Basic Concepts of Mathematical Analysis

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1

SET THEORY

1.1 Concept of sets

Sets are usually denoted by capital letters such as A, B ... X, etc., while their elements use lowercase letters a, b ... $x,$ etc. But in the representation of the set, we can also use different symbols and punctuation elements, such as points, numbers, geometric figures, etc. For example, set A with elements a, b ... etc. is denoted by:

$$A = \{a, b, \cdots\}.$$

Sets can also be described. For example:

$$A = \{x\colon\ x \text{ has property } P\} \text{ or } A = \{x\ /P(x)\}.$$

Another example:

- Set $A = \{x\ /\ x = 2k, k - \text{integer number}\}$, express the set of even numbers;
- Set $B = \{x\ /\ sinx = 0\}$, express the set of solutions of the function sin(x), etc.

There are finite and infinite sets.
For example:

- $\{a_1, a_2 \cdots\}$ is an infinite set;
- $\{b_1, b_2, \cdots,\ b_n\}$ is a finite set with $n -$ elements.

So, very often we consider sets with one element $\{p\}$, where p could be any object, for example $\{1,2,3\}$. The empty set, i.e. the set without any elements, will be denoted by Φ.

1.2 Quantifications $\exists$ and $\forall$

When an element a exists from the set A, then it will be denoted by $\exists\ a$ from set A. For a set A, we say that it is a subset of set B if all the elements of set A also belong to set B, and this is symbolically denoted by $A \subset B$, which means that $\forall x \in A$, then it follows that $x \in B,$ in cases where the inclusion of those sets is rigorous (this means that there are elements in set B that are not in the set A), and $A \subseteq B$ in cases where the inclusion is not rigorous.

We say that two sets A and B are equal to each other, and denoted by $A = B$, if the following relations hold: $A \subset B$ and $B \subset A$.

1.3 Set operations

Definition 1.3.1 Union of the sets

The union of two sets A and B, symbolically A ∪ B, is called the set which deals with the collection of the elements of those two sets into a single set without repeating them:

$$A \cup B = \{x\colon x \in A \vee x \in B\},$$

where the symbol $\vee$ is read as ‘or’.

Next, we see that the union of sets can also be defined for a larger number of sets. By the union of $n -$ sets, we mean the new set:

$$\bigcap_{i=1}^{n} A_i = \{x/\forall i \in \{1,2,3, \ldots, n\}, x \in A_i\},$$

Analogously, the union of sets for a countable number of sets is expressed using the relation:

$$\bigcap_{i=1}^{\infty} A_i = \{x/\forall i \in \mathbb{N}, x \in A_i\},$$

which means that the element x belongs to the countable union of sets if it belongs to at least one of those sets.

Definition 1.3.2 Intersection of sets

The intersection or common part of the sets A and B, denoted $A \cap B,$ is the set

$$A \cap B = \{x|x \in A \wedge x \in B\}.$$

The symbol $\wedge$ is read as ‘and’.

This means that an element x belongs to the sets A and B if it belongs to both of them together.

With regards to the cutting of sets, we can consider cutting a finite number of them, and it is defined as follows:

$$\bigcap_{i=1}^{n} A_i = \{x/\forall k \in \{1,2,3,\ldots.,n\}, x \in A_k\},$$

which means an element x belongs to the cutting of $n-$ sets if it belongs to each of the considered sets. As in the case of the union of sets, we can consider cutting the countable sets, and it is defined as follows:

$$\bigcap_{i=1}^{\infty} A_i = \{x: \forall k \in \mathbb{N}, x \in A_k\}.$$

Definition 1.3.3 The difference between sets

The difference between set A and B, which symbolically is denoted by $A \setminus B$, is a set that consists of elements that are in the first set, but not in the second set:

$$A \setminus B = \{x|x \in A \wedge x \notin B\}.$$

Example 1.3.4 Find the difference between sets:

$$A = \{1,2,3,4,5\} \text{ and } B = \{3,6,8,9\}.$$

$A \setminus B = \{x|x \in A \wedge x \notin B\} = \{1,2,3,4,5\} \setminus \{3,6,8,9\} = \{1,2,4,5\}.$

If $A \subseteq B$, then the difference $B \setminus A$ is called the complement of set A into B and is denoted in one of the following forms: $C_B A$, $\overline{B}$, or B^C.

In the case where set $\boldsymbol{B}$ and is denoted in one of the following forms: $\overline{A}$ or A^C and we call it the compliment of A. It is obvious $C(CA) = A$ or $\overline{\overline{A}} = A$.

Taking into consideration the difference between sets, we can define the symmetric difference of sets.

The symmetric difference of sets A and B is defined by the relation:

$$A\Delta B = (A\backslash B) \cup (B\backslash A),$$

which expresses the union between sets $A\backslash B$ and $B\backslash A$.

