A Journey into Ambiguous Set Theory

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Exploring Ambiguity

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I humbly dedicate this book at the feet of Maa Saraswati, the Goddess of Knowledge.

"Everything that comes to us that belongs to us if we create the capacity to receive it." By Rabindernath Tagore

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PREFACE

A Journey into Ambiguous Set Theory: Exploring Ambiguity offers a comprehensive exploration of the principles, methodologies, and practical applications of ambiguous set theory in diverse domains. Written by experts in the field, this book provides a thorough understanding of ambiguous set theory and its relevance in addressing complex real-world problems characterized by uncertainty and ambiguity.

The book begins by introducing the foundational concepts of ambiguous set theory, including fuzzy sets, uncertainty modeling, and ambiguity representation. It then delves into the mathematical formulation of ambiguous set theory, covering set operations, membership functions, and aggregation operators. Through clear explanations and illustrative examples, readers gain insight into the mathematical underpinnings of ambiguous set theory and its computational implications.

Building upon this theoretical foundation, the book explores various applications of ambiguous set theory in decision-making processes, control system design, medical imaging analysis, and pattern recognition. Readers learn how to apply ambiguous set theory algorithms to analyze CT scans, detect edges in images, and make informed decisions in uncertain environments. Real-world case studies and practical examples provide valuable insights into the practical implementation of ambiguous set theory in diverse contexts. In addition to its theoretical and practical coverage, *A Journey into Ambiguous Set Theory: Exploring Ambiguity* offers guidance on implementing ambiguous set theory algorithms, enabling readers to develop proficiency in applying ambiguous set theory to solve complex problems.

With its rigorous treatment of theory and practical insights into applications, *A Journey into Ambiguous Set Theory: Exploring*

Ambiguity serves as an invaluable resource for anyone interested in leveraging ambiguous set theory to address complex problems in uncertain environments. Whether you are a student, researcher, or practitioner, this book equips you with the knowledge and skills needed to navigate ambiguity and make informed decisions in a variety of domains.

The present book has been accomplished at National Taipei University of Technology (Taiwan), Jagiellonian University (Poland), and Central University of Rajasthan (India). All experiments were conducted at National Taipei University of Technology (Taiwan), Jagiellonian University (Poland), and Central University of Rajasthan (India). The empirical results presented in the book were published by esteemed journals.

Rajasthan, January, 2025
Pritpal Singh

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Prima facea, I am grateful to the God for the good health and well-being that were necessary to complete this book.

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Last but not least, I would like to thank almighty for everything.

Chapter 1
Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception

"We do not see things as they are, we see them as we are." Anaïs Nin

Abstract[1] Recently, precise measurement of uncertainty of data with fuzzy attributes is considered as one of the main problems. For this purpose, fuzzy sets, intuitionistic fuzzy sets, and neutrosophic set theory are extensively introduced. The problem arises in computing the complement of true or false membership values in the case of indeterminacy. Instead of indeterminacy as discussed in neutrosophic set theory, it can be completely true, partially true, or partially false. To deal with this, the theory of ambiguous sets has recently been introduced. A real-time example of dealing with unconsciousness and ambiguousness in relation to human perception is discussed, illustrating the motivation of the ambiguous set theory.

Keywords Fuzzy set, intuitionistic fuzzy set, neutrosophic set, ambiguous set, unconsciousness, uncertainty.

1.1 Introduction

Recently, one of the major issues for data science researchers is the precise assessment of uncertainty and vagueness of attributes. To deal with this issue, fuzzy set (FS) theory was developed by Zadeh (Zadeh,

[1]Based on: (Singh, 2023b)

1965). The FS theory employs a non-probabilistic approach to manage with the event's vagueness. This theory provides a single value membership to assess both belonging and non-belonging degrees of an event g in a particular FS in the interval $[0, 1]$. This membership degree is considered the true membership degree ($\Delta t(g)$) in the case of FS theory. As a result, non-belonging degree of an event is inevitably complement to 1 of the $\Delta t(g)$. However, some of the events may contain a certain non-belonging degree as independent rather than complement only. However, it is difficult to express the non-belonging degree of an event in FS.

Atanassov (Atanassov, 1986) developed an intuitionistic fuzzy set (IFS), which can express the belonging and non-belonging degrees to an event together. In the IFS theory, these belonging and non-belonging membership degrees are called true membership degree ($\Delta t(g)$) and false membership degree ($\Delta f(g)$), respectively. In the IFS, the ($\Delta f(g)$) of an event is always complement to 1 of the $\Delta t(g)$, i.e., $\Delta t(g) = 1 - \Delta f(g)$. Here, $\Delta t(g) \in [0, 1]$ and $\Delta f(g) \in [0, 1]$ with the condition $0 \leq \Delta t(g) + \Delta f(g) \leq 1$. A third component in the IFS is employed to describe the uncertainty between the $\Delta t(g)$ and $\Delta f(g)$, can be called hesitant membership degree ($\Delta h(g)$). The $\Delta h(g)$ of an event is always complement to 1 of the $\Delta t(g)$ and $\Delta f(g)$, i.e., $\Delta h(g) = 1 - \Delta t(g) - \Delta f(g)$. In the case of IFS, $\Delta t(g)$, $\Delta f(g)$, and $\Delta h(g)$ are linearly dependent on one another, and express the uncertainty of an event in the interval $[0, 1]$. That is, if $\Delta t(g)$ increases, then $\Delta f(g)$ must decrease and vice versa, thus creating the situation of indeterminacy. Nevertheless, in many situations, this might not be feasible. For illustration, consider a tennis tournament between players from Australia and Argentina, in which Australia supporters celebrate their team's victory while Argentina supporters lament their team's defeat.

