Solved Problems of Classical Mechanics

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By

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Foreword

These are our personal notes of some problems of the Theoretical Mechanics course that has been taught at the Faculty of Physics of the University of Havana since the 1960s, that is, a long time ago. We also use this collection in undergraduate and graduate courses at the Autonomous University of the State of Morelos, Cuernavaca, Mexico, since 2000.

We have chosen for this material a few dozen problems that we believe are significant, although we are aware that every selection has a great subjective burden, so we already plan to add as many problems in the future. In its first moments there were entire songs that were missing; for example: Dispersion Theory, Non-Inertial Reference Systems, Lagrangian and Hamiltonian Continuous Field Theories, Dynamic Systems, Hamilton-Jacobi Procedure, etc. Even the topics included may have some disproportion to each other.

We want to draw attention to the fact that some problems are solved in detail, but in others we only give the general outlines of the solution.

Of course, many of these problems can be found -solved or not- in texts of very diverse dissemination. On many occasions we have reworked the statements to adapt them to our purposes. Eventually we note the citation of the texts where these same problems are analyzed. When we do not give credit in this regard, it is either because it is a problem that has been raised and perhaps solved in many places, or because it is our own initiative.

We hope that this collection is an adequate complement to the study of this discipline due to the main texts used in its teaching.

One of the most important objectives we set for ourselves was to present the calculations in some detail so that the still inexperienced student can follow them.

We appreciate in advance that you let us know of any errors that may have been *sneaked in without our permission*, so that we can correct these humble notes.

Enjoy.

RPA and MEMR

Cuernavaca, Mexico. August 2024
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Chapter 1

Newtonian Mechanics of a Particle

1.1 Motion with Constant Jerk

Find the equations of motion with constant jerk.

Solution:

Let $j(t) = j =$ constant, and by integrating repeatedly, we arrive at

$$\begin{aligned} j(t) &= j = \text{constant} && (1.1)\\ a(t) &= a_0 + j\,(t - t_0) && (1.2)\\ v(t) &= v_0 + a_0(t - t_0) + \frac{j(t - t_0)^2}{2} && (1.3)\\ x(t) &= x_0 + v_0(t - t_0) + \frac{a_0(t - t_0)^2}{2} + \frac{j(t - t_0)^3}{6}\ . && (1.4) \end{aligned}$$

1.2 Velocity and Acceleration in Polar Coordinates

Find the expressions for velocity and acceleration in polar coordinates.

Solution:

These are the equations for the transformation between polar and Cartesian coordinates:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \varphi = \arctan\left(\frac{y}{x}\right)\ . \end{cases} \tag{1.5}$$

The position in polar coordinates is defined as:

$$\vec{r} = \rho\,\vec{e}_\rho\ , \tag{1.6}$$

where ρ is the distance to the origin, and $\vec{e}_\rho$ is the radial unit vector, with components along the x and y axes as follows:

$$\vec{e}_\rho = \vec{e}_\rho(\varphi) = \cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y\ . \tag{1.7}$$

The expression for velocity is found by differentiating Equation (1.6):

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\rho}\vec{e}_\rho + \rho\dot{\vec{e}}_\rho \ . \tag{1.8}$$

However:

$$\frac{d\vec{e}_\rho}{dt} = \frac{d\vec{e}_\rho}{d\varphi}\dot{\varphi} \ , \tag{1.9}$$

and from Equation (1.7), it follows that:

$$\frac{d\vec{e}_\rho}{d\varphi} = -\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y = \vec{e}_\varphi \ . \tag{1.10}$$

It can be easily verified that $\vec{e}_\varphi$ is a unit vector perpendicular to $\vec{e}_\rho$, and it points in the region where φ increases.

