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This book first published 2025

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN: 978-1-0364-5067-0
ISBN (Ebook): 978-1-0364-5068-7

Safarov I.I., Teshaev M.Kh., Usmonov B.Sh. "Propagation of eigenwaves in plate and cylindrical viscoelastic bodies". Monograph, 251 p.

The monograph research aims to develop the theory and develop a scientific basis for wave propagation in extended plate and cylindrical viscoelastic bodies connected with a medium. The monograph develops methods for solving and an algorithm for the spectral problem of reduction to a system of ordinary differential equations with complex coefficients, and also develops strategies for characterizing the damping property of a wave over a dissipative mechanical system. It is determined that, during oscillations of a dissipative inhomogeneous viscoelastic cylindrical shell with a viscous liquid, its dissipative processes proceed more intensely, the closer the natural frequencies and localization of the oscillation amplitudes near the shell, is also determined that the significance of phase velocities of waves in a viscoelastic medium (real parts of complex phase velocities) is reduced by 10–15% compared to an elastic medium.

For researchers, doctoral students, and students specializing in continuum mechanics.

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Reviewers:
Doctor of Physical and Mathematical Sciences, professor B.M. Mardonov
Doctor of Physical and Mathematical Sciences, professor I.K. Mirzaev

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INTRODUCTION

Modeling and research of deformation waves in geometrically and physically nonlinear elastic and viscoelastic rods and plates is carried out in many works [12,57,58,59]. In seismic exploration, elastic waves are artificially excited [6,28,29,30,39,49,60,70,71,72], which makes it possible to identify underground deposits of minerals (hydrocarbons, ores, water, geothermal waters, etc.), archaeological sites and to obtain geological information for engineering purposes. Wave propagation in continuous multilayer systems (with different rheological properties) attracts the attention of numerous researchers in our country and abroad [27,37,57, 85,94]. This is because, in many areas of science and technology, we increasingly have to deal with the need to calculate stress and strain fields arising in layered bodies with different rheological properties under the influence of various types of dynamic loads [11,50,51,52,67,75,89]. Dynamic problems of dissipative (viscoelastic) dynamic systems are solved by methods of mathematical physics [14,23,25,34,64,69,74,78,79,101,102].

The damping capacity of a material, its property to absorb energy during repeated deformation due to irreversible processes within itself, was used in the development of vibration-absorbing coatings and vibration-absorbing structural materials [25,108,109,110,111,112,116]. The task of such coatings is to reduce the level of resonant vibrations, to reduce the sound level regardless of its source. Such coatings have been created in several countries [125,126,128,133,137] and are used in various fields of engineering - in aviation, construction, shipbuilding, transport engineering, etc. [81,83,98,104]. The main attention is paid to the study of the laws of wave propagation in bodies of various geometric shapes with homogeneous initial states. Despite the large number of mathematical models of a mechanical system (waveguides), mathematical methods for solving problems have been developed mainly for systems such as acoustic systems, whose elastic motions are described by linear differential equations [124,149,150,158]. In [26,92,106,107,151,152], an attempt was made to determine and optimize the dissipative characteristics and the stress-strain state of mechanical systems. The above-mentioned works consider two modes of system operation: natural and forced oscillations. Natural oscillations are understood to be motions in which all points of the system oscillate with the same frequencies and damping indices (but with different amplitudes). In the mathematical formulation of the problem of excitation and propagation of waves in an ideally elastic waveguide, certain difficulties arise with setting conditions at infinity, which should play the same role as the radiation condition in space [33,35,68,115,117,119]. After all, for a half-space, it is necessary to specify not only a cylindrical wave running to infinity but also the condition of the near-surface disturbances – the Rayleigh wave. The requirements formulated in this case excluded the standing Rayleigh wave from the general idea of the solution. A condition of a similar type should be set in the case of normal waves, taking into account additional difficulties – the geometric dispersion of modes in the waveguide [18,31,32,61,62,84,105]. Setting such conditions in elastic waveguides is complicated by the fact that in some cases these modes have opposite signs of the phase and group velocities. Therefore, to clarify the meaning of the specified conditions, it is advisable to first study the characteristics of those elementary states of the waveguide (normal modes), the superposition of which is the wave field in the general case. Theoretical results of studying the properties of normal modes in elastic waveguides have shown the presence of many interesting features that have no analog for modes in acoustic and electromagnetic waveguides. In this work, the problem of propagation of free waves in layered (two- and three-layer) bodies with different rheological characteristics is studied. It is assumed that there are no external influences during the propagation of free waves. The propagation of waves occurs under the action of moving loads. A mechanical system consisting of N deformable elements constrained to each other by surfaces with full (rigid contact) or incomplete (sliding contact) contact was investigated [65,66,86,87,88,91]. The deformable elements of the system are made of elastic (or viscoelastic) bodies. The physical properties of the materials are described by the rheological model of a standard viscoelastic body [67,83,85,137]. In some cases, the physical properties of viscoelastic materials are described by linear hereditary Boltzmann-Volterra relations with integral differences in the kernels of heredity. Wave propagation occurs under the action of moving loads. A mechanical system consisting of N deformable elements connected by surfaces with full (rigid contact) or incomplete (sliding contact) contact has been investigated [65,66,86,87,88,91]. The deformable elements of the system are made of elastic (or viscoelastic) bodies. The physical properties of the materials are described by the rheological model of a standard viscoelastic body [67,83,85,137]. In some cases, the physical properties of viscoelastic materials are described by linear hereditary Boltzmann-Volterra relations with integral differences of the heredity kernels.