Smarandache (Smarandache, 1999) introduced the concept of neutrosophic set (NS) theory. In this set theory, the true, false and hesitant part of IFS became independent in $[0, 1]$. It means the NS can be considered as a generalization of IFS theory. In the NS, the uncertainty of an event is expressed as similarly to IFS expressing the membership degrees in terms of $\Delta t(g)$, $\Delta f(g)$, and $\Delta h(g)$. However, $\Delta h(g)$ is referred to by Smarandache as the indeterministic membership degree ($\Delta i(g)$) in the interval $[0, 1]$. Smarandache explicitly mentioned that $\Delta t(g)$, $\Delta f(g)$, and $\Delta i(g)$ are not interdependent when the uncertainty of an event is expressed by NS. Here, $\Delta t(g)$, $\Delta f(g)$, and

$\Delta i(g)$ must satisfy the condition of $^{-}0 \leq \Delta t(g) + \Delta i(g) + \Delta f(g) \leq 3^{+}$, where individually $\Delta t(g)$, $\Delta f(g)$, and $\Delta i(g)$ must belong to the interval $]^{-}0, 1^{+}[$. This mathematical representation of NS creates an issue while defining the complement of $\Delta i(g)$ in the case of partially true or partially false or human consciousness, where two truths exist. In this case, the complement of $\Delta i(g)$ may not be only indeterminate. It may be partially true, partially false or unknown. This leads to a real-time problem while dealing with the uncertainty of a real time event containing partially true or partially false membership values. In case of unknown uncertainty, the event is liberalized which needs human consciousness for exploration. The problem becomes crucial while representation of the partially true and partially false behavior of an uncertain event creates unclear boundaries among: (a) true membership degree and partially true membership degree, and (b) false membership degree and partially false membership degree. Hence, to deal with partially true and partially false membership degrees, Singh et al. (Singh et al., 2019) have introduced an ambiguous set (AS) theory. This theory deals entirely with the unconscious behavior of human perception, in which the representation of true and false is inherently ambiguous. The AS theory provides the representation to uncertain event with respect to four membership degrees as: true membership degree ($\pi t(g)$), false membership degree ($\pi f(g)$), true-ambiguous membership degree ($\pi ta(g)$), and false-ambiguous membership degree ($\pi fa(g)$). Here, ($\pi t(g)$), ($\pi f(g)$), ($\pi ta(g)$), and ($\pi fa(g)$) are dependent on one another, and they have well-defined dimensions for unclear boundaries in the AS. Therefore, AS is applicable to the rare case of an uncertain event in which the human unconscious plays an important role.

1.2 Recent works on ambiguous set

Recently, Singh and Bose (Singh and Bose, 2021b) examined various properties of ambiguous sets. Singh and Huang (Singh and Huang, 2023b) explored set-theoretic properties of ambiguous sets and their applications in decision-making. Singh (Singh, 2023b) proposed a formula to measure the ambiguity of an event, termed ambiguous entropy. Singh (Singh, 2023a) generalized the concept of ambiguous sets to single-valued ambiguous numbers (SVANs) and proposed several comparison functions, operational laws, the ambiguous weighted geometric operator (AWGO), the ambiguous ordered weighted geometric operator (AOWGO), and the ambiguous hybrid geometric operator (AHGO) for SVANs. Singh (Singh, 2023c) introduced the concepts of partial ambiguous order sets (PAOS) and lattice ambiguous sets (LAS) for SVANs. Singh and Huang (Singh and Huang, 2023a) discussed various laws supporting different algebraic operations for comparing propositions in terms of the four membership degrees of ambiguous sets. Singh and Huang (Singh and Huang, 2024) presented a new ambiguous edge detection method (AEDM) for identifying the edges and boundaries of different regions in CT scans of COVID-19 cases using ambiguous sets. Motivated from above studies, the author tried to focus on explaining the basics of AS and its related concepts with an illustrative examples.

1.3 Background of the study

In the following, we provide definitions for FS, IFS, NS, and AS.

Definition 1.3.1. (FS) (Zadeh, 1965). Assume g be any event, which is defined in the fixed universe $\mathbb{G}$, i.e., $g \in \mathbb{G}$. Then, a FS $\tilde{F}$ for $g \in \mathbb{G}$ can be defined as:

$$\tilde{F} = \{\langle g, \Delta t(g)\rangle \mid g \in \mathbb{G}\} \tag{1.3.1.1}$$

Here, $\Delta t : \mathbb{G} \to [0, 1]$ denotes the membership degree function for the FS $\tilde{F}$, and $\Delta t(g) \in [0, 1]$ is referred to the membership degree of $g \in \mathbb{G}$ in $\tilde{F}$.

Definition 1.3.2. (IFS) (Atanassov, 1986). An IFS I for any event $g \in \mathbb{G}$ on the universe, which is fixed, can be expressed as:

$$I = \{\langle g, \Delta t(g), \Delta f(g)\rangle \mid g \in \mathbb{G}\} \tag{1.3.1.2}$$

where, $\Delta t : \mathbb{G} \to [0, 1]$ and $\Delta f : \mathbb{G} \to [0, 1]$ are called the membership degree and non-membership degree, respectively.