Using the vectors $\vec{e}_\rho$ and $\vec{e}_\varphi$ and Equations (1.7)-(1.10), we obtain:

$$\vec{v} = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi \ . \tag{1.11}$$

Let's find the expression for acceleration by differentiating the previous expression:

$$\vec{a} = \dot{\vec{v}} = \ddot{\rho}\vec{e}_\rho + \dot{\rho}\dot{\vec{e}}_\rho + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \rho\ddot{\varphi}\vec{e}_\varphi + \rho\dot{\varphi}\dot{\vec{e}}_\varphi \ . \tag{1.12}$$

Similarly:

$$\frac{d\vec{e}_\varphi}{dt} = \frac{d\vec{e}_\varphi}{d\varphi}\dot{\varphi} \tag{1.13}$$

$$\frac{d\vec{e}_\varphi}{d\varphi} = -\cos\varphi\vec{e}_x - \sin\varphi\vec{e}_y = -\vec{e}_\rho \ . \tag{1.14}$$

Substituting the appropriate expressions for $\vec{e}_\rho$ and $\vec{e}_\varphi$ and grouping terms, we obtain:

$$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\vec{e}_\varphi \ . \tag{1.15}$$

1.3 Velocity and Acceleration in Cylindrical Coordinates

Find the expressions for velocity and acceleration in cylindrical coordinates.

Solution:

Cylindrical coordinates are a combination of a Cartesian axis and a polar coordinate system in a plane perpendicular to that axis (Fig. 1.1). The transformation equations between Cartesian and cylindrical coordinates are:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \varphi = \arctan\left(\frac{y}{x}\right) \\ z = z\ . \end{cases}$$

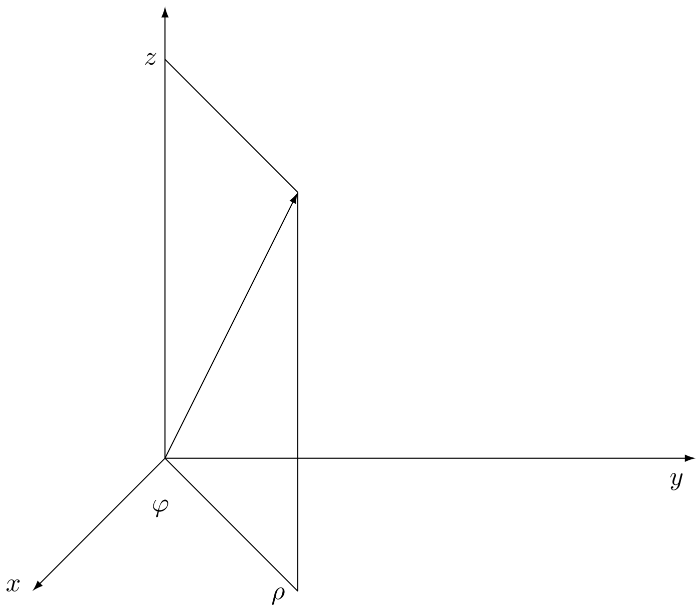

Figure 1.1: Cylindrical coordinate system (ρ, φ, z).

The expressions for the position, velocity, and acceleration of a particle in cylindrical coordinates are then:

$$\vec{r} = \rho\vec{e}_\rho + z\vec{e}_z \quad (1.16)$$
$$\vec{v} = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\vec{e}_z \quad (1.17)$$
$$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\vec{e}_\varphi + \ddot{z}\vec{e}_z\ . \quad (1.18)$$

1.4 Velocity and Acceleration in Spherical Coordinates

Find the expressions for velocity and acceleration in spherical coordinates.

Solution:

In this system, a point is determined by the distance from the origin and two angles. The distance r determines a sphere in which the point lies, and the angles θ and φ determine the latitude and longitude of the point, respectively. See Figure 1.2 and formulas (1.19).

The transformation equations between Cartesian and spherical coordinates are:

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta\ . \end{aligned} \quad (1.19)$$

It should be noted that the angles are restricted as follows:

$$0 \leq \theta \leq \pi \quad (1.20)$$
$$0 \leq \varphi < 2\pi\ , \quad (1.21)$$

to establish a one-to-one relationship between points and coordinates.