Wave propagation occurs under the action of moving loads. A mechanical system consisting of N deformable elements connected by surfaces with full (rigid contact) or incomplete (sliding contact) contact has been investigated [65,66,86,87,88,91]. The deformable elements of the system are made of elastic (or viscoelastic) bodies. The physical properties of the materials are described by the rheological model of a standard viscoelastic body [67,83,85,137]. In some cases, the physical properties of viscoelastic materials are described by linear hereditary Boltzmann-Volterra relations with integral differences of the heredity kernels.

In the works [93,159,160,161] the general basis of the deformation mechanism of real bodies experiencing very small deformations is presented. The corresponding relationships between stresses and strains and the resulting equations for the propagation of small-amplitude seismic vibrations are given. The consequences of these equations are considered. Experimental data are discussed. The propagation of mechanical vibrations in solids is accompanied by their attenuation [7,10,145]. For a wide class of solids, a linear dependence of the attenuation coefficient on frequency is observed based on the generalized equation of a Maxwell Ian body. In the work [5,168,169,170,171] Maxwell's representation is used to study the propagation of seismic waves and determine the rheological properties of the earth's crust and soils. An elastic-plastic medium is considered to experience not only shear deformation but also all-around compression or expansion. Differential equations of motion of an elastic-plastic medium, formulas determining the propagation velocities of longitudinal and transverse waves, and relationships for the attenuation coefficient and relaxation time of the medium are obtained. The attenuation coefficient of transverse waves is inversely proportional to the viscosity coefficient of the medium. Consequently, in a medium with high viscosity, transverse waves attenuate more slowly and prevail over longitudinal waves. Conversely, in a medium with low viscosity, longitudinal waves dominate, for which the attenuation coefficient is directly proportional to viscosity. This is confirmed by instrumental observations. Based on the data of instrumental observations of the propagation of seismic waves, it is possible to determine the attenuation coefficient and then the relaxation time. The works [140,165,166,167] are devoted to investigating the attenuation of elastic waves in rock samples under the combined action of uniform pressure and uniaxial compression. The samples were made in the form of cylinders with a diameter of 2 cm and a length of 10 cm. The uniform pressure is created by a multiplier; diesel oil is used as a pressure-transmitting medium. The uniaxial load on the sample is created by a press. It was found that the most intense increase in the velocity of longitudinal waves occurs under hydrostatic pressure. Several important units and components of modern technical devices operate in sharply dynamic modes due to rapid changes in time of the external forces acting on them [121,122,123,129,130, 139,146,147]. In this case, dynamic stresses arise in the structures, which must be taken into account when assessing the strength and performance, as well as when choosing the optimal functional conditions of certain elastic elements. The latter is especially important for technical devices whose operating principle is based on the use of non-stationary wave fields and related mechanical effects. The scientific basis for such calculation is the theory of oscillations and waves in elastic bodies. With a rapid change in loads in an elastic body, processes occur that have a pronounced wave character. In this raw material, the phenomenon is adequate to physics. The initial conditions in the harmonic problem are set. Mechanical displacements, stresses, and deformations are represented by functions of coordinates and time. The study of general laws of wave propagation in various media is the subject of the theory of oscillations, which has currently received great development [1,16,19,22,54,63,113,118,127]. Structures in the form of plates and shells interacting with an elastic (liquid) medium have found wide applications in engineering and construction. In particular, problems arising in the design of underground and underwater tanks and pipelines, linings of subway tunnels and capital mine workings, airfield pavements; elements of solid-fuel engines, etc. are reduced to such a calculation scheme. When constructing refined theories of three-layer shells, the motion of the filler should be described by three-dimensional equations of elasticity theory, then the bearing layers are shells on an elastic foundation.