Both $\Delta t(g)$ and $\Delta f(g)$ must satisfy the following condition as:

$$0 \leq \Delta t(g) + \Delta f(g) \leq 1 \tag{1.3.1.3}$$

A hesitation membership degree $\Delta h(g)$ is always considered in IFS I. Hence, an IFS I with respect to $\Delta h(g)$ can be expressed as:

$$I = \{\langle g, \Delta t(g), \Delta f(g), \Delta h(g)\rangle \mid g \in \mathbb{G}\} \tag{1.3.1.4}$$

In Eq. 1.3.1.4, $\Delta t(g)$, $\Delta f(g)$, and $\Delta h(g)$ must satisfy the following condition as:

$$\Delta t(g) + \Delta f(g) + \Delta h(g) = 1 \tag{1.3.1.5}$$

Definition 1.3.3. (NS) (Smarandache, 1999). A NS N for any event $g \in \mathbb{G}$ on the universe, which is fixed, can be expressed as:

$$N = \{\langle g, \Delta t(g), \Delta i(g), \Delta f(g)\rangle \mid g \in \mathbb{G}\} \tag{1.3.1.6}$$

where, Δt, Δi, and Δf are called the truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively. Here, $\Delta t, \Delta i, \Delta f \ : \ \mathbb{G} \ \rightarrow]^{-}0, 1^{+}[$, and $^{-}0 \leq \Delta t(g) + \Delta i(g) + \Delta f(g) \leq 3^{+}$.

In the following, we provide definition for the AS as:

Definition 1.3.4. (AS) (Singh and Huang, 2024). Assume g be any event, which is defined in the fixed universe $\mathbb{G}$, i.e., $g \in \mathbb{G}$. Then, an AS $\hat{S}$ for $g \in \mathbb{G}$ can be defined as:

$$\hat{S} = \{g, \pi t(g), \pi f(g), \pi ta(g), \pi fa(g) \mid g \in \mathbb{G}\} \tag{1.3.1.7}$$

Here, $\pi t, \pi f, \pi ta, \pi fa : \mathbb{G} \rightarrow [0, 1]$ with the following condition:

$$0 \leq \pi t(g) + \pi f(g) + \pi ta(g) + \pi fa(g) \leq 2 \tag{1.3.1.8}$$

Here, πt, πf, πta, and πfa represent four different membership functions, designated as true, false, true-ambiguous, and false-ambiguous, respectively. In Eq. 1.3.1.7, $\pi t(g), \pi f(g), \pi ta(g), \pi fa(g) \in [0, 1]$ are termed as the true, false, true-ambiguous, and false-ambiguous membership degrees of $g \in \mathbb{G}$ in $\hat{S}$, respectively. Here, πt, πf, πta, and πfa are real standard values or non-standard subsets of $[0, 1]$. These four membership degrees are thus a representation of the ambiguities of $g \in \mathbb{G}$ in terms of $\hat{S}$.

1.4 Example for the ambiguous set

Consider the following perception of human cognition while eating pizza in a restaurant to illuminate the idea of AS:

- ⋆ P1: Pizza is *very tasty*.

In the case of a FS, the above perception P1 can be considered true, and it is assigned a true membership degree (i.e., Δt(very tasty). In the case of a IFS, if perception P1 has a Δt(very tasty), there must be a false perception, which can be defined as:

- ⋆ P2: Pizza is *not very tasty.*

Here, perception P2 is a contradiction to perception P1, which can be regarded as false and to which a false membership degree , i.e., Δf(not very tasty) is assigned.

Human perception cannot fully distinguish between *very tasty* and *not very tasty* in perceptions P1 and P2 because there is an indeterminate unconsciousness between perceptions P1 and P2. Such a perception can be defined as:

- ⋆ P12: Pizza is *either very tasty or not very tasty.*

In the case of NS, the perceptions P1 and P2 can be represented by the Δt(very tasty) and Δf(not very tasty). Perception P12, however, can be considered indeterministic, and it is assigned an indeterministic membership degree, i.e., Δi(either very tasty or not very tasty). Thus, in the case of NS, indeterministic perception always leads to confusion in decision-making and final opinion. Another problem with NS is that the perceptions P1–P12 are independent, i.e., Δt(very tasty), Δf(not very tasty), and Δi(either very tasty or not very tasty) are also independent. In this example, however, the three perceptions, namely P1–P12, are defined over the same perception. So, it is obvious that the membership degrees are interdependent. But, these membership degrees have unclear margins that make them indistinguishable from each other. For example, there is an unclear consciousness between Δt(very tasty) and Δf(not very tasty) in perception P12. To solve this problem of unclear margin between Δt(very tasty) and Δf(not very tasty), the following two additional perceptions can also be made with respect to perceptions P1 and P2:

- ⋆ P3: Pizza is *a little tasty.*
- ⋆ P4: Pizza is *not a little tasty.*

Perceptions P3–P4 may have different membership degrees in addition to the two membership degrees, i.e., Δt(very tasty) and

Δf(not very tasty). Perception P3 is very closely related to perception P1, and it inherits unconsciousness from perception P1. Therefore, perception P3 can be considered true-ambiguous, and represented by a true-ambiguous membership degree. Similarly, perception P4 is very closely related to perception P2, and it inherits unconsciousness from perception P2. Therefore, perception P4 can be considered as false-ambiguous, and represented by a false-ambiguous membership degree. To solve the problem of including these four membership degrees in the analysis of perception or uncertain events, the AS theory was introduced.