The position vector in spherical coordinates is:

$$\vec{r} = r\vec{e}_r\ , \quad (1.22)$$

where $\vec{e}_r$ is a unit vector expressed as:

$$\vec{e}_r = \sin\theta\cos\varphi\vec{e}_x + \sin\theta\sin\varphi\vec{e}_y + \cos\theta\vec{e}_z\ . \quad (1.23)$$

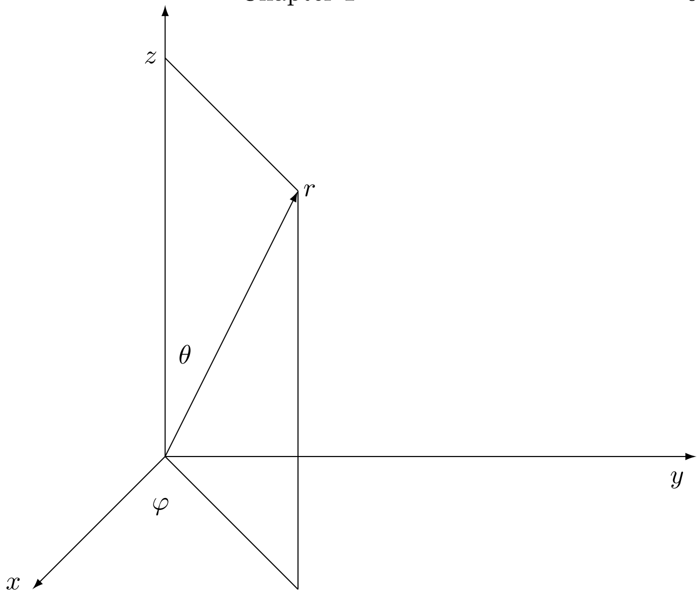

Figure 1.2: Spherical coordinate system (r, φ, θ).

Let's find the expression for velocity:

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r \ . \tag{1.24}$$

However,

$$\frac{d\vec{e}_r}{dt} = \frac{\partial \vec{e}_r}{\partial \theta}\dot{\theta} + \frac{\partial \vec{e}_r}{\partial \varphi}\dot{\varphi} \ . \tag{1.25}$$

By the chain rule, and by partially differentiating the expression for $\vec{e}_r$, we obtain:

$$\begin{aligned} \frac{\partial \vec{e}_r}{\partial \theta} &= \cos\theta\cos\varphi\vec{e}_x + \cos\theta\sin\varphi\vec{e}_y - \sin\theta\vec{e}_z \\ &= \vec{e}_\theta \ . \end{aligned} \tag{1.26}$$

This vector $\vec{e}_\theta$ is unitary, normal to $\vec{e}_r$, and oriented in the increasing direction of θ. This result is similar to that of polar coordinates, as seen in the figure, where if $\varphi =$ constant (the equation defining a plane containing the z-axis), in that plane, the remaining spherical coordinates r, θ are also the polar coordinates of that plane.

$$\frac{\partial \vec{e}_r}{\partial \varphi} = \sin\theta(-\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y) = \sin\theta\vec{e}_\varphi \ . \quad (1.27)$$

$\vec{e}_\varphi$ is a unit vector that has no component along the z-axis and is also normal to $\vec{e}_r$ and $\vec{e}_\theta$. This set of unit vectors forms a right-handed triad that moves with the point. Therefore,

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\dot{\varphi}\sin\theta\vec{e}_\varphi \ . \quad (1.28)$$

The expression for acceleration in spherical coordinates requires knowing the partial derivatives of $\vec{e}_\theta$ and $\vec{e}_\varphi$ with respect to θ and φ. These are:

$$\frac{\partial \vec{e}_\theta}{\partial \theta} = -\vec{e}_r \quad (1.29)$$

$$\frac{\partial \vec{e}_\theta}{\partial \varphi} = \cos\theta\vec{e}_\varphi \quad (1.30)$$

$$\frac{\partial \vec{e}_\varphi}{\partial \theta} = 0 \quad (1.31)$$

$$\frac{\partial \vec{e}_\varphi}{\partial \varphi} = -\sin\theta\vec{e}_r - \cos\theta\vec{e}_\theta \ . \quad (1.32)$$

Finally, the acceleration is expressed as follows:

$$\begin{aligned}\vec{a} = &(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\vec{e}_r + \\ &(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\vec{e}_\theta + \\ &(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\vec{e}_\varphi \ . \quad (1.33)\end{aligned}$$

In summary,

$$\vec{r} = r\vec{e}_r \quad (1.34)$$

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\dot{\varphi}\sin\theta\vec{e}_\varphi \quad (1.35)$$

$$\begin{aligned}\vec{a} = &(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\vec{e}_r + \\ &(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\vec{e}_\theta + \\ &(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\vec{e}_\varphi \ . \quad (1.36)\end{aligned}$$

Note in the above equations that:

(a) if $\varphi =$ constant, the expressions reduce to those of polar coordinates. The plane in which we define polar coordinates is precisely $\varphi =$ constant. r becomes ρ, and correspondingly $\vec{e}_r$ becomes $\vec{e}_\rho$. Now, θ is the polar angle, and thus, $\vec{e}_\theta$ plays the role of the polar $\vec{e}_\varphi$.