In practically used design elements, with the advent of materials with low attenuation of ultrasonic waves and the development of methods for efficient excitation of a wave field, conditions have been created for an extremely wide application of the phenomenon of waveguide propagation not only in delay lines but also in some new radio engineering devices. In this regard, a new branch of science and technology, called acoustoelectronics, was formulated and is rapidly developing [17,24,40, 41,55,103,131]. Of course, the issues arising in this area cannot be completely resolved by studying the properties of normal waves in isotropic cylinders and plates. However, knowledge of these properties is the basis for analyzing and systematizing data related to practically used systems [47,76,77,80,99,134,157]. The study of the properties of waveguide modes is also important in connection with the development of a technique for using acoustic emission to assess the level of stress in design elements

[135,142,143]. Wide beams of surface waves were widely used in radio-electronic engineering.

Until recently, these waveguides have not been widely used for three main reasons. One of these reasons is related to two disadvantages that are usually attributed to waveguides: losses and low excitation efficiency. These disadvantages are unfounded [46,114]. The second reason for the comparatively rare use of waveguides is related to the question of whether they help to obtain devices that meet modern requirements. At present, the use of waveguides in delay lines for long delay times is seriously considered, mainly to eliminate beam broadening. However, this application is very narrow and requires low losses in the waveguide and negligible dispersion. The third reason is that there is still no waveguide that would allow a significant increase in the capacity of memory devices using beams. This is explained by the residual dispersion, and it is this dispersion that is the main obstacle to the use of waveguides in information storage devices. Considering the wave processes in a waveguide, two types of problems can be distinguished. In the problems of the right type, we are not interested in the sources of wave motion and are looking for only possible states of the waveguide, in essence, we are talking about searching for some resonance situations - such solutions of the equations of motion, for which provide zero boundary conditions concerning a certain number of static and kinematic factors. These particular solutions are called normal modes or normal waves in the waveguide. The second type of problem is associated with the study of forced wave motions in the waveguide. Due to the presence of an infinite set of possible states (normal modes) in the waveguide, the problems arising here differ from similar problems for a half-space by greater complexity. In the mathematical formulation of the problem of propagation of waves in an ideally elastic waveguide, certain difficulties arise with setting conditions at infinity, which should play the same role as the condition of study in the case of space.

Also, problems of hydro elasticity are divided into stationary and non-stationary. This paper considers non-stationary problems, which in turn are divided into oscillation problems and wave problems. According to the solution methods, they are divided into coupled and uncoupled. In uncoupled problems, the equations of fluid dynamics are solved first, and then the wave motion of an elastic body, with priority given to determining the parameters of the fluid - velocity and pressure. This approach is used in the study of the mechanics of living organisms, i.e. biomechanics. Another approach considers the motion of a fluid interacting with a solid. The stress acting from the fluid on the solid, friction, and pressure are determined. This means that it is assumed that the deformations of the shell do not affect the fluid flow [136]. Then they are substituted into the equations of the dynamics of the body as elastic and the displacements, longitudinal and normal, are found, i.e. deflection. This allows us to determine the stress-strain state of the elastic structure, which is a priority in an uncoupled problem. This approach is applicable, for example, in determining the strength of the wings and fuselage of an aircraft. For the coupled problem, the equations of the dynamics of the elastic body and the fluid are solved simultaneously, taking into account the corresponding boundary conditions on the impermeable surfaces. This approach has been applied, for example, to the study of hydroelastic oscillations [73], as well as in the present study of nonlinear deformation waves in elastic shells containing an incompressible viscous fluid.