The symmetric difference between sets can also be defined as follows:

$$A\Delta B = (A \cup B)\backslash(A \cap B).$$

Based on these definitions and the properties of the operations of the sets, we can prove the following relations:

Proposition 1.3.5 For given sets A, B and C, the relations are:

$$\begin{gathered} A\backslash B = A \cap \overline{B} \\ A\Delta B = \overline{A}\Delta\overline{B} \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ \overline{(A \cup B)} = \overline{A} \cap \overline{B} \\ \overline{(A \cap B)} = \overline{A} \cup \overline{B}. \end{gathered}$$

Proof: We will now prove one of these relations. For example, in the first case, we know we have to prove that the left-hand side of the above relation should be included in the right-hand side and vice versa.

$$x \in A\backslash B \Rightarrow x \in A \wedge x \notin B \Rightarrow x \in A \wedge x \in \overline{B} \Rightarrow x \in \left(A \cap \overline{B}\right)$$

proves one side of the relation. We will now prove the other side of the relation:

$$x \in \left(A \cap \overline{B}\right) \Rightarrow x \in A \wedge x \in \overline{B} \Rightarrow x \in A \wedge x \notin B \Rightarrow x \in A\backslash B.$$

Hence, the first relation is proved. We can prove the other relations given in the above proposition in a similar way.

1.4 The partitive set - Boole's algebra

The main idea here is that from a set M another set $P(M)$ is formed, the elements of which make the whole of set M.

Definition 1.4.1 The set of all parts (subsets) of the set M, (including M and Φ), is called the set of parts (partitives) of M. So,

$$P(M) = \{X | \forall X \subseteq M\}.$$

Example 1.4.2 Find the set of parts for M={a,b,c}.

Solution:

$$P(M) = \{\Phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

If M is an arbitrary set and $P(M)$ is the set of its parts, we apply the operations cut, union, and complement to the elements of $P(M)$. These operations are closed in $P(M)$ and the ordered system $(P(M), \cup, \cap, \setminus)$ is called the Boolean algebra (after George Boole) of the set M.

1.5 Direct (Cartesian) production of sets

Let the sets A and B be given. By the Cartesian product of those sets, we mean the set:

$$A \times B = \{(a, b): a \in A \wedge b \in B\}.$$

Example 1.5.1 The sets A={1,2,3} and B={5,6} are given. Find their Cartesian product.

Solution: From the definition of the Cartesian product, we have:

$$A \times B = \{(a, b): a \in A \wedge b \in B\} = \{1,2,3\} \times \{5,6\}$$
$$= \{(1,5), (1,6), (2,5), (2,6), (3,5), (3,6)\}.$$

We know that in sets the relation $\{x, y\} = \{y, x\}$ is valid. However, this does not apply to the listed pairs. If we want to specify the order of the elements in the sets, then, for example, instead of $\{x, y\}$ we write (x, y) where x is the first element, y the second element, and $(x, y) \neq (y, x)$ holds, except for when $x = y$. In this case, (x, y) is called an ordered pair.

For two ordered pairs (a, b) and (c, d), we say that they are mutually equal if and only if: $a = c$ and $b = d$. For $A = B$, we have $A \times A = A^2$, which is called the Cartesian square of set A, so that:

$$A^2 = \{(x, y) | x, y \in A\}.$$

The set $\{(x,x): x \in A\} = diag\ A^2$ is called the diagonal of the square A^2.

The Cartesian product is related to the operations ∪,∩,\ by these equations:

1. $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
2. $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
3. $(A \backslash B) \times C = (A \times C) \backslash (B \times C)$.

Next, we look at one of these relations - the first relationship.

Let it be that:

$(x,y) \in (A \cup B)xC \Rightarrow x \in (A \cup B) \wedge y \in C \Rightarrow (x \in A \vee x \in B) \wedge y \in C \Rightarrow (x,y) \in (AxC) \vee (x,y) \in (BxC) \Rightarrow (A \cup B)xC \subseteq (AxC) \cup (BxC)$.

Here, one relation of inclusion is shown, and analogously, the other relation of inclusion is also shown, which shows that relation 1 is valid for the given sets.

The ordered triple (x,y,z) is defined as $((x,y),z)$. In general, the associative property of the action does not apply, i.e. $((x,y),z) \neq (x,(y,z))$, but, by agreement, it is assumed that:

$$((x,y),z) = (x,(y,z)) = (x,y,z).$$

Here, too, we have that:

$$(x,y,z) = (x',y',z') \Leftrightarrow x = x', y = y', z = z'.$$

So we have:

$$A \times B \times C = \{(x,y,z) | x \in A, y \in B, z \in C\}.$$

And in general:

$$A_1 \times \ldots \times A_n = \{(x_1, \ldots, x_n) | x_i \in A_i\} = \bigotimes_{i=1}^{n} A_i.$$

When there is at least one $A_j = \emptyset, j \in \{1, \ldots, n\} = I$, then $\bigotimes_{i=1}^{n} A_i = \emptyset$. If $A_i = A$, for all i, we have the $n-$th Cartesian power:

$$A \times \ldots \times A = A^n.$$

1.6 Understanding the relationship

Definition 1.6.1 Every subset $\rho \subseteq A \times B$ is called a relation defined in the pair of sets A and B. If $(a\ b) \in \rho,\ a \in A, b \in B,$ we say that a is in relation ρ to b.

Instead of $(a, b) \in \rho,$ we denote $a\rho b$. When $B = A$, ρ is called a binary relation (or a relation of length 2) of set A.