To make a clear distinction in the representation of the membership degrees of FS, IFS and NS, the designations true membership degree, false membership degree, true-ambiguous membership degree, and fasle-ambiguous membership degree of AS are used (Definition 1.3.4). According to these designations, the membership degrees for the perceptions P1, P2, P3, and P4 can be defined with AS as:

- $\star$ $\pi t(\text{very tasty}) \in [0, 1]$,
- $\star$ $\pi f(\text{not very tasty}) \in [0, 1]$,
- $\star$ $\pi ta(\text{a little tasty}) \in [0, 1]$, and
- $\star$ $\pi fa(\text{not a little tasty}) \in [0, 1]$,

The above four representations of membership degrees solve the problem of uncertain margins arises through human unconsciousness. In the case of AS, the ambiguous membership functions (AMFs) define the four membership degrees in such a way that it must satisfy the condition (Eq. 1.3.1.8) as:

$$0 \leq \pi t(\text{very tasty}) + \pi f(\text{not very tasty}) + \pi ta(\text{a little tasty}) + \pi fa(\text{not a little tasty}) \leq 2 \quad (1.4.1.1)$$

Suppose two customers A and B visit a restaurant and order a pizza. After eating the pizza, both customers may judge the taste of the pizza differently. The perceptions of customers A and B regarding the taste of the pizza can be denoted by ASs $\hat{S}_1$ and $\hat{S}_2$, and defined in Eqs. 1.4.1.2 and 1.4.1.3, respectively, as:

$$\hat{S}_1 = \{\text{tasty}, 0.46, 0.47, 0.42, 0.43 \mid g \in \mathbb{G}\} \quad (1.4.1.2)$$

$$\hat{S}_2 = \{\text{tasty}, 0.55, 0.38, 0.51, 0.35 \mid g \in \mathbb{G}\} \quad (1.4.1.3)$$

Here, Eqs. 1.4.1.2 and 1.4.1.3 can be read as:

$$\{\text{tasty}, \pi t(\text{very tasty}) + \pi f(\text{not very tasty}) + \\ \pi ta(\text{a little tasty}) + \pi fa(\text{not a little tasty})\} \quad (1.4.1.4)$$

Chapter 2
Ambiguous membership functions

"Probability is not a mere computation of odds on the dice or cards, but a tool to be learned and used in analyzing life's myriad complexities." By Leonard Mlodinow

Abstract[1] To deal with uncertainty in a very precise way, the ambiguous set theory has been proposed. This theory supports the representation of ambiguities in terms of four membership degrees, namely "true", "false", "true-ambiguous", and "false-ambiguous". These membership degrees are linearly dependent on each other, because they are defined over the same event. However, these four membership degrees have the difficulty of unclear margins that prevent them to be separated from each other in the ambiguities representation of an event. To address this problem of unclear margins in the ambiguous set, the membership degrees are defined using four different functions, and called ambiguous membership functions (AMFs). These membership degrees are defined in such a way that their sum should be less than or equal to 2, and the individual membership degree should be in the range $[0, 1]$. To visually represent the uncertain margins for the membership degrees, the concept of an ambiguous region (AR) is introduced.

Keywords Ambiguous membership functions (AMFs), ambiguous region (AR).

[1]Based on: (Singh and Huang, 2023b)

2.1 Introduction

The ambiguities of any event $g \in \mathbb{G}$ in the ambiguous set $\hat{S}$ (Definition 1.3.4) can be featured with four membership degrees, namely true $(\pi t(g))$, false $(\pi f(g))$, true-ambiguous $(\pi ta(g))$, and fasle-ambiguous $(\pi fa(g))$. To address the problem of unclear margin in the $\hat{S}$, the membership degrees are defined using four different functions, and called ambiguous membership functions (AMFs). This AMFs provide margins to the membership degrees in the $\hat{S}$. In this chapter, four different types of AMFs are discussed, namely T1AMFs, T2AMFs, T3AMFs, and T4AMFs. Their descriptions are provided next.

2.2 Description of T1AMFs

T1AMFs can be defined as:

Definition 2.2.1. (T1AMFs). Let $\mathbb{G} = \{g_1, g_2, \ldots, g_n\}$ be the fixed universe for any event $g_i (i = 1, 2, \ldots, n)$. Then, T1AMFs can be defined for any $g_i \in \mathbb{G}$ as:

$$\pi t(g_i) = G(g_i)\,[1 - \alpha_{min} \times \alpha_{max}] \tag{2.2.2.1}$$

$$\pi f(g_i) = 1 - \pi t(g_i) - \beta_{Tmin} \times \beta_{Tmax} \tag{2.2.2.2}$$

$$\begin{aligned}\pi ta(g_i) = \pi t(g_i) - \pi t(g_i)\,[1 - A_D(g_i) \times \beta_{Tmin} \\ - A_D(g_i) \times \beta_{Tmax}]\end{aligned} \tag{2.2.2.3}$$

$$\begin{aligned}\pi fa(g_i) = \pi f(g_i) - \pi f(g_i)\,[1 - A_D(g_i) \times \beta_{Fmin} \\ - A_D(g_i) \times \beta_{Fmax}]\end{aligned} \tag{2.2.2.4}$$

In Eqs. 2.2.2.3 and 2.2.2.4, $A_D(g_i)$ is referred to as the *ambiguous distance function*, which can be defined with respect to $\pi t(g_i)$ and $\pi f(g_i)$:

$$A_D(g_i) = \sqrt{\pi t(g_i)^2 + \pi f(g_i)^2 - 2 \times \pi t(g_i) \times \pi f(g_i)} \tag{2.2.2.5}$$

