frequency. The strain tensor associated with (18) is given by

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}(x_i)\exp\,(\mathrm{i}\omega t)[1-\exp\,(-i\omega s)]. \quad (19)$$

Clearly, the strain vanishes at zero lag ($s = 0$). This conforms to condition (2) for simple fluids.

Substituting (18) into (5) or (19) into (4) we obtain the rheological law for periodic flows of small amplitude in the Newtonian form

$$\sigma_{ij} = 2F(\omega)\dot{\varepsilon}_{ij}, \quad (20)$$

$$F(\omega) = \int_0^\infty \Pi(s)\exp(-\mathrm{i}\omega s)ds = \Pi^*(i\omega), \quad (21)$$

where $\Pi^*(i\omega)$ is the Laplace transform of the memory kernel $\Pi(s)$. The function $F(\omega)$ is the effective Newtonian viscosity depending on frequency. In the case of the Andrade model and, hence, in RMM, which reduces to the Andrade model at small strains, $F(\omega)$ takes the following form:

$$F(\omega) = A(i\omega)^{m-1}/m\Gamma(m). \quad (22)$$

Consider now a periodic small-amplitude flow imposed on a basic stationary flow associated with large strains. Strain and stress are represented in this case as

$$\varepsilon_{ij} = \bar{\varepsilon}_{ij} + \varepsilon'_{ij}, \qquad \varepsilon_{ij} = \bar{\sigma}_{ij} + \sigma'_{ij}, \tag{23}$$

where overbars and primes mark basic and imposed flows, respectively. Following from eq. (16), the effective viscosity of basic flow in RMM, equivalent to the power-law model for stationary flows, takes the form

$$\bar{\eta} = \frac{A s_m^{1-m}}{(1-m)\,\Gamma(m)\,\Gamma(1-m)}\,, \qquad s_m = \varepsilon_{tr}/\dot{\bar{\varepsilon}}\,. \tag{24}$$

The rheological relationship for the imposed periodic flow is obtained by substituting (23) into (13), using (18) and (19), and linearizing in small strain $\dot{\varepsilon}_{ij}$:

$$\sigma'_{ij} = 2\eta_{ijkl}\dot{\varepsilon}'_{kl}, \tag{25}$$

where the tensor of the fourth order defining the anisotropic viscosity in RMM is of the form

$$\eta_{ijkl} = \frac{[\gamma(1-m,\ -\omega s_M) - (i\omega s_M)^{1-m} H(\omega s_M)]\delta_{ik}\delta_{jl} A(i\omega)^{m-1}}{\mathrm{m}\,\Gamma(m)\,\Gamma(1-m)} - \frac{2(1-m)\bar{\eta} H(\omega s_M)\dot{\bar{\varepsilon}}_{ij}\dot{\bar{\varepsilon}}_{kl}}{\dot{\bar{\varepsilon}}^2}\,. \tag{26}$$

Here σ_{ik} is the identity tensor, $\dot{\bar{\varepsilon}}$ is the second invariant of the strain rate tensor in the basic flow,

$$H(\omega s_M) = [1 - \exp(-i\omega s_m)]/i\omega s_m \tag{27}$$

and γ is the incomplete gamma function,

$$\gamma(\alpha, x) = \int_0^x y^{\alpha-1} e^{-y} dy, \mathcal{R}e\ \alpha > 0.$$

The incomplete gamma function has the following asymptotic properties:

$$\gamma(\alpha, x) = \Gamma(x), \quad |x| \gg 1,$$

$$\gamma(\alpha, x) = x^{\alpha}/\alpha, \quad |x| \ll 1, \tag{28}$$

and in the present case $\alpha = 1 - m,\ x = i\omega s_M$. Observing that $H(\omega s_M)$ is equal to 1 at $\omega s_M \ll 1$ and using the asymptotics (28) in eq. (26), we find

$$\eta_{ijkl} = \bar{\eta}\left[\delta_{ik}\delta_{jl} - \frac{2(1-m)\dot{\bar{\varepsilon}}_{ij}\dot{\bar{\varepsilon}}_{kl}}{\dot{\bar{\varepsilon}}^2}\right], \qquad \omega s_m \ll 1, \tag{29}$$

$$\eta_{ijkl} = [A(i\omega)^{m-1}/\mathrm{m}\,\Gamma(m)]\delta_{ik}\delta_{jl}\,, \quad \omega s_m \gg 1\,. \tag{30}$$

Eq. (29) shows that the imposed flow is fully described by the power-law model when the period of oscillation is great compared to the memory depth in the basic stationary flow. We can arrive at eqs (25) and (29) by substituting (23) into eq. (10) which describes the power-law model.