Definition 1.6.2 If ρ is a relation given in the pair of sets A and $B,$ then the relation ρ^{-1} defined in the pair of sets B and A, is called an inverse relation, if:

$$b\rho^{-1}a \Leftrightarrow a\rho b \text{ or } \rho^{-1} = \{(b, a) \colon (a, b) \in \rho, a \in A, b \in B\}.$$

Definition 1.6.3 Let ρ be the relation given in the pair of sets $A,\ B,$ and σ the relation in sets B, C. The product of relations ρ and $\sigma,$ is the relation $\rho \cdot \sigma,$ for short $\rho\sigma$, given in $A,\ C$ and it holds that $a\ (\rho\sigma)c\ (a \in A, b \in B)$ if and only if $x \in B$ exists, such that $a\rho x$ and $x\sigma c$. So,

$$a\ (\rho\sigma)\ c \Leftrightarrow \underset{x}{\exists}\ (a\rho x) \wedge (x\sigma c).$$

When the relation ρ is given in the sets A and B, while σ is in the sets C and D for $B \neq C$, the product $\rho\sigma$ cannot be defined.

The equality relation on the set A, which we denote by I_A, or I for short, actually consists of the diagonal of A^2. Regarding the binary relations ρ, σ, τ and the relation I in set A, the following applies.

Theorem 1.6.4 If ρ, σ and τ are relations, then the following identities apply:

$$\rho I = I\rho, \rho(\sigma\tau) = (\rho\sigma)\tau, (\rho\sigma)^{-1} = \sigma^{-1}\rho^{-1}.$$

We note that from the equation

$$\rho I = I\rho = \rho,$$

which shows that I has the role of the neutral element concerning the production of relations, it follows that $\rho \cdot \rho^{-1} = I,$ for every binary relation ρ.

1.7 Equivalence and partial order

We see the binary relation ρ in the set $A \neq \Phi$, which has any of the following properties:

1° $a\rho a, \forall a \in A$ (reflexive property)
2° $a\rho b \Rightarrow b\rho a, \forall a, b \in A$ (symmetric property)
3° $a\rho b \wedge b\rho c \Rightarrow a\rho c, \forall a, b, c \in A$ (transitive property)
4° $a\rho b \wedge b\rho a \Rightarrow a = b, \forall a, b \in A$ (antisymmetric property).

Definition 1.7.1 The binary relation ρ in the set A is called an equivalence relation or equivalence, if it has the properties 1°, 2°, and 3°. It is marked with an $\sim$.

The relation $\sim$, defined on the set A, is related to the decomposition of the set A into disjointed parts (classes and layers) with empty cuts.

A system S of nonempty subsets of A, is called a decomposition of A, if every element of A is found in one and only one class of S.

In other words, the system S consists of classes S_a, such that:

$\cup S_\alpha = A$, $S_\alpha \cap S_\beta = \Phi$ for $\alpha \neq \beta$ and $\alpha, \beta \in I$, where I is a set of indices.

Theorem 1.7.2 Every decomposition S defines in A an equivalence and conversely, every equivalence in A defines an S decomposition of A.

Proof: First, let's see that a decomposition defines an equivalence. For this, we assume that we have the system of nonempty subsets S_α, such that:

$$\cup S_\alpha = A, S_\alpha \cap S_\beta = \Phi, \text{ for } \alpha \neq \beta.$$

We define the binary relation ρ in the set A in this way: $x\rho y \Leftrightarrow$ if x and y are in the same decomposition class S, $\forall x, y \in A$ and $\alpha, \beta \in I$, where I is a set of indices.

It can be seen that with ρ an equivalence relation is defined in A. Indeed, for every $a \in A$, there is at least one $\alpha \in I$ such that $a \in S_\alpha$. From this, we see that $a\rho a$, i.e. the reflexive property, applies. Furthermore, if $\forall x, y \in A$, such that $x\rho y$, then $\beta \in I$ is found such that $x, y \in S_\beta$. From this, $y, x \in S_\beta$

(or $y\rho x$) holds the symmetric property. Finally, let us assume that $x, y, z \in A$, such that $x\rho y$ and $y\rho z$, then there is at least one $\alpha \in I$, such that $x, y \in S_\alpha$ and at least one $\beta \in I$, such that $y, z \in S_\beta$. From these last relations, it can be seen that $S_\alpha \cap S_\beta = \{b\}$, which means that $S_\alpha = S_\beta$ or $(x\rho z)$. With this, it is shown that ρ defines an equivalence in A.

Conversely, to show that every equivalence ρ in A defines a decomposition, for any $\in A$, we write:

$$C_\alpha = \{x/x\rho a, x \in A\}.$$

The subset C_a is called the class of the element a, according to the equivalence ρ. We will show that C_a are classes of a decomposition of A. For each $a \in A$, we have it that $a\rho a$, since ρ is an equivalence relation in A, i.e., it is also reflexive. This means that each element belongs to a class, so $a \in C_a$. With this, we show that $A \subseteq \bigcup_{x\in A} C_x$. On the other hand, for $a \in \bigcup_{x\in A} C_x$, we have it that at least one $y \in A$ is found, so that $a \in C_y \subseteq A$. With this, we show that $A = \bigcup_{x\in A} C_x$. In the following, we show that: from $C_a \cap C_b = \Phi$, we will have it that $a \neq b$. Indeed, if $a = b$, then because $a \in C_a$, it follows that $b \in C_a$, namely that $C_a \cap C_b \neq \Phi$, which is contrary to the assumption made above. With this, it is shown that the equivalence classes C_a define a decomposition for the set A.