In Eq. 2.2.2.1, α_{min} and α_{max} can be defined by:

$$\alpha_{min} = \min[G(g_1|\bar{g}, \sigma), G(g_2|\bar{g}, \sigma), \ldots, G(g_n|\bar{g}, \sigma)] \tag{2.2.2.6}$$

$$\alpha_{max} = \max[G(g_1|\bar{g}, \sigma), G(g_2|\bar{g}, \sigma), \ldots, G(g_n|\bar{g}, \sigma)] \tag{2.2.2.7}$$

Here, $G(e_i) \in [0,1]$ is the gaussian membership of $g_i \in \mathbb{U}$, which can be defined as:

$$G(g_i) = \exp\left(-\frac{(g_i - \bar{g})^2}{2\sigma^2}\right);\ i = 1, 2, \ldots, n \tag{2.2.2.8}$$

Here, σ and $\bar{g}$ denote the standard deviation and mean of events $\{g_1, g_2, \ldots, g_n\} \in \mathbb{G}$, respectively.
In Eqs. 2.2.2.2 and 2.2.2.3, β_{Tmin} and β_{Tmax} can be defined by:

$$\beta_{Tmin} = \min[\pi t(g_1), \pi t(g_2), \ldots, \pi t(g_n)] \tag{2.2.2.9}$$
$$\beta_{Tmax} = \max[\pi t(g_1), \pi t(g_2), \ldots, \pi t(g_n)] \tag{2.2.2.10}$$

In Eq. 2.2.2.4, β_{Fmin} and β_{Fmax} can be defined as:

$$\beta_{Fmin} = \min[\pi f(g_1), \pi f(g_2), \ldots, \pi f(g_n)] \tag{2.2.2.11}$$
$$\beta_{Fmax} = \max[\pi f(g_1), \pi f(g_2), \ldots, \pi f(g_n)] \tag{2.2.2.12}$$

Here, "min" and "max" represent the minimum and maximum operations, respectively.

Example 1. (T1AMFs). Assume the gaussian memberships of each of the events belong to $\mathbb{G} = \{g_1, g_2, \ldots, g_9\}$ as: $G(g_1) = 0.1, G(g_2) = 0.2, \ldots, G(g_9) = 0.9$, respectively. Determine α_{min} and α_{max} from gaussian memberships as:

$$\begin{aligned}
\alpha_{min} &= \min[G(g_1), G(g_2), \ldots, G(g_9)] = \min[0.1, 0.2, \ldots, 0.9] = 0.1 \\
\alpha_{max} &= \min[G(g_1), G(g_2), \ldots, G(g_9)] = \min[0.1, 0.2, \ldots, 0.9] = 0.9
\end{aligned}$$

Now, obtain $\pi t(g_1)$, $\pi f(g_1)$, $\pi ta(g_1)$, and $\pi fa(g_1)$ of $g_1 \in \mathbb{G}$ as:

$$\begin{aligned}
\pi t(g_1) &= G(g_1)[1 - \alpha_{min} \times \alpha_{max}] = 0.1 \times [1 - 0.1 \times 0.9] = 0.0910 \\
\pi f(g_1) &= 1 - \pi t(g_1) - \beta_{Tmin} \times \beta_{Tmax} \\
&= 1 - 0.0910 - 0.0910 \times 0.8190 = 0.8345 \\
\pi ta(g_1) &= \pi t(g_1) - \pi t(g_1)[1 - A_D(g_1) \times \beta_{Tmin} - A_D(g_1) \times \beta_{Tmax}] \\
&= 0.0910 - 0.0910[1 - 0.7435 \times 0.0910 - 0.7435 \times 0.8190] \\
&= 0.0616 \\
\pi fa(g_1) &= \pi f(g_1) - \pi f(g_1)[1 - A_D(g_1) \times \beta_{Fmin} - A_D(g_1) \times \beta_{Fmax}] \\
&= 0.8345 - 0.8345[1 - 0.7435 \times 0.1065 - 0.7435 \times 0.8345] \\
&= 0.5838
\end{aligned}$$

Table 2.1: Representation of the ambiguous sets $\hat{S}_i$ for the events g_i in terms of T1AMFs.

g_i	$G(g_i)$	$\pi t(g_i)$	$\pi f(g_i)$	$A_D(g_i)$	$\pi ta(g_i)$	$\pi fa(g_i)$	$\hat{S}_i$
g_1	0.1	0.0910	0.8345	0.7435	0.0616	0.5838	$\hat{S}_1 = \{g_1, 0.09, 0.83, 0.06, 0.58\}$
g_2	0.2	0.1820	0.7435	0.5615	0.0930	0.3928	$\hat{S}_2 = \{g_2, 0.18, 0.74, 0.09, 0.39\}$
g_3	0.3	0.2730	0.6525	0.3795	0.0943	0.2330	$\hat{S}_3 = \{g_3, 0.27, 0.65, 0.09, 0.23\}$
g_4	0.4	0.3640	0.5615	0.1975	0.0654	0.1043	$\hat{S}_4 = \{g_4, 0.36, 0.56, 0.06, 0.10\}$
g_5	0.5	0.4550	0.4705	0.0155	0.0064	0.0068	$\hat{S}_5 = \{g_5, 0.45, 0.47, 0.00, 0.00\}$
g_6	0.6	0.5460	0.3795	0.1665	0.0827	0.0595	$\hat{S}_6 = \{g_6, 0.54, 0.37, 0.08, 0.05\}$
g_7	0.7	0.6370	0.2885	0.3485	0.2020	0.0946	$\hat{S}_7 = \{g_7, 0.63, 0.28, 0.20, 0.09\}$
g_8	0.8	0.7280	0.1975	0.5305	0.3515	0.0986	$\hat{S}_8 = \{g_8, 0.72, 0.19, 0.35, 0.09\}$
g_9	0.9	0.8190	0.1065	0.7125	0.5310	0.0714	$\hat{S}_9 = \{g_9, 0.81, 0.10, 0.53, 0.07\}$