When the period of oscillation is small compared with the memory depth, the rheology of the imposed flow is isotropic and the same as that of a small-amplitude periodic flow in the absence of basic stationary flow. This is verified by observing that eqs (25) and (30) are identical with (20) and (22). Note that eq. (30) follows from (26) as the zero-order approximation in the small parameter $(\omega s_M)^{-m}$. The rheological anisotropy of the imposed flow appears in the first-order approximation in this parameter. The ratio of the effective viscosity of the imposed flow (22) to the effective viscosity of the basic flow (24) is expressed as

$$F(\omega) = (i\omega s_M)^{m-1}(1-m)\,\Gamma(1-m)/\mathrm{m}. \tag{31}$$

This expression shows that the effective viscosity of the imposed flow is significantly less than that of the basic flow when $\omega s_M \gg 1$.

In the mantle rheological model proposed earlier (Birger 1991, 1995), the rheology of the imposed flow depends on the duration of the basic steady-state creep flow. When this duration tends to infinity (the case of stationary flow), the effective viscosity of the imposed flow tends to zero; that is, the stationary flow is always unstable. This shortcoming of the previous model is related to the excessively long memory depth. The rheological model introduced here (RMM) has a shorter memory. As follows from eq. (13), the memory fading in RMM is determined not only by the decreasing function *R(s)*, but also by the abrupt termination of memory when ε attains the value ε_{tr}.

Elasticity and diffusion rheology

The strain tensors ε_{ij} and e_{ij} in the above equations are are associated with the dislocation micromechanism. The total strain E_{ij} is the sum of strains due to mechanisms of dislocation, diffusion, and elasticity. To take into account elasticity and the diffusion component of rheology, it is sufficient to substitute

$$\dot{\varepsilon}_{ij}=\dot{E}_{ij}-\frac{\dot{\sigma}_{ij}}{2\mu}-\frac{\sigma_{ij}}{2\eta} \quad (32)$$

into (13). Here μ is the elastic shear modulus and η is the Newtonian viscosity due to the diffusion rheology, which is described by the Newtonian model (3). The mechanical analogy of the rheological model (32) consists of three elements that are connected in series and represent elasticity, diffusion rheology, and dislocation rheology. The reciprocal effective viscosity of this model equals the sum of reciprocal viscosities of three such elements. Thus, to neglect the diffusion rheology it is necessary to assume the dislocation effective viscosity to be much lower than the diffusion viscosity η. It follows from (16) that the diffusion component in RMM can be neglected in the case of stationary flow if the following condition holds:

$$\dot{\bar{\varepsilon}} >> \varepsilon_{tr}(A/\eta)^{1/(1-m)}. \quad (33)$$

From eqs (22) and (30) it follows that, in the case of periodic flow as well as imposed periodic flow, the diffusion rheology can be neglected under the condition

$$\omega \gg (A/\eta)^{1/(1-m)}. \quad (34)$$

The Newtonian viscosity η associated with diffusion rheology is assumed here to be high enough for eqs (33) and (34) to be satisfied, even with the slowest geophysical processes. Nevertheless, it is important to introduce the Newtonian viscosity η to establish a link with the theory of simple fluids with fading memory. According to this theory and to creep tests, a material behaves like a Newtonian fluid in the limit of 'slow motion'($\dot{\varepsilon} \to 0$). For the rheological model given by eqs (13) and (32), η is the viscosity corresponding to 'slow motion'. It follows from eq. (24) that the effective viscosity $\bar{\eta}$, taken without the diffusion component, tends to infinity in the case of very slow flow when $\dot{\bar{\varepsilon}}$ approaches zero. The diffusion component of rheology sets the upper limit η for the effective viscosity.

Elastic effects can be ignored in the case of periodic flows if the following condition holds:

$$\omega \ll (\frac{\mu}{A})^{1/m}. \quad (35)$$

The condition (35) is obtained in a similar manner to (34) by comparison of the respective effective viscosities.