Definition 1.7.3 The set of all equivalence classes $\sim$ in A, which is denoted:

$$A/_\sim = \{C_a : a \in A\}$$

is called the factor-set of A according to $\sim$.

Definition 1.7.4 The relation r, which is reflexive, antisymmetric, and transitive, is called an order. If the relation r also has the property that for the two elements a and b of A, arb or bra, then we say that the relation r is a relation of linear order.

An example of the ordering relation is presented in the set of natural numbers (and also in the set of integer and rational numbers) where the relation is smaller (<). This is a relation in the set of natural numbers that satisfies all three conditions to be a relation of linear order.

1.8 The meaning of the function

Let the sets A and B and their Cartesian product

$$A \times B = \{(a,\ b): a \in A,\ b \in B\}$$

be given.

We know that in a relation, each element of set A is associated with one, or several elements of set B. Cases when each element of set A is associated with only one element of set B lead us to a very important mathematical concept, that of the function.

Definition 1.8.1 For the subset f of the Cartesian product $A \times B$, we say that it is a function of set A on set B if the following conditions are met:

1. If $(\forall a \in A)$ it follows that $f(a) \in B$

2. If $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

In other words, f from A to B is a function if every element of set A is associated with one and only one element of set B.

Symbolically, we denote this as $f: A \to B$. The set A is called the area of the definition of the function or the domain of the function, while the set B is called the area of values of the function or the codomain of the function.

We denote the domain of the function in most cases by D_f, and only as D when the function is known.

The set $R_f = f(A)$, which contains all the elements $y \in B$, and for which $y = f(x)$ and $\in A$, is called the set of values of the function f or the range of the function f. The range of the function in the general case is the subset of the codomain of the function.

If we mark the elements of the set A with the variable x, while y is the elements of the set B, then symbolically we mark:

$$f: x \to y \text{ or } x \xrightarrow{f} y \text{ or } y = f(x),$$

which is called the analytic form of the function assignment. We call the variable x the independent variable of the function f or the argument of the function f, while we call the variable y the dependent variable of the function f.

Definition 1.8.2 The function, whose rank is the set of real numbers or a subset thereof, is called a real-valued function. Moreover, if its domain is also the set of real numbers, then we call it a real function.

1.9 Ways of giving the function

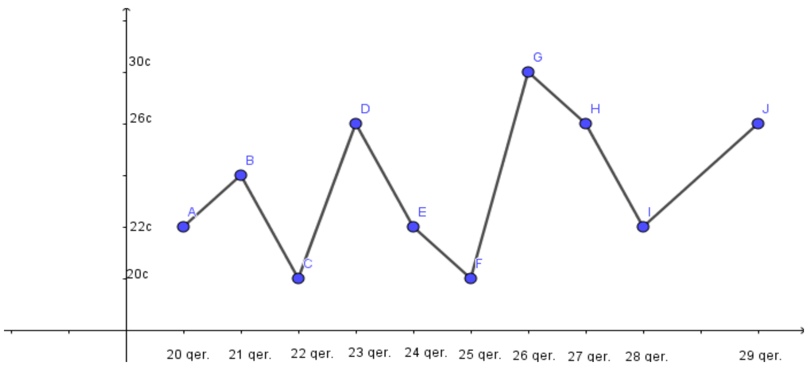
1.9.1 Graphical method

Let $f: A \to B$ be a function defined in set A and in the values in set B, so that each element $x \in A$ is accompanied by an element $f(x) \in B$. Thus, using some pair of $(x, f(x))$ we obtain the set of points of the plane:

$$G_f = \{(x, f(x)): x \in A\}.$$

Definition 1.9.2 The set G_f of all ordered pairs $(x, f(x))$ is called the graph of the function f.

Example 1.9.3 Data from June 20, 2019, to June 29, 2019, from the city of Pristina, as given in a graph:



We see that a function has been given using this graph because each day corresponds to a temperature value.

1.10 Explicit and implicit ways

It is said that the function is given in the explicit form (solved where one variable is expressed through the other) if its analytical expression is written in the form:$y = f(x)$, i.e. the variables are separated, as was the case in the examples:

$$y = 2x + 3, y = x, y = ax^2 + bx + c.$$

A function is said to be given in implicit (or unresolved) form if:

$$F(x, y) = 0.$$

For example:

$$F(x, y) \equiv x^3 + 3xy + y^2 - 3y + 2 = 0.$$

The transition from the explicit form to the implicit one is always possible. But the opposite is not always possible. This means that it is not always possible to solve the equation according to the variables that participate in that relation.

1.11 Parametric forms of function assignment

The function can also be given in the parametric form.

The variables x and y are expressed through a new variable t using the equations:

$$\left.\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}\right\},$$

where t is the auxiliary variable, and is called a parameter.

The transition from the parametric form to the explicit, respectively implicit form, is done by eliminating the parameter.