The accuracy of the scheme for approximately considering fluid compressibility for solving problems of natural oscillations of a structure with an ideal fluid, described in [82,90,95] and adapted to the problem of calculating natural oscillations of elastic structures with a viscous compressible fluid, is investigated.

The state of the issue of the dynamic interaction of shells and plates with continuous media (air, water, soil) in recent decades has been the subject of reviews of work, from which it follows [96,97,100] that despite numerous studies on the indicated problems, some issues have not been sufficiently studied. In particular, calculations of structures for the action of various types of moving loads are very important for practice. The simplest examples of systems with moving loads are: railway rails under the action of moving trains, asphalt, concrete, and other surfaces under the action of moving vehicles, in particular airplanes, pipelines under the action of cleaning devices and moving liquid, plates, and shells under the action of moving liquid, gas, etc. A significant number of works are devoted to this issue, a detailed analysis of which is contained in reviews [42,43,120, 132]. However, general methods for calculating a plate of the variable cross-section have not yet been developed. Problems of elastic wave propagation in structural elements such as shells and plates began to be considered in the last century. In 1889, Rayleigh investigated wave propagation in a plate, and later phase and group velocities were determined for the plate. The history of this issue and the corresponding bibliography of works up to 1965 are given in the review by L. Ainona and W. Nigula, as well as in the work [2,3,13]. For cylindrical shells, the characteristic equation for the exact three-dimensional theory for axially symmetric deformations is given in [5,21,36,38], however, in [20] the phase curve for the first mode of axially symmetric motion is given. In 1960, D. Miklovits published an extensive list of works by foreign authors on the propagation of elastic waves (mainly stationary) in rods, plates, cylindrical shells, half-space, and unlimited space [153]. Studies of shell vibrations with a filler (elastic or liquid) and the analysis of the propagation of free waves in the shell-inertial medium system began much later. In the first works of this kind, the main object of study was the shell itself, and the filler was assumed to be inertial and its presence was taken into account using the Winkler-Zimmerman model, according to which the filler's reaction was considered proportional to the deflection of the shell. In the case of a shell with liquid, the simplest model was an incompressible ideal liquid; different pressures were also used. A fairly detailed analysis of the state of the art in the field of vibrations of shells and plates interacting with a fluid medium is contained in [4,15,44,45,48,53,56] and for shells with an elastic filler [138]. The problem of propagation of free waves in the plate–elastic foundation or shell–filler systems is the subject of [141,148]. In [54], the construction of dispersion curves for free waves in a layer and a support half-space with different elastic properties is considered, the plate equation is used, and a comparison is made with the results obtained by applying the elasticity theory to the layer. It is noted that for a relatively rigid layer, good agreement is achieved over the entire range of wave numbers for which free waves can exist. In [155], the problem of propagation of free waves in three-layer plates is considered in a refined formulation, where the motion of the filler is described by the Lyane equations with inertial terms, and the Kirchhoff–Love hypothesis is used for skins. Dispersion equations are obtained and phase velocities are determined for symmetric and antisymmetric waves. Axially symmetric free waves in an infinitely long cylindrical shell with an elastic filler are investigated in the papers [156,162,163,164,172]. The study of wave propagation patterns in such elastic themes for which not only the interaction of boundaries plays a significant role in the formation of the field is carried out. An infinite elastic plate or a strip of variable thickness are used as objects that are considered in this connection.

To summarize the literature review, the following conclusions can be drawn.

1. As before, the study of wave processes in viscoelastic (dissipative homogeneous and inhomogeneous) layered bodies within the framework of flat and three-dimensional mechanics of a deformable body remains relevant.

2. Also, the study of wave processes in deformable waveguides of non-circular cross-section within the framework of three-dimensional viscoelasticity is relevant.

CHAPTER I

PROPAGATION OF NATURAL WAVES IN DISSIPATIVELY INHOMOGENEOUS LAYERED BODIES

In the first chapter, the propagation of natural waves in layered (two- or three-layered) planar and cylindrical bodies is considered in the exact formulation. When analyzing numerical results, mechanical effects for dissipatively inhomogeneous mechanical systems are found [172].

1.1. Mechanical problem

The problem of propagation of waves in a waveguide in the form of rods, plates, or pipes has been studied in many works [173,174] (Fig. 1.1). These waves are similar in their characteristics to Lamb waves and the obtained SH-polarization waves.

In many aspects, the problem of non-destructive testing is connected with the formulation and analysis of quantitative data on the propagation of harmonic waves. These problems arise, for example, when determining the shape, volume, orientation, and location of defects inside an elastic body. Particularly complex and interesting wave problems arise in connection with the use of the phenomenon of acoustic emission to predict the durability of structures [175].

In longitudinal axially symmetric waves, similar to symmetric Lamb waves, the motion occurs relative to the axis z and predominantly longitudinal (axial) is observed.

In non-axially symmetric bending waves, similar to Lamb's antisymmetric waves, the axis z undergoes bending. Here, predominantly the transverse component of displacement is observed.

In torsional waves, similar to SH-polarized transverse waves in plates, there is only one azimuthal displacement component $\Delta\varphi$ here the movement is symmetrical about the axis. z and represents the rotation of the cross-section of the rod (pipe) relative to this axis.

Waves propagating in pipes (rods) are classified as longitudinal (L-fashion), bending (F-fashion), and twisting (Γ-fashion). The L and F modes are close to Lamb waves with non-plane displacement, in particular, the L-modes are identical axially symmetric modes, and the T-mode is similar to the SH-wave. Each of these three modes is described using the mode parameters n and m as follows: $L(0,m)$, $F(n,m)$, $T(n,m)$ $(n,m = 1, 2, 3,...j$ VT - fashion $n = 0, 1, 2, 3,...)$.

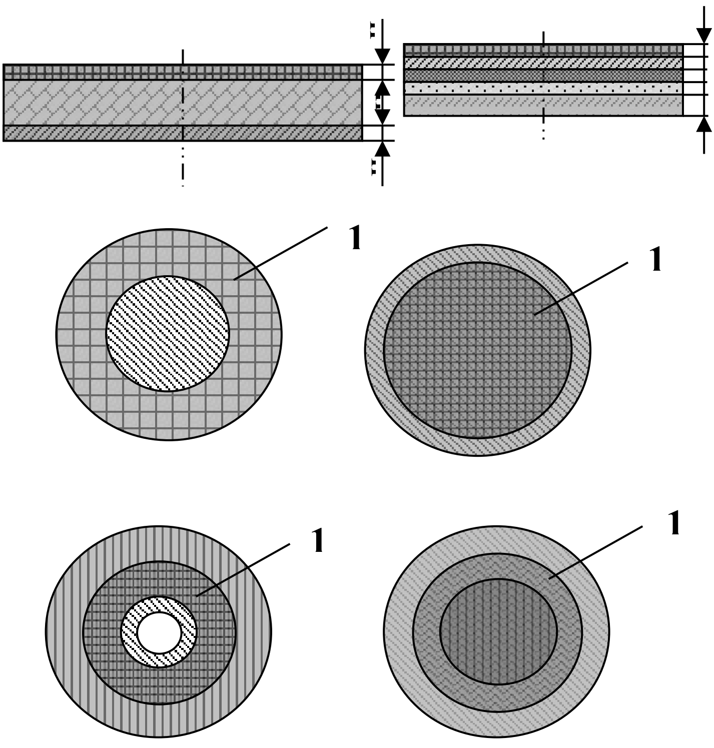

Fig.1.1 Calculation schemes of mechanical engineering structures consisting of flat and cylindrical bodies.

Parameter n shows the mode by the pipe circumference, m - by the wall thickness. Normal waves of all types are used in ultrasonic flaw detection - testing of wires, rods, sheet structures, pipes, tanks, pipelines, etc. The use of normal waves in a plate is an independent and rather complex task.

The problems of wave propagation in a plate with constant thickness are solved in the works [15,21,31,75,86]. In these works the main attention is paid to the description of resonance phenomena in extended bodies. The study of the existence of natural frequency and the form of oscillations depending on geometric parameters (plate thickness) represents the most interesting problem of stationary dynamics of an elastic body.

Taking into account internal friction in waveguides complicates the tasks set. In this paper, the relationship between stress and strain (or viscosity) is taken into account using the hereditary theory [61].

The main problem is to discover new types of free waves that propagate in extended bodies with variable cross-sections.

The equations of motion of the deformable layer in the absence of mass forces have the form [69]:

$$\tilde{\mu}_j \nabla^2 \vec{u} + (\tilde{\lambda}_j + \tilde{\mu}_j)\, grad\ div\ \vec{u} = \rho_j \frac{\partial^2 \vec{u}}{\partial t^2}, \text{ (j =1, 2, 3..)} \quad (1.1)$$

Here $\vec{u}(u_x, u_y, u_z)$-vector of displacement of points of the environment; ρ_j - material density; u_i – displacement components; ν_j - Poisson's ratio;

$$\tilde{\lambda}_j = \frac{\nu_j \tilde{E}_j}{(1+\nu_j)(1-2\nu_j)}; \qquad \tilde{\mu}_j = \frac{\nu_j \tilde{E}_j}{2(1+\nu_j)},$$

where

$\tilde{E}$ – the operator modulus of elasticity, which has the form [23,61]:

$$\widetilde{E}\varphi(t)=E_{01}\left[\varphi(t)-\int_{0}^{t}R_E(t-\tau)\varphi(t)d\tau\right] \qquad (1.2)$$

$\varphi(t)$–an arbitrary function of time; $R_E(t-\tau)$– relaxation core; E_{01}– instantaneous modulus of elasticity;

We take the integral terms in (1.2) to be small, then the functions $\varphi(t)=\psi(t)e^{-i\omega_R t}$, where $\psi(t)$-a slowly changing function of time, ω_R - is a real constant. Further, applying the freezing procedure [138], we note that relations (1.2) are approximated by

$$\overline{E}\varphi=E\left[1-\Gamma^C(\omega_R)-i\Gamma^S(\omega_R)\right]\varphi,$$

where $\Gamma^C(\omega_R)=\int_0^\infty R(\tau)\cos\omega_R\tau\, d\tau$, $\Gamma^S(\omega_R)=\int_0^\infty R(\tau)\sin\omega_R\tau\, d\tau$,

respectively, the cosine and sine Fourier images of the relaxation kernel of the material. As an example of a viscoelastic material, we take a three-parametric relaxation kernel $R(t)=Ae^{-\beta t}/t^{1-\alpha}$. The influence function is subject to the usual requirements of integrability, continuity (except for *t*=0), the sign of definiteness, and monotonicity:

$$R>0, \qquad \frac{dR(t)}{dt}\leq 0,\ \ 0<\int_0^\infty R(t)dt<1.$$

$\vec{u}$ - vector of displacements of the environment of the j-th layer.

If the displacement vector is represented as a potential and solenoidal part, then the wave equation in Cartesian (x, y, z) and cylindrical (r,θ, z) coordinate systems respectively have the form $\vec{u}=grad\ \ \varphi+rot\vec{\psi}$,

where φ is the potential of longitudinal waves; $\vec{\psi}\,(\psi_t,\psi_\theta,\psi_z)$ - the potential of transverse waves. Potentials of the function in Cartesian coordinate systems satisfy the following wave equations:

$$\nabla^2\varphi-\frac{1}{c_p^2\Gamma_\kappa}\frac{\partial^2\varphi}{\partial t^2}=0;\ \ \nabla^2\psi_x-\frac{1}{c_S^2\Gamma_\kappa}\frac{\partial^2\psi_x}{\partial t^2}=0; \tag{1.3}$$

$$\nabla^2\psi_y-\frac{1}{c_S^2\Gamma_\kappa}\frac{\partial^2\psi_y}{\partial t^2}=0;\ \ \nabla^2\psi_Z-\frac{1}{c_S^2\Gamma_\kappa}\frac{\partial^2\psi_Z}{\partial t^2}=0,$$

where $\Gamma_\kappa=1-\Gamma^C(\omega_R)-i\Gamma^S(\omega)$;

$c_p^2=(\lambda+2\mu)/\rho;\ \ c_s^2=\mu/\rho-$ respectively, the speed of propagation of longitudinal and transverse waves in an elastic body.

Based on the known displacement potentials, the displacement of the medium is determined:

$$u_x=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi_z}{\partial y}-\frac{\partial\psi_y}{\partial z};u_y=\frac{\partial\varphi}{\partial y}+\frac{\partial\psi_x}{\partial z}-\frac{\partial\psi_z}{\partial x};$$

$$u_z=\frac{\partial\varphi}{\partial z}+\frac{\partial\psi_y}{\partial x}-\frac{\partial\psi_x}{\partial y};\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

The displacement vectors, deformations, and displacement potentials in cylindrical coordinate systems are expressed as:

$$u_r=\frac{\partial\varphi}{\partial r}+\frac{1}{r}\frac{\partial\psi_z}{\partial\theta}-\frac{\partial\psi_\theta}{\partial z};$$

$$u_\theta=\frac{1}{r}\frac{\partial\varphi}{\partial\theta}+\frac{\partial\psi_r}{\partial z}-\frac{\partial\psi_z}{\partial r};$$

$$u_z=\frac{\partial\varphi}{\partial z}+\frac{\partial\psi_\theta}{\partial r}+\frac{\psi_\theta}{r}-\frac{1}{r}\frac{\partial\psi_r}{\partial\theta}; \tag{1.4}$$

$$\nabla^2\varphi-\frac{1}{c_p^2\Gamma_\kappa}\frac{\partial^2\varphi}{\partial t^2}=0;\ \ \nabla^2\psi_Z-\frac{1}{c_S^2\Gamma_\kappa}\frac{\partial^2\psi_Z}{\partial t^2}=0;$$

$$\nabla^2\psi_\theta - \frac{\psi_\theta}{r^2} + \frac{2}{r^2}\frac{\partial\psi_r}{\partial\theta} - \frac{1}{c_s^2\Gamma_\kappa}\frac{\partial^2\psi_\theta}{\partial t^2} = 0$$

$$\nabla^2\psi_r - \frac{\psi_r}{r^2} - \frac{2}{r^2}\frac{\partial\psi_\theta}{\partial\theta} - \frac{1}{c_s^2\Gamma_\kappa}\frac{\partial^2\psi_r}{\partial t^2} = 0\,;$$

The propagation of waves in dissipative mechanical systems consisting of extended flat or cylindrical bodies is considered. The main objective of this work is to study the dissipative properties of extended cylindrical mechanical systems during the propagation of harmonic waves.
In many cases, the propagation of elastic waves in a bounded body is a dispersion phenomenon, in which the propagation velocity of infinite monochromatic sinusoidal waves is a function of the wavelength (or frequency). In the study of impulsive stress waves, dispersion plays an important role. The impulse can be considered as a Fourier integral of sinusoidal components (with different frequencies), each of which will travel at a different speed; it follows that the shape of the impulse will change as it moves away from the point of origin, or other words, dispersion is accompanied by distortion. In analyzing the phenomenon of dispersion, it is first necessary to determine the change in the propagation velocity (phase velocity C) depending on the wavelength λ; the result of such an investigation can be depicted as a dispersion curve, which represents the dependence of the value of C on λ [67,76,99,100,151]. This part of the analysis consists of finding solutions to the general equations of motion of a deformable body (1.1), harmonically changing in time and satisfying certain boundary conditions. In practice, the equations representing the boundary conditions are so complex that using exact equations this stage can be carried through to completion only for a small number of cases.

Speed *C*, appearing in relations (1.14) – (1.15) is called the phase velocity since it represents the distance traveled per unit of time by any point of constant phase, say maxima or minima. It does not necessarily coincide with the speed of propagation of the pulse, which is called the group velocity and will be denoted by V. The group velocity V can be determined by drawing an envelope of the pulse and measuring the distance traveled by this envelope per unit time. It can be shown that the group velocity V is equal to $V = C - \lambda\frac{dc}{d\lambda} = C + \omega\frac{dc}{d\omega}$, where C, λ, ω, dc/dλ, and dc/dω are averages over the frequency range that makes up the main part of the pulse.

If C increases with frequency, C<V, the envelope propagates faster than the individual frequency components. If C decreases with frequency, the opposite is true. Dispersion is important for several reasons, perhaps most significantly that the pulse energy propagates with velocity V. In addition, the dispersion of body waves follows from most theories proposed to account for wave absorption, but in practice, it is not observed. Most rocks simply show minor variations in velocity with frequency in the seismic range. In [142], in borehole studies on tonal signals to a depth of about 800 m, the same speed was obtained at 35 and 55 Hz. In the works [6,139] it is given that, in normal situations, seismic body waves can be expected to have low dispersion. However, dispersion is significant for surface waves and some other phenomena. Many of the postulated mechanisms predict that the quality factor Q (Q = 2π / fraction of energy lost per period) depends on the frequency (in liquids it is proportional to the frequency). Intensity loss in dB = 10 lg (I_0 /I) = 20 lg (A_0 /A) = 0.3 ω (X_S – 200)/ 2000, where X_S is the distance to the explosion point. The increase in absorption at high frequencies leads to a change in the waveform with increasing distance from the source.

1.2. About the propagation of Eigen waves in dissipatively inhomogeneous layered flat bodies

In the spatial Cartesian coordinate system (x, y, z) a sequence of parallel viscoelastic planes is given (Fig. 1.2). The main goal is to study the propagation of natural waves in multilayer bodies. The general appearance of multilayer bodies is shown in Fig. 1.2.

Below we will consider a plane problem where the displacements do not depend on the Z coordinate. Mathematically, this problem is formulated as follows:

$$\rho_j \frac{\partial^2 u_j}{\partial t^2} = \frac{\partial \sigma_{xx}^{(j)}}{\partial x} + \frac{\partial \sigma_{xy}^{(j)}}{\partial y}, \quad \rho_j \frac{\partial^2 \vartheta_j}{\partial t^2} = \frac{\partial \sigma_{yy}^{(j)}}{\partial y} + \frac{\partial \sigma_{xy}^{(j)}}{\partial x}, \qquad (1.5)$$

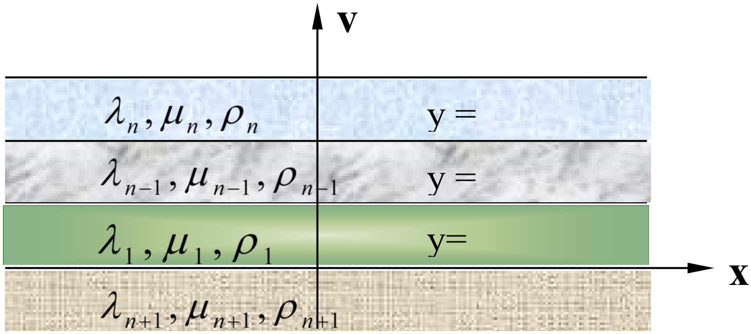

Fig. 1. 2. Calculation scheme: body on a half-space

where ρ_j is the density of the material; u_j and ϑ_j respectively, displacements in the x and y directions; (j=1,2,…….,N), j is the layer number. Two types of conditions can be set at the boundary of two bodies:

1. In the case of rigid contact at the interface, the condition of continuity of the corresponding components of the stress tensor and the displacement vector is set, i.e.

$$\sigma_{nn}^{(1)} = \sigma_{nn}^{(2)}; \quad \sigma_{ns_1}^{(1)} = \sigma_{ns_1}^{(2)}; \sigma_{ns_2}^{(1)} = \sigma_{ns_2}^{(2)};$$
$$u_n^{(1)} = u_m^{(2)}; \quad u_{s_1}^{(1)} = u_{s_1}^{(2)}; u_{s_2}^{(1)} = u_s^{(2)}. \qquad (1.6,a,)$$

If there is no friction at the interface, then

$$\sigma_{nn}^{(1)} = \sigma_{nn}^{(2)}; \quad \sigma_{ns_1}^{(1)} = \sigma_{ns_1}^{(2)} = \sigma_{ns_2}^{(1)} = \sigma_{ns_2}^{(2)} = 0; \quad u_n^{(1)} = u_m^{(2)}; \qquad (1.6,b)$$

2. On the free surface, the condition of freedom from stress is set, i.e.

$$\sigma_{nn}^{(1)} = 0; \quad \sigma_{ns_1}^{(1)} = 0; \sigma_{ns_2}^{(1)} = 0. \qquad (1.6, s)$$

Solution methods