(b) if $\theta =$ constant $= \pi/2$, these spherical coordinates coincide with the polar ones in this plane. $r/\vec{e}_r$ again becomes $\rho/\vec{e}_\rho$. The variable θ exits the scene, while the spherical φ coincides with the polar φ.

1.5 Trajectory, Velocity, and Acceleration of a Particle with the Equation of Motion $x = a\cos\omega t; y = b\sin\omega t; z = 0$

The equation of motion for a particle is given by

$$\begin{aligned} x &= a\cos\omega t \\ y &= b\sin\omega t \\ z &= 0\,, \end{aligned}$$

where a, b, and ω are constants. Find the trajectory, linear and angular velocities, as well as the acceleration.
Oljovskii, Example 1.1, Page 22.
Irodov, Problem 1.1, Page 33.

Solution:

It is evident that

$$\begin{aligned} \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= \cos^2\omega t + \sin^2\omega t \equiv 1 \qquad (1.37) \\ z &= 0\,. \end{aligned}$$

Therefore, the trajectory is an ellipse with semi-axes a and b in the plane $z = 0$.

The velocity, on the other hand, is given by

$$\begin{aligned} \vec{v} &= \dot{\vec{r}} && (1.38) \\ &= \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + \dot{z}\vec{e}_z && (1.39) \\ &= -a\omega\sin\omega t\;\vec{e}_x + b\omega\cos\omega t\;\vec{e}_y + 0\;\vec{e}_z\,. && (1.40) \end{aligned}$$

Both position and velocity follow trigonometric laws, all with angular frequency ω.

The acceleration is the derivative of velocity. Therefore,

$$\begin{aligned}
\vec{a} &= \dot{\vec{v}} &(1.41)\\
&= \dot{v}_x\vec{e}_x + \dot{v}_y\vec{e}_y + \dot{v}_z\vec{e}_z &(1.42)\\
&= -a\omega^2\cos\omega t\ \vec{e}_x - b\omega^2\sin\omega t\ \vec{e}_y + 0\ \vec{e}_z &(1.43)\\
&= -\omega^2\ \vec{r}\,. &(1.44)
\end{aligned}$$

1.6 Particle on a Circle

A particle is constrained to move with constant speed on the circle $r = a$. Find the Cartesian and polar coordinates of its velocity and acceleration.

Corben, Problem 1.1, Page 25.

Solution:

In general,

$$\vec{r} = \rho\vec{e}_\rho\ . \qquad (1.45)$$

Then

$$\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} &(1.46)\\
&= \dot{\rho}\vec{e}\rho + \rho\frac{d\vec{e}_\rho}{dt} &(1.47)\\
&= \dot{\rho}\vec{e}_\rho + \rho\frac{d(\cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y)}{dt} &(1.48)\\
&= \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}(-\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y) &(1.49)\\
&= \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi\ , &(1.50)
\end{aligned}$$

and

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} && (1.51)\\
&= \ddot{\rho}\vec{e}_\rho + \dot{\rho}\frac{d\vec{e}_\rho}{dt} + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \rho\ddot{\varphi}\vec{e}_\varphi + \rho\dot{\varphi}\frac{d\vec{e}_\varphi}{dt} && (1.52)\\
&= \ddot{\rho}\vec{e}_\rho + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \rho\ddot{\varphi}\vec{e}_\varphi + \rho\dot{\varphi}(-\dot{\varphi}\vec{e}_\rho) && (1.53)\\
&= (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\vec{e}_\varphi\ . && (1.54)
\end{aligned}$$