Example 1.11.1 When the function given in the parametric form

$$x = \cos t, \quad y = \sin t, \quad t \in [0, 2\pi],$$

by eliminating the parameter t, we obtain

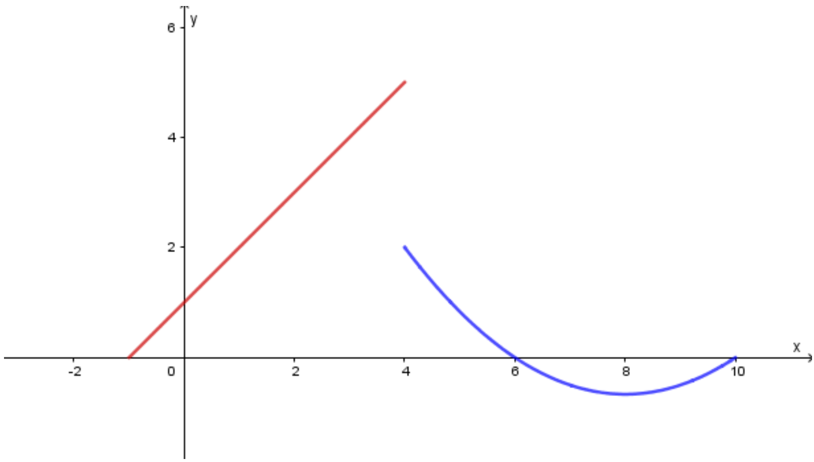
$$x^2 + y^2 = sin^2t + cos^2t = 1.$$

Here, $x^2 + y^2 = 1$, representing the canonical equation of the circle with the center at the origin and radius 1.

In addition to the above representation of functions, there are cases where the function can be given with more expressions (or with more branches), such as:

$$f(x) = \begin{cases} x + 1, & -1 < x < 4; \\ \dfrac{x^2 - 16x + 60}{6}, & 4 < x < 10; \end{cases}$$

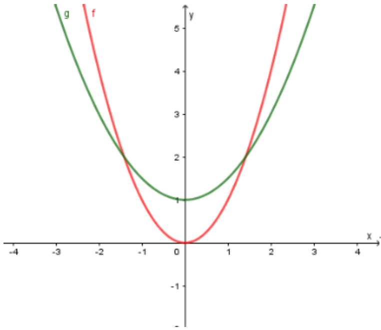
whose graph is:

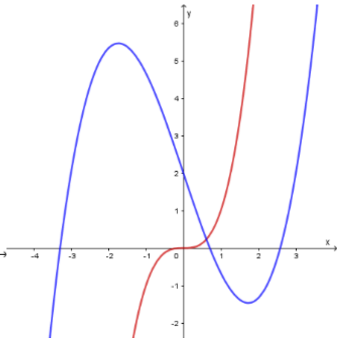


1.12 Classification of functions

Power functions:

The function: $f(x) = ax^n$, where a is a constant, while n is any natural number, is called a power function. Depending on whether n is an even or odd number, its graph will change.





Polynomial functions: The form:

$$f(x) = a_0 + a_1 \cdot x^1 + a_2 \cdot x^2 + a_3 \cdot x^3 + \cdots + a_n \cdot x^n$$

is a polynomial function.

For $n = 0$, we get the constant functions, $f(x) = a_0$, where a_0 is a constant.

For $n = 1$, we get the linear function (straight lines), $f(x) = a_0 + a_1 x$, and a_0 and a_1 are real numbers, where $a_1 \neq 0$.

The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x, \forall x \in \mathbb{R}$, is called an identical function. Usually, the identical function is denoted by $I_{\mathbb{R}}$, so in this case, we have $I: \mathbb{R} \to \mathbb{R}$. So, the linear function in which we have $a_0 = 0$ and $a_1 = 1$ is called an identical function.

If $n = 2$, then we get the quadratic function, $f(x) = a_0 + a_1 x + a_2 x^2$, where $a_0, a_1,$ and a_2 are real numbers and $a_2 \neq 0$.

Rational functions (fractional): This is the class of functions obtained by dividing two polynomial functions by themselves, and it has the general form:

$$f(x) = \frac{a_0 + a_1 \cdot x^1 + a_2 \cdot x^2 + a_3 \cdot x^3 + \cdots + a_n \cdot x^n}{b_0 + b_1 \cdot x^1 + ba_2 \cdot x^2 + b_3 \cdot x^3 + \cdots + b_m \cdot x^m}.$$

Functions:

$$f(x) = \frac{x}{1+2x}; \; g(x) = \frac{x^2+3x-2}{x+1}; \; r(x) = \frac{1}{x^4-4x+1}$$

are rational functions.

Irrational functions (root): This class consists of functions of:

$$f(x) = \sqrt[n]{g(x)},$$

where the function $g(x)$ can be one of the functions seen above.

Functions:

$$f(x) = 3\sqrt{x+2};\ g(x) = \frac{x+3}{\sqrt[3]{x+4}};\ r(x) = 1 + \sqrt{x^2 + 3x - 2}$$

are irrational functions.

Exponential functions: This class of functions is obtained from the above classes, but where instead of the argument x we have an exponential function of the form a^x or another similar form.

Functions are given by the expressions:

$$f(x) = 4^x\ ;\ g(x) = 2^{2x} - 3 \cdot 2^x + 5;\ r(x) = \frac{a^x+2}{a^{2x}-3};$$
$$t(x) = \sqrt[3]{2^x - 3}$$

are exponential functions.