In the above calculation, $A_D(g_1)$ is obtained with respect to $\pi t(g_1)$ and $\pi f(g_1)$ as:

$$\begin{aligned} A_D(g_1) &= \sqrt{\pi t(g_1)^2 + \pi f(g_1)^2 - 2 \times \pi t(g_1) \times \pi f(g_1)} \\ &= \sqrt{0.0910^2 + 0.8345^2 - 2 \times 0.0910 \times 0.8345} \\ &= 0.7435 \end{aligned}$$

For the above calculation, β_{Tmin} and β_{Tmax} are obtained as:

$$\begin{aligned} \beta_{Tmin} &= \min[\pi t(g_1), \pi t(g_2), \dots, \pi t(g_n)] \\ &= \min[0.0910, 0.1820, \dots, 0.8190] = 0.0910 \\ \beta_{Tmax} &= \max[\pi t(g_1), \pi t(g_2), \dots, \pi t(g_n)] \\ &= \max[0.0910, 0.1820, \dots, 0.8190] = 0.8190 \end{aligned}$$

Similarly, β_{Fmin} and β_{Fmax} are determined as:

$$\begin{aligned} \beta_{Fmin} &= \min[\pi f(g_1), \pi f(g_2), \dots, \pi f(g_n)] \\ &= \min[0.8345, 0.7435, \dots, 0.1065] = 0.1065 \\ \beta_{Fmax} &= \max[\pi f(g_1), \pi f(g_2), \dots, \pi f(g_n)] \\ &= \min[0.8345, 0.7435, \dots, 0.1065] = 0.8345 \end{aligned}$$

Hence, the ambiguous set for the event $g_1 \in \mathbb{G}$ can be defined in terms of T1AMFs as: $\hat{S}_1 = \{g_1, 0.0910, 0.8345, 0.0616, 0.5838\}$. Ambiguous sets for other events are defined in a similar way. The ambiguous sets $\hat{S}_i$ for the events $g_i \in \mathbb{G}$ in terms of T1AMFs are shown in Table 2.1, where $i = 1, 2, \dots, 9$.

2.3 Description of T2AMFs

T2AMFs can be defined as:

Definition 2.3.1. (T2AMFs). Let $\mathbb{G} = \{g_1, g_2, \ldots, g_n\}$ be the fixed universe for any event $g_i (i = 1, 2, \ldots, n)$. Then, T2AMFs can be defined for any $g_i \in \mathbb{G}$ as:

$$\pi t(g_i) = G(g_i)[1 - \alpha_{min} \times \alpha_{max}] \tag{2.3.2.1}$$

$$\pi f(g_i) = 1 - \pi t(g_i) - \beta_{Tmin} \times \beta_{Tmax} \tag{2.3.2.2}$$

$$\pi ta(g_i) = \pi t(g_i) - \pi t(g_i)[1 - A_D(g_i) \times \beta_{Tmin}] \tag{2.3.2.3}$$

$$\pi fa(g_i) = \pi f(g_i) - \pi f(g_i)[1 - A_D(g_i) \times \beta_{Fmin}] \tag{2.3.2.4}$$

Description of all the parameters used in Eqs. 2.3.2.1-2.3.2.4 are provided in the explanation of T1AMFs.

Example 2. (T2AMFs). Assume the gaussian memberships of each of the events belong to $\mathbb{G} = \{g_1, g_2, \ldots, g_9\}$ as: $G(g_1) = 0.1, G(g_2) = 0.2, \ldots, G(g_9) = 0.9$, respectively. Determine α_{min} and α_{max} from gaussian memberships according to Example 1.

Now, obtain $\pi t(g_1)$, $\pi f(g_1)$, $\pi ta(g_1)$, and $\pi fa(g_1)$ of $g_1 \in \mathbb{G}$ as:

$$\begin{aligned}
\pi t(g_1) &= G(g_1)[1 - \alpha_{min} \times \alpha_{max}] \\
&= 0.1[1 - 0.1 \times 0.9] = 0.0910 \\
\pi f(g_1) &= 1 - \pi t(g_1) - \beta_{Tmin} \times \beta_{Tmax} \\
&= 1 - 0.0910 - 0.0910 \times 0.8190 = 0.8345 \\
\pi ta(g_1) &= \pi t(g_1) - \pi t(g_1)[1 - A_D(g_1) \times \beta_{Tmin}(g_1)] \\
&= 0.0910 - 0.0910[1 - 0.7435 \times 0.0910] = 0.0062 \\
\pi fa(g_1) &= \pi f(g_1) - \pi f(g_1)[1 - A_D(g_1) \times \beta_{Fmin}] \\
&= 0.8345 - 0.8345[1 - 0.7435 \times 0.1065] = 0.0661
\end{aligned}$$

In the above calculation, α_{min}, α_{max}, β_{Tmin}, β_{Tmax}, $A_D(g_1)$, and β_{Fmin} are calculated according to Example 1. Hence, the ambiguous set for the event $g_1 \in \mathbb{G}$ can be defined in terms of T2AMFs as: $\hat{S}_1 = \{g_1, 0.0910, 0.8345, 0.0062, 0.0661\}$. Ambiguous sets for other events are defined in a similar way. The ambiguous sets $\hat{S}_i$ for the events $g_i \in \mathbb{G}$ in terms of T2AMFs are shown in Table 2.2, where $i = 1, 2, \ldots, 9$.