The parameter A of RMM is related to the rheological parameter B obtained in laboratory studies of steady-state creep in mantle rocks by eq. (17). At the typical values of m = 1/3 and ε_{tr} = 0.1 for mantle rocks, eq. (17) reduces to

$$A \approx 5B^{-1/3}. \quad (36)$$

Data from laboratory studies (Kirby & Kronenberg 1987) and eq. (36) lead to an estimation for the mean value of A over the continental lithosphere of $A \approx 10^{12}$ Pa $s^{1/3}$. The shear modulus μ is estimated to be 10^{11} Pa, and hence the right-hand side of (35) is of the order of 10^{-3} s^{-1} and the elastic effect can be neglected when oscillations with large periods are considered. Convective oscillations of the lithosphere with periods of the order of 100 Ma will be studied in this paper.

For the adopted values of m and A, condition (34) holds even for the slowest geophysical processes if $\eta \approx 10^{24}$ Pa s. The right-hand side of (34) is about 10^{-18} $s^{-1} \approx 3\times10^{-11}$ yr^{-1} with this value of η, and we can ignore the diffusion component in the universal rheological relationship (32).

Stability analysis of the lithosphere

The buoyant nature of continental crust prevents large-scale subduction in contrast to large-scale convective circulation beneath oceans. Fleitout & Yuen (1984) have solved the problem of steady-state thermal convection numerically in the upper mantle beneath old continental cratons using a temperature- and pressure-dependent power-law fluid rheological model (see also Christensen 1984). Beneath the upper boundary layer (continental lithosphere), there is small-scale convective motion with velocities of the order of several centimeters per year and temperature gradients close to zero. The temperature drop through the boundary layer is about 1300 K, the thickness of boundary layer is about 200 km, and velocities are equal to zero within the layer.

Thus, the continental lithosphere behaves as an immobile lid over the underlying convective layer. Following on from (10), the effective viscosity for a power-law fluid (n = 3) is given by

$$\eta_{eff} = (2B)^{-1/3}\dot{\varepsilon}^{-2/3}.$$

Since $\dot{\varepsilon} = 0$ for the boundary layer, η_{eff} goes to infinity in the continental lithosphere. Hence, within the framework of the power-law fluid model, this layer is convective-stable in spite of the presence of a large temperature gradient. Here $\dot{\varepsilon}$ and η_{eff} are the strain rate and the effective viscosity associated with the thermal convection rather than with the other processes in the lithosphere.

In the present study, the problem of thermal convection beneath continents solved by Fleitout & Yuen (1984) is considered in a more adequate rheological model of the mantle (RMM). Since RMM reduces to the power-law fluid model in the case of stationary flow, the state of steady-state convection in RMM is identical to the solution found by Fleitout & Yuen. However, the stability analysis of the state of stationary convection in RMM differs greatly from such an analysis in the power-law model.

It is clear that the stability of stationary convection is determined by the stability of the boundary layer (the most unstable part of a convective cell). A local stability analysis of the upper boundary layer will be done below. In fact, this In fact, this approximate approach to the stability problem implies the introduction of simplified conditions at the boundary between the layer under consideration and the rest of the stationary convection domain.

If we solve the problem of the stability of stationary convection without using the local stability approximation, then we should consider the imposed flows in the mantle underlying the boundary layer. The rheology of imposed flows is determined by eqs (23) – (31). These relationships are not relevant to the local stability analysis: the rheology of the boundary layer (lithosphere) is described by eqs (20) and (22).

The upper cold boundary layer of basic stationary convection is modeled as a homogeneous infinite layer. Its physical properties are given by the following parameters: density $\rho \approx 3.3 \times 10^3$ kg m^{-3}; thermal diffusivity $k \approx 10^{-6}$ m^2 s^{-1}; thermal expansion $\alpha \approx 4 \times 10^{-5}$ K^{-1}; gravitational acceleration $g \approx 10$ m s^{-2}; elastic shear modulus $\mu \approx 10^{11}$ Pa; temperature drop between the lower and the upper isothermal surfaces of the layer $\Delta T \approx 10^3$ K; and the thickness of the layer $d \approx 2 \times 10^5$ m. We do not take into account the thickening of the boundary layer within a convective cell.