Logarithmic functions: This class of functions is obtained from the classes above, but where instead of the argument x we have a logarithmic function of the basic form $\log_a x$, or even forms of it.

Functions given by relations:

$$f(x) = \log_4 x\ ;\ g(x) = (\log_2(2x+1))^2 - 3 \cdot \log_2(2x+1) + 3;$$
$$r(x) = \frac{\log_x(x+1)+2}{\log_x(x+1)-3};\ \ t(x) = \sqrt[3]{\log_3(1+x) - 3}$$

are logarithmic functions.

Trigonometric functions: In this class, we have all the above functions, but where instead of the argument x we have any of the trigonometric functions.

Functions:

$$f(x) = sin^3x + 2sinx - 1, : g(x) = \sqrt[3]{tan(x+1)}$$

are trigonometric functions.

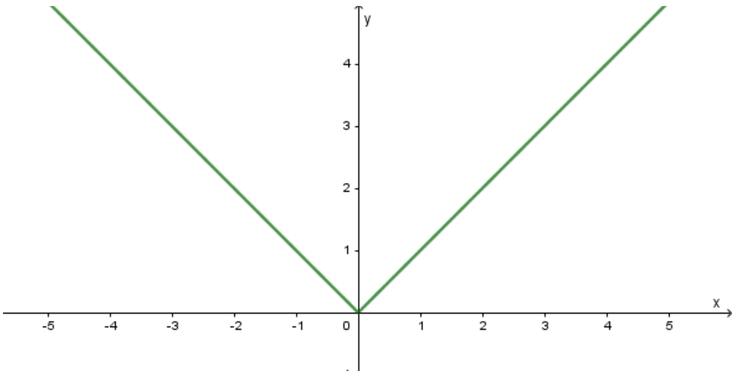
1.13 Certain special functions

Absolute valued function

The function absolute value of the real number x, $g(x) = |x|$, is defined by the expression:

$$g(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}.$$

From the definition, we get $|3| = 3$, while $|-2| = -(-2) = 2$. The graph of this function is:



Proposition 1.13.1 The following properties apply to the absolute value function:

$$|a + b| \leq |a| + |b|$$
$$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$$
$$|a - b| \leq |a| + |b|$$
$$|a + b| \geq |a| - |b|$$
$$|a \cdot b| = |a| \cdot |b|$$
$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$$
$$\big||a| - |b|\big| \leq |a - b|.$$

Proof: Next, we will see the proof of any of these absolute value properties. Let's start with the first property; for this, we start from the definition of the absolute value, which for a given value a, is defined as follows: $-|a| \leq a \leq |a|$. Such a relation also applies to the other number b, i.e., that $-|b| \leq b \leq |b|$. If we put these two relations side-by-side, then we will get:

$$-|a| - |b| \leq a + b \leq |a| + |b| \Rightarrow -(|a| + |b|) \leq a + b \leq |a| + |b| \Rightarrow$$
$$|a + b| \leq |a| + |b|.$$

Analogously, the other relations also apply.

Whole part functions

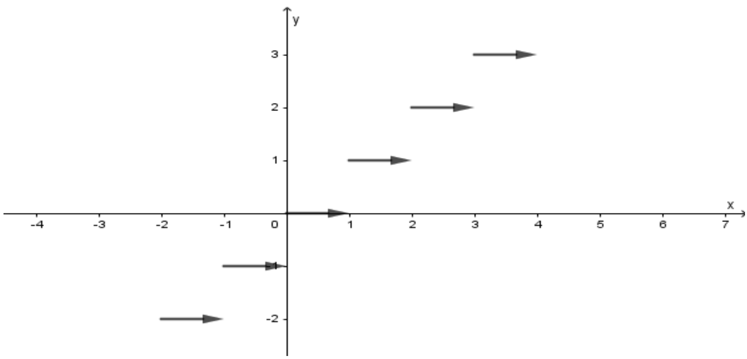
Another function that is of great use is the integer part function, which is defined as:

$$f(x) = [x] = k, k \in \mathbb{Z}, \text{ if } k \leq x < k + 1.$$

From the definition we have:

$$[1.2] = 1, [2.6] = 2, [-1.1] = -2.$$

For $x \in [0,1)$ we have $[x] = 0$, for $x \in [1,2)$ we have $[x] = 1$, for $x \in [2,3)$ we have $[x] = 2$, for $x \in [-1,0)$ we have $[x] = -1$, and so on, so we will have the graph:



We see that the following properties of the part-integer function also apply.

Proposition 1.13.2 The following properties hold for the integral part function:

$$[x+1]=[x]+1,\forall x\in\mathbb{R}$$
$$[x+n]=[x]+n,\forall x\in\mathbb{R},n\in\mathbb{N}$$
$$[x]+[y]\le[x+y]\le[x]+[y]+1$$
$$\left[x+\frac{1}{2}\right]=[2x]-[x]$$
$$[x]-2\left[\frac{x}{2}\right]=0\vee 1$$
$$\left[\frac{x}{n}\right]=\left[\frac{[x]}{n}\right],n\in\mathbb{N}$$
$$[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=[nx]$$
$$\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\ldots+\left[\frac{n+2^k}{2^{k+1}}\right]+\ldots=n,n\in\mathbb{N}.$$

Proof: We look at the first property and then any of the following properties.