2.4 Description of T3AMFs

T3AMFs can be defined as:

Definition 2.4.1. (T3AMFs). Let $\mathbb{G} = \{g_1, g_2, \dots, g_n\}$ be the fixed universe for any event $g_i(i = 1, 2, \dots, n)$. Then, T3AMFs can be defined for any $g_i \in \mathbb{G}$ as:

$$\pi t(g_i) = G(g_i)[1 - \alpha_{min} \times \alpha_{max}] \tag{2.4.2.1}$$

$$\pi f(g_i) = 1 - \pi t(g_i) - \beta_{Tmin} \times \beta_{Tmax} \tag{2.4.2.2}$$

$$\pi ta(g_i) = \pi t(g_i) - \pi t(g_i)[1 - A_D(g_i) \times \beta_{Tmax}] \tag{2.4.2.3}$$

$$\pi fa(g_i) = \pi f(g_i) - \pi f(g_i)[1 - A_D(g_i) \times \beta_{Fmax}] \tag{2.4.2.4}$$

Description of all the parameters used in Eqs. 2.4.2.1-2.4.2.4 are provided in the explanation of T1AMFs.

Example 3. (T3AMFs). Assume the gaussian memberships of each of the events belong to $\mathbb{G} = \{g_1, g_2, \dots, g_9\}$ as: $G(g_1) = 0.1, G(g_2) = 0.2, \dots, G(g_9) = 0.9$, respectively. Determine α_{min} and α_{max} from gaussian memberships according to Example 1.

Now, obtain $\pi t(g_1)$, $\pi f(g_1)$, $\pi ta(g_1)$, and $\pi fa(g_1)$ of $g_1 \in \mathbb{G}$ as:

$$\begin{aligned}
\pi t(g_1) &= G(g_1)[1 - \alpha_{min} \times \alpha_{max}] \\
&= 0.1[1 - 0.1 \times 0.9] = 0.0910 \\
\pi f(g_1) &= 1 - \pi t(g_1) - \beta_{Tmin} \times \beta_{Tmax} \\
&= 1 - 0.0910 - 0.0910 \times 0.8190 = 0.8345 \\
\pi ta(g_1) &= \pi t(g_1) - \pi t(g_1)[1 - A_D(g_1) \times \beta_{Tmax}(e_1)] \\
&= 0.0910 - 0.0910[1 - 0.7435 \times 0.8190] = 0.0554 \\
\pi fa(g_1) &= \pi f(g_1) - \pi f(g_1)[1 - A_D(g_1) \times \beta_{Fmax}] \\
&= 0.8345 - 0.8345[1 - 0.7435 \times 0.8345] = 0.5177
\end{aligned}$$

In the above calculation, α_{min}, α_{max}, β_{Tmin}, β_{Tmax}, $A_D(g_1)$, and β_{Fmax} are calculated according to Example 1. Hence, the ambiguous set for the event $g_1 \in \mathbb{G}$ can be defined in terms of T3AMFs as: $\hat{S}_1 = \{g_1, 0.0910, 0.8345, 0.0554, 0.5177\}$. Ambiguous sets for other events are defined in a similar way. The ambiguous sets $\hat{S}_i$ for the events $g_i \in \mathbb{G}$ in terms of T3AMFs are shown in Table 2.3, where $i = 1, 2, \dots, 9$.

2.5 Description of T4AMFs

T4AMFs can be defined as:

Definition 2.5.1. (T4AMFs). Let $\mathbb{G} = \{g_1, g_2, \dots, g_n\}$ be the fixed universe for any event $g_i (i = 1, 2, \dots, n)$. Then, T4AMFs can be defined for any $g_i \in \mathbb{G}$ as:

$$\pi t(g_i) = G(g_i)[1 - \alpha_{min} \times \alpha_{max}] \tag{2.5.2.1}$$

$$\pi f(g_i) = 1 - \pi t(g_i) - \beta_{Tmin} \times \beta_{Tmax} \tag{2.5.2.2}$$

$$\pi ta(g_i) = \pi t(g_i)[1 - \beta_{Tmin} \times \beta_{Tmax}] \tag{2.5.2.3}$$

$$\pi fa(g_i) = \pi f(g_i)[1 - \beta_{Fmin} \times \beta_{Fmax}] \tag{2.5.2.4}$$

Description of all the parameters used in Eqs. 2.5.2.1-2.5.2.4 are provided in the explanation of T1AMFs.

Example 4. (T4AMFs). Assume the gaussian memberships of each of the events belong to $\mathbb{G} = \{g_1, g_2, \dots, g_9\}$ as: $G(g_1) = 0.1, G(g_2) = 0.2, \dots, G(g_9) = 0.9$, respectively. Determine α_{min} and α_{max} from gaussian memberships according to Example 1.