In our case, $\rho = a, \dot{\rho} = 0, \ddot{\rho} = 0$, and therefore the expressions simplify to

$$\begin{aligned}
\vec{r} &= a\vec{e}_\rho && (1.55)\\
\vec{v} &= a\dot{\varphi}\vec{e}_\varphi && (1.56)\\
\vec{a} &= -a\dot{\varphi}^2\vec{e}_\rho + a\ddot{\varphi}\vec{e}_\varphi\ . && (1.57)
\end{aligned}$$

If the velocity is constant, it follows that $v = |\vec{v}| = a\dot{\varphi}$ is constant. Therefore, $\dot{\varphi} = \omega$ is constant, and $\ddot{\varphi} = 0$. As a result, the expressions become

$$\begin{aligned}
\vec{r} &= a\vec{e}_\rho && (1.58)\\
\vec{v} &= a\omega\vec{e}_\varphi && (1.59)\\
\vec{a} &= -a\omega^2\vec{e}_\rho\ . && (1.60)
\end{aligned}$$

To express these formulas in Cartesian coordinates, consider that

$$\begin{aligned}
\vec{e}_\rho &= \cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y && (1.61)\\
&= \frac{x}{\sqrt{x^2+y^2}}\vec{e}_x + \frac{y}{\sqrt{x^2+y^2}}\vec{e}_y && (1.62)\\
\vec{e}_\varphi &= -\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y && (1.63)\\
&= -\frac{y}{\sqrt{x^2+y^2}}\vec{e}_x + \frac{x}{\sqrt{x^2+y^2}}\vec{e}_y\ . && (1.64)
\end{aligned}$$

Therefore,

$$\vec{r} = a\left(\frac{x}{\sqrt{x^2+y^2}}\vec{e}_x + \frac{y}{\sqrt{x^2+y^2}}\vec{e}_y\right) \tag{1.65}$$

$$= x\vec{e}_x + y\vec{e}_y \tag{1.66}$$

$$\vec{v} = a\omega\left(-\frac{y}{\sqrt{x^2+y^2}}\vec{e}_x + \frac{x}{\sqrt{x^2+y^2}}\vec{e}_y\right) \tag{1.67}$$

$$= \omega\left(-y\vec{e}_x + x\vec{e}_y\right) \tag{1.68}$$

$$\vec{a} = -\omega^2\left(x\vec{e}_x + y\vec{e}_y\right) . \tag{1.69}$$

1.7 Particle with Velocity $\vec{v} = \vec{v}_0\left(1 - \frac{t}{\tau}\right)$

Starting from $t = 0$, the particle moves with a velocity that depends on time according to the law

$$\vec{v} = \vec{v}_0\left(1 - \frac{t}{\tau}\right) . \tag{1.70}$$

τ and $\vec{v}_0$ are constants. Find the position and the displacement as functions of time t.
Irodov, problem 1.2, page 34.

Solution:

Direct integration of the velocity as a function of time gives us the position, also as a function of time, namely:

$$\vec{r} = \vec{v}_0\, t\left(1 - \frac{t}{2\tau}\right) . \tag{1.71}$$

For simplicity, we have set the origin of coordinates at the initial position, so that $\vec{r}(0)$ is $\vec{0}$. Now, since $\vec{v}_0$ is constant, the motion occurs in a straight line whose directional vector is precisely $\vec{v}_0$. To calculate the displacement, we will consider that for $t < \tau$, the motion is in one direction and the displacement coincides with the absolute value (modulus) of the position, so that $s(t) = |\vec{r}(t)| = v_0\, t\, \left(1 - \frac{t}{2\tau}\right)$. This is the case until the particle reaches the turning point, where $t = \tau$ and the displacement $s = s_\tau = |\vec{r}(\tau)| = v_0\tau/2$. Meanwhile, for $\tau < t$, the velocity is reversed, and the displacement is $s_\tau = v_0\tau/2$

plus the displacement from the turning point to the current position, which is:

$$
\begin{aligned}
s &= s_\tau + |\vec{r}(\tau) - \vec{r}(t)| \qquad \tau < t && (1.72)\\
&= \frac{v_0\tau}{2} + \left|\vec{v}_0\,\tau\left(1-\frac{\tau}{2\tau}\right) - \vec{v}_0\,t\left(1-\frac{t}{2\tau}\right)\right| \\
&= \frac{v_0\tau}{2} + v_0\left|\frac{\tau}{2} - t\left(1-\frac{t}{2\tau}\right)\right| \\
&= \frac{v_0\tau}{2} + \frac{v_0\tau}{2}\left|1 - \frac{2t}{\tau}\left(1-\frac{t}{2\tau}\right)\right| \\
&= \frac{v_0\tau}{2} + \frac{v_0\tau}{2}\left(1 - \frac{2t}{\tau} + \frac{t^2}{\tau^2}\right) \\
s &= \frac{v_0\tau}{2}\left[1 + \left(1-\frac{t}{\tau}\right)^2\right] . && (1.73)
\end{aligned}
$$

In summary,

$$
s = \begin{cases} v_0\,t\,\left(1-\frac{t}{2\tau}\right) & t \le \tau \\ \frac{v_0\tau}{2}\left[1+\left(1-\frac{t}{\tau}\right)^2\right] & t \ge \tau\ . \end{cases} \qquad (1.74)
$$

It is worth noting that

$$
s(\tau-) = s(\tau+) = \frac{v_0\tau}{2}\ . \qquad (1.75)
$$

In Figure 1.3, the coincidence and difference between position and displacement in this problem are illustrated.

1.8 Particle with Acceleration $a = a_0 - bx$

A train moves in a straight line between stations A and B in such a way that its acceleration varies according to the law

$$
a = a_0 - bx\ , \qquad (1.76)
$$

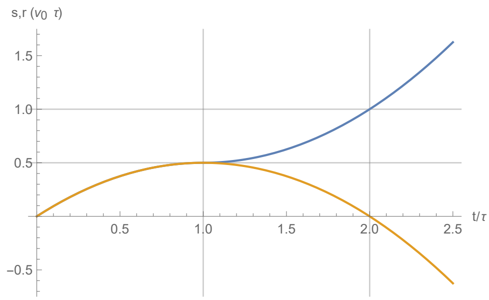

Figure 1.3: Figure of problem 1.7. Note that for $0 \leq t \leq \tau$, the position r and the displacement s coincide. At $t = \tau$, the velocity reverses, and the particle approaches the starting point; at $t = 2\tau$, it passes through it and continues in the opposite direction to when the motion began. The displacement s, on the other hand, never stops growing.

where a_0 and b are constants, and x is the position measured from station A. Find the distance between these stations and the maximum velocity that the train reaches during its journey.
Irodov, Problem 1.3, Page 35.

Solution:

$$a = a_0 - bx \tag{1.77}$$
$$\frac{dv}{dt} = a_0 - bx \tag{1.78}$$
$$dv = (a_0 - bx)dt \tag{1.79}$$
$$vdv = (a_0 - bx)vdt\ . \tag{1.80}$$

So,

$$v, dv = (a_0 - bx)dx$$
$$\frac{v^2}{2} = a_0 x - b\frac{x^2}{2} + C \tag{1.81}$$
$$v^2 = 2a_0 x - bx^2 + v^2(0) \tag{1.82}$$
$$v = \sqrt{(2a_0 - bx)x + v^2(0)}\ . \tag{1.83}$$

From (1.82), it is immediately clear that the maximum velocity v_{max} is reached at the point x_{max} such that

$$x_{max} = \frac{a_0}{b} , \tag{1.84}$$

and its value is

$$v_{max} = \sqrt{\frac{a_0^2}{b} + v^2(0)} . \tag{1.85}$$

Now, if we assume that the train starts from rest ($v_0 = 0$) and comes to a stop at station B, from equation (1.82), we can deduce that the distance between the stations is equal to

$$L = \frac{2a_0}{b} . \tag{1.86}$$

At this point, one might wonder about the equation of motion $x = x(t)$. This is more complicated and must be derived as follows:

$$\begin{aligned}
\frac{dv}{dt} &= (a_0 - bx) \\
v\frac{dv}{dt} &= (a_0 - bx)\frac{dx}{dt} \\
vdv &= (a_0 - bx)dx \\
\frac{v^2}{2} &= a_0 x - b\frac{x^2}{2} + C && (1.87) \\
v^2 &= 2a_0 x - bx^2 + v^2(0) && (1.88) \\
dt &= \frac{dx}{\sqrt{(2a_0 - bx)x + v^2(0)}} && (1.89) \\
t &= \int_0^x \frac{dz}{\sqrt{(2a_0 - bz)z + v^2(0)}} . && (1.90)
\end{aligned}$$