For $[x]=k,\Rightarrow k\le x<k+1/+1\Rightarrow k+1\le x+1<k+2\Rightarrow$ $[x+1]=k+1=[x]+1$.

In the following, we also see the relation:

$[x]+[y]\le[x+y]\le[x]+[y]+1$. Let's mark this with $[x]=a$ and $[y]=b$, then from the definition of the function it is assumed that the left side of this relation is directly seen to be valid, and the right side of this relation remains to be proved. If we assume that the above numbers are positive real numbers (proved analogously when the numbers are negative), then we have these two real possibilities, which can be presented for the above numbers:

- $a\le x<a+0.5; a+0.5\le x<a+1$
- $b\le y<b+0.5; b+0.5\le y<b+1$.

If we are dealing with the first cases of these numbers, then we get:

- ✓ $a\le x<a+0.5\wedge b\le y<b+0.5\Rightarrow a+b\le x+y<a+b+1\Rightarrow[x+y]=a+b=[x]+[y]\le[x]+[y]+1$, with which the relationship was shown.
- ✓ If $a\le x<a+0.5\wedge b+0.5\le y<b+1\Rightarrow a+b+0.5\le x+y<a+b+1.5/+0.5\Rightarrow a+b+1\le x+y+0.5<a+b+1+1\Rightarrow[x+y+0.5]=a+b+1=[x]+[y]+1$, and this holds for every real number x and y, i.e. that: $[x+y]\le[x+y+0.5]=[x]+[y]+1$,

then the relationship as a whole is proven.

Let $[2x] = k$, where k is even, then by definition we have that:

$k \leq 2x < k + 1 \Rightarrow \frac{k}{2} \leq x < \frac{k}{2} + \frac{1}{2} \Rightarrow \frac{k}{2} \leq x < \frac{k}{2} + 1.$ Here: $[x] = \frac{k}{2}$ applies, while on the other side, we have it that: $\frac{k}{2} \leq x < \frac{k}{2} + \frac{1}{2}$. If we put these side-by-side we get:

$$\frac{k}{2} < \frac{k}{2} + \frac{1}{2} \leq x + \frac{1}{2} < \frac{k}{2} + 1 \Rightarrow \left[x + \frac{1}{2}\right] = \frac{k}{2}.$$

From the collection of the last two relations, we obtain:

$$[x] + \left[x + \frac{1}{2}\right] = \frac{k}{2} + \frac{k}{2} = k = [2x].$$

If k is even, we have the following: $[2x] = k$. Then we get:

$k \leq 2x < k + 1 \Rightarrow \frac{k}{2} \leq x < \frac{k+1}{2} \Rightarrow \frac{k}{2} + \frac{1}{2} \leq x + \frac{1}{2} < \frac{k}{2} + 1 < \frac{k+1}{2} + 1.$

Hence $\left[x + \frac{1}{2}\right] = \frac{k+1}{2}$. On the other side, it is assumed that:

$$k \leq 2x < k + 1 \Rightarrow \frac{k}{2} \leq x < \frac{k+1}{2} \Rightarrow \frac{k-1}{2} \leq \frac{k}{2} \leq x < \frac{k-1}{2} + 1 \Rightarrow [x] = \frac{k-1}{2}.$$

From these relations, we obtain:

$$\left[x + \frac{1}{2}\right] + [x] = \frac{k+1}{2} + \frac{k-1}{2} = k = [2x].$$

Sign function (signum function)

The function $f: \mathbb{R} \to \mathbb{R}$ given by

$$y = \begin{cases} 1, & if \quad x > 0 \\ 0, & if \ x = 0 \\ -1, & if \ x < 0 \end{cases}$$

is called the signum function.

Characteristic function

Let us assume that we have a set A and the universal set U (this is meant as the set that contains all other considered sets). Then the characteristic function of the set A is defined by the following relation:

$$\chi_A^U(x) = \begin{cases} 1, x \in A \\ 0, x \in U \backslash A = \overline{A} \end{cases}$$

Proposition 1.13.3 For the characteristic function, the following properties hold:

$$\chi_U^U(x) = 0$$
$$\chi_\Phi^U(x) = 1$$
$$\chi_{U\backslash A}^U(x) = 1 - \chi_A^U(x)$$
$$\chi_{A\cup B}^U(x) = \chi_A^U(x) \cdot \chi_B^U(x)$$
$$\chi_{A\cap B}^U(x) = \chi_A^U(x) + \chi_B^U(x) - \chi_A^U(x) \cdot \chi_B^U(x)$$
$$\chi_{A\backslash B}^U(x) = 1 - \chi_B^U(x) + \chi_{A\cup B}^U(x)$$
$$A = \bigcup_{i\in I} A_i \Rightarrow \chi_A^U(x) = \min_{i\in I} \chi_{A_i}^U(x)$$
$$A = \bigcap_{i\in I} A_i \Rightarrow \chi_A^U(x) = \max_{i\in I} \chi_{A_i}^U(x).$$

Proof: In the following, we will look at any of the above properties, starting for example with the property $\chi_{U\backslash A}^U(x) = 1 - \chi_A^U(x)$. Indeed, from the definition of the characteristic function of the set we have:

$$\chi_{U\backslash A}^U(x) = \begin{cases} 0, x \in U\backslash A \\ 1, x \in U(U\backslash A) = U\backslash\overline{A} = A \end{cases} = 1 - \begin{cases} 1, x \in U\backslash A \\ 0, x \in A \end{cases} = 1 - \chi_A^U(x).$$

We also look at the case $\chi_{A\cup B}^U(x) = \chi_A^U(x) \cdot \chi_B^U(x)$. We mark this with $f(x) = \chi_{A\cup B}^U(x)$, $f_1(x) = \chi_A^U(x)$ and $f_2(x) = \chi_B^U(x)$.