Now, obtain $\pi t(g_1)$, $\pi f(g_1)$, $\pi ta(g_1)$, and $\pi fa(g_1)$ of $g_1 \in \mathbb{G}$ as:

$$\begin{aligned}
\pi t(g_1) &= G(g_1)[1 - \alpha_{min} \times \alpha_{max}] \\
&= 0.1[1 - 0.1 \times 0.9] = 0.0910 \\
\pi f(g_1) &= 1 - \pi t(g_1) - \beta_{Tmin} \times \beta_{Tmax} \\
&= 1 - 0.0910 - 0.0910 \times 0.8190 = 0.8345 \\
\pi ta(g_1) &= \pi t(g_1)[1 - \beta_{Tmin}(g_1) \times \beta_{Tmax}(g_1)] \\
&= 0.0910[1 - 0.0910 \times 0.8190] = 0.0842 \\
\pi fa(g_1) &= \pi f(g_1)[1 - \beta_{Fmin}(g_1) \times \beta_{Fmax}(g_1)] \\
&= 0.8345[1 - 0.1065 \times 0.8345] = 0.7603
\end{aligned}$$

In the above calculation, α_{min}, α_{max}, β_{Tmin}, β_{Tmax}, β_{Fmin}, and β_{Fmax} are calculated according to Example 1. Hence, the ambiguous set for the event $g_1 \in \mathbb{G}$ can be defined in terms of T4AMFs as: $\hat{S}_1 = \{g_1, 0.0910, 0.8345, 0.0842, 0.7603\}$. Ambiguous sets for other events are defined in a similar way. The ambiguous sets $\hat{S}_i$ for the events $g_i \in \mathbb{G}$ in terms of T4AMFs are shown in Table 2.4, where $i = 1, 2, \dots, 9$.

Table 2.2: Representation of the ambiguous sets $\hat{S}_i$ for the events g_i in terms of T2AMFs.

g_i	$G(g_i)$	$\pi t(g_i)$	$\pi f(g_i)$	$A_D(g_i)$	$\pi ta(g_i)$	$\pi fa(g_i)$	$\hat{S}_i$
g_1	0.1	0.0910	0.8345	0.7435	0.0062	0.0661	$\hat{S}_1 = \{g_1, 0.0910, 0.8345, 0.0062, 0.0661\}$
g_2	0.2	0.1820	0.7435	0.5615	0.0093	0.0445	$\hat{S}_2 = \{g_2, 0.1820, 0.7435, 0.0093, 0.0445\}$
g_3	0.3	0.2730	0.6525	0.3795	0.0094	0.0264	$\hat{S}_3 = \{g_3, 0.2730, 0.6525, 0.0094, 0.0264\}$
g_4	0.4	0.3640	0.5615	0.1975	0.0065	0.0118	$\hat{S}_4 = \{g_4, 0.3640, 0.5615, 0.0065, 0.0118\}$
g_5	0.5	0.4550	0.4705	0.0155	0.0006	0.0008	$\hat{S}_5 = \{g_5, 0.4550, 0.4705, 0.0006, 0.0008\}$
g_6	0.6	0.5460	0.3795	0.1665	0.0083	0.0067	$\hat{S}_6 = \{g_6, 0.5460, 0.3795, 0.0083, 0.0067\}$
g_7	0.7	0.6370	0.2885	0.3485	0.0202	0.0107	$\hat{S}_7 = \{g_7, 0.6370, 0.2885, 0.0202, 0.0107\}$
g_8	0.8	0.7280	0.1975	0.5305	0.0351	0.0112	$\hat{S}_8 = \{g_8, 0.7280, 0.1975, 0.0351, 0.0112\}$
g_9	0.9	0.8190	0.1065	0.7125	0.0531	0.0081	$\hat{S}_9 = \{g_9, 0.8190, 0.1065, 0.0531, 0.0081\}$

2.6 Additional definitions related to ambiguous set

The assignment of memberships to any observation in the ambiguous set is known as *ambiguousness*.

Definition 2.6.1. (Ambiguousness). The process of allocating membership degrees to the events $g_i (i = 1, 2, \dots, n)$ of the universe of discourse $\mathbb{G}$ using T1AMFs–T4AMFs is called an *ambiguousness* operation. An *ambifier* $\nabla = (\pi t, \pi f, \pi ta, \pi fa)$ is a 4-tuple of membership functions $\pi t, \pi f, \pi ta, \pi fa : \mathbb{G} \to [0, 1]$. When applied to $\mathbb{G}$, the ambifier ∇ yields an ambiguous set $\hat{S}$ in $\mathbb{G}$ as:

$$\nabla(\mathbb{G}) = \{g_i, \pi t(g_i), \pi f(g_i), \pi ta(g_i), \pi fa(g_i) \mid g_i \in \mathbb{G}\} \qquad (2.6.2.1)$$

An ambiguous set forms a region in a space based on T1AMFs–T4AMFs, viz., πt, πf, πta, and πfa. This region is called *ambiguous region (AR)* (Singh and Huang, 2023b). ARs formed by different ambiguous sets $\hat{S}_1 - \hat{S}_4$ based on T1AMFs–T4AMFs are shown in Fig. 2.1(a)-(d), respectively.

For any event, T1AMFs–T4AMFs create two line segments on x- and y-axes (Fig. 2.1), which are called *ambiguous-true segment (ATS)* and *ambiguous-false segment(AFS)*, respectively. These two segments are defined as follows.

Definition 2.6.2. (ATS). For $g_i \in \mathbb{G}$, the ATS, denoted by $TS(g_i)$, is defined as $TS(g_i) = [X_1(g_i), X_2(g_i)]$, where $X_1(g_i) = \min(\pi t(g_i), \pi ta(g_i))$ and $X_2(g_i) = \max(\pi t(g_i), \pi ta(g_i))$.

Definition 2.6.3. (AFS). For $g_i \in \mathbb{G}$, the AFS, denoted by