The origin of the coordinate system is situated on the lower boundary of the layer. The z-axis is directed vertically upwards. Only 2-D convective modes are considered. The linearized equations governing the stability of the boundary layer are

$$\frac{\partial v'_{x,}}{\partial x} + \frac{\partial v'_z}{\partial z} = 0, \quad (37)$$

$$-\frac{\partial p'}{\partial x} + \frac{\partial \sigma'_{xx}}{\partial x} + \frac{\partial \sigma'_{xz}}{\partial z} = 0, \quad (38)$$

$$-\frac{\partial p'}{\partial z} + \frac{\partial \sigma'_{xz}}{\partial x} + \frac{\partial \sigma'_{zz}}{\partial z} + Ra\theta' = 0, \quad (39)$$

$$\frac{\partial \theta'}{\partial t} - v'_{z,} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\theta' = 0, \quad (40)$$

where $\theta', p', v'_{i,}, \sigma'_{ij}$ are the non-dimensionalized perturbations of temperature, pressure, velocity, and components of deviatoric stress tensor. The length scale is the thickness of the layer $d \approx 2 \times 10^5$ m, the velocity scale $k/d \approx 5 \times 10^{-12}$ m s^{-1}, the timescale $d^2 / k \approx 4 \times 10^{16}$ s, the pressure and stress scale $\mu \approx 10^{11}$ Pa, and the temperature scale $\Delta T \approx 10^3$ K. The Rayleigh number is defined as

$$R\alpha = \alpha \rho g \Delta T d / \mu$$

This dimensionless parameter is estimated as $R\alpha \approx 3\times10^{-3}$ for the boundary layer. If we introduce the reference viscosity $\eta = \mu\, d^2 / k$, then the definition of the Rayleigh number is identical to that used for a Newtonian fluid. In writing eqs (37)–(40) we neglect the inertia terms in the momentum eqs (38) and (39), since the Prandtl number is very large, and use the Boussinesq approximation.

To complete the governing equations we must add a rheological relationship. Since strains associated with the basic stationary convective flow vanish in the boundary layer $(\bar{\varepsilon}_{ij} = 0)$ and strain perturbations are assumed to be small in∞ linear stability analysis $(\varepsilon'_{ij} < \varepsilon_{tr})$, RMM used in the problem reduces to the Andrade model, governed by (4), (5), (11) and (12), for the boundary layer.

We seek a solution for the governing eqs (37) - (40) in the form of an oscillation:

$$b'_1 = B_1(z)\exp(\mathrm{i}\omega t)\cos kx, \qquad b'_2 = B_2(z)\exp(\mathrm{i}\omega t)\sin kx, \quad (41)$$

where b'_1 is one of variables $v'_z, \theta', \sigma'_{xx}, \sigma'_{zz}$ and b'_2 is one of v'_x, σ'_{xz}. The dimensionless wavenumber k describes periodicity in x. The dimensionless frequency ω is a complex number. $B_1(z)$ and $B_2(z)$ are the complex amplitudes of the corresponding variables.

The Andrade rheological model for periodic flows is governed by eqs (20) - (22). Using non-dimensionalized variables, we rewrite (22) in the form

$$F(\omega) = \beta(i\omega)^{m-1}/\mathrm{m}\,\Gamma(m), \tag{42}$$

where β is the dimensionless rheological parameter

$$\beta = (A/\mu)(\kappa d^2)^m. \tag{43}$$

Only the value $m = 1/3$ is used below. For this value, $m\Gamma(m) \approx 1$ in eq. (42). The complex dimensionless function $F(\omega)$ can be called the complex viscosity. Eqs (42) and (43) determine the complex viscosity for the Andrade model.

Substituting (41) into eqs (37), (38) and using the rheological relation (20), we find

$$V_x = -\frac{1}{k} D V_z, \tag{44}$$

$$P = \frac{1}{k^2} F(\omega)(D^3 - k^2 D)V_z, \tag{45}$$

$$\Sigma_{zz} = -\Sigma_{xx} = 2F(\omega) D V_z, \tag{46}$$

$$\Sigma_{xz} = -\frac{1}{k} F(\omega)(D^2 + k^2)V_z, \tag{47}$$

where D is the differential operator $D = d/dz$.

Substituting (41) into eqs (39) and (40) and using (44)–(47), we obtain the governing equations for stability analysis:

$$F(\omega) L^4 V_z - Ra k^2 \Theta = 0, \tag{48}$$

$$(i\omega - L^2)\Theta - V_z = 0, \tag{49}$$

where L is the differential operator: $L^2 = D^2 - k^2$.

Let us suppose that the vertical velocity, the tangential stress and the temperature perturbation vanish on the lower and upper surfaces of the layer:

$$z = 0;\ z = 1;$$

$$v_z' = 0, \quad \sigma_{xz}' = 0. \quad \theta' = 0. \tag{50}$$