We show that in each possible case, we get $f = f_1 \cdot f_2$. To prove this property, we distinguish the following cases:

I) Let $x \in A$ be given, then from the notes above it is obtained that $f_1(x) = \chi_A^U(x) = 0$, which means that an even $f = f_1 \cdot f_2 = 0$.

II) Let us now assume that $x \notin A$. Then $f_1(x) = \chi_A^U(x) = 1$. On the other hand since $x \notin A$, i.e. $x \in B \backslash A \subset B \Rightarrow f_2 = 0$, we assume that again $f = f_1 \cdot f_2 = 0$. The same relation is shown to hold when $x \in B$ and in the case when $x \notin B$.

III) Let $x \in (A \cup B) \Rightarrow f = 0$. We see below that $f = f_1 \cdot f_2 = 0$. Indeed from $x \in (A \cup B)$, we will distinguish the subcases, firstly when $x \in A$ and $x \notin B \Rightarrow f_1 = 0$ and $f_2 = 1$, from which it is obtained that $f = f_1 \cdot f_2$. In the case analogous to this, when $x \notin A$ and $x \in B$, we will have: $f_1 = 1$ and $f_2 = 0 \Rightarrow f_1 \cdot f_2 = 0$. In the following, we also distinguish the last case when $x \in A$ and $x \in B$, where $f_1 = 0$ and $f_2 = 0 \Rightarrow f_1 \cdot f_2 = 0$.

IV) In this case we get that $x \notin (A \cup B) \Rightarrow x \in \overline{A} \wedge x \in \overline{B}$, from which we get that $f_1 = 1$ and $f_2 = 1$, respectively $f = f_1 \cdot f_2 = 1$.

Analogously, other properties of the characteristic function are proved.

1.14. Definition area (domain) and function value area (codomain)

We will examine functions whose domains and codomains are the set of real numbers ℝ or subsets of the set of real numbers. We call these functions with a real variable (argument).

From what has been said above, as well as from the definition of the function, we have it that the domain of a function is the set of all real values, for which the function has meaning.

Example 1.14.1 Find the area of definition and area of values of the function

$$f(x) = x^2 + 4x + 6.$$

Solution: The function $f(x) = x^2 + 4x + 6$ is defined for every $x \in \mathbb{R}$. So, $D(f) = \mathbb{R}$, while the codomain is the set of numbers expressed by the relation: $x \in [2\ \infty)$ because the given function can also be written in this form:

$$f(x) = (x + 2)^2 + 2$$

and the latter is always greater or equal to 2, because $(x + 2)^2 \geq 0$, for every $x \in \mathbb{R}$.

The domain of any polynomial function is the entire set of real numbers. So, for the constant, linear, quadratic, cubic, etc. functions, the domain of definition is the whole set of real numbers $\mathbb{R} = (-\infty,\ +\infty)$.

Example 1.14.2 Find the domain of the function

$g(x) = \frac{x^3}{x+2}$.

Solution The given function is rational. The area of definition of these functions is the set of real values for which the denominator is different from zero. So in the general case, for

$f(x) = \frac{a_0+a_1\cdot x^1+a_2\cdot x^2+a_3\cdot x^3+\cdots+a_n\cdot x^n}{b_0+b_1\cdot x^1+ba_2\cdot x^2+b_3\cdot x^3+\cdots+b_m\cdot x^m}$, it must be that

$$b_0 + b_1 \cdot x^1 + b_2 \cdot x^2 + b_3 \cdot x^3 + \cdots + b_m \cdot x^m \neq 0.$$

Therefore the function $g(x) = \frac{x^3}{x+2}$ is defined for every $x \in \mathbb{R}$, except for $x \neq -2$, i.e. for denominator values other than zero:

$$D(g) = \mathbb{R}\backslash\{-2\} = (-\infty, -2) \cup (-2, +\infty).$$

Remark The area of definition of rational (fractional) functions is taken from the set of real numbers excluding the values for which the denominator of the fraction becomes zero.

Example 1.14.3 Find the function definition area of

1) $h(x) = \sqrt{1-x}$ and 2) $r(x) = \sqrt[3]{\frac{x}{x-1}}$.

Solution The given function is an irrational function of the form $f(x) = \sqrt[n]{g(x)}$. Regarding the definition area of such functions, we distinguish two cases:

1) If the indicator of the root n is an odd number, then the area of the definition of the function $f(x)$, is like the area of the definition of the function under the sign of the root, i.e. like the area of the definition of the function $g(x)$, because even if we have negative value within the root, e.g. $\sqrt[3]{-8}$, we know that $\sqrt[3]{-8}$ is the real number -2.