1.9 Calculating Acceleration from the Passage Times through Two Intervals

A body moving in a straight line with uniform acceleration passes through two consecutive intervals of equal length, a, at times t_1 and t_2. Prove that its acceleration is given by:

$$\frac{2a(t_1 - t_2)}{t_1 t_2 (t_1 + t_2)} \,.$$

Corben, problem 1.15, page 26.

Solution:

First, let's change the notation to work with more common names for the variables. Let's call a the acceleration and D the distance mentioned in the problem. Then,

$$x(t) = x_0 + v_0 t + \frac{at^2}{2} \,.$$

Here, x_0/v_0 is the position/velocity at time $t = 0$. Apply this expression to the endpoints of the two intervals mentioned in the problem. That is,

$$x_1 = x_0 + v_0 t_1 + \frac{at_1^2}{2} \quad (1.91)$$

$$x_2 = x_0 + v_0 t_2 + \frac{at_2^2}{2} \quad (1.92)$$

$$x_3 = x_0 + v_0 t_3 + \frac{at_3^2}{2} \,. \quad (1.93)$$

$$x_2 - x_1 = v_0(t_2 - t_1) + \frac{a}{2}(t_2^2 - t_1^2) \quad (1.94)$$

$$x_3 - x_2 = v_0(t_3 - t_2) + \frac{a}{2}(t_3^2 - t_2^2) \quad (1.95)$$

$$(1.96)$$

$$D = v_0(t_2 - t_1) + \frac{a}{2}(t_2^2 - t_1^2) \quad (1.97)$$

$$D = v_0(t_3 - t_2) + \frac{a}{2}(t_3^2 - t_2^2) \,. \quad (1.98)$$

$$\frac{D}{(t_2 - t_1)} = v_0 + \frac{a}{2}(t_2 + t_1) \quad (1.99)$$

$$\frac{D}{(t_3 - t_2)} = v_0 + \frac{a}{2}(t_3 + t_2) \,. \quad (1.100)$$

$$
\begin{aligned}
\frac{D}{(t_3-t_2)}-\frac{D}{(t_2-t_1)} &= \frac{a}{2}(t_3+t_2)-\frac{a}{2}(t_2+t_1) \\
&= \frac{a}{2}(t_3-t_2+t_2-t_1)\,. \qquad (1.101)
\end{aligned}
$$

$$
\begin{aligned}
a &= \frac{2}{(t_3-t_2+t_2-t_1)}\left[\frac{D}{(t_3-t_2)}-\frac{D}{(t_2-t_1)}\right] \\
&= \frac{2}{(\tau_2+\tau_1)}\left[\frac{D}{\tau_2}-\frac{D}{\tau_1}\right] \\
&= \frac{2D}{\tau_2\tau_1}\,\frac{\tau_1-\tau_2}{\tau_2+\tau_1}\,. \qquad (1.102)
\end{aligned}
$$

The formula requested by the problem is obtained with the clarification that the times appearing here are not instants but rather the intervals that the body takes to travel the first (τ_1) and the second (τ_2) stretch.

1.10 Estimation of the Height of a Building by Dropping an Object

A student drops a water-filled balloon from the tallest building in the neighborhood, attempting to hit their very fast companion below. The first student hears the balloon impact on the ground 4.021 seconds after releasing it. If the speed of sound is 331 m/s, determine the height of the building, neglecting air resistance.

Solution:

Let d be the height of the building, $v_s = 331$ m/s be the speed of sound, and $t = 4.021$ s be the time elapsed between the first student releasing the balloon and hearing the impact on the ground. Then,

$$
d = \frac{g(t-d/v_s)^2}{2}\,. \qquad (1.103)
$$

Therefore,

$$
\begin{aligned}
2dv_s^2 &= g(v_s t-d)^2 \qquad (1.104)\\
gd^2-2v_s(gt+v_s)d+gv_s^2t^2 &= 0\,. \qquad (1.105)
\end{aligned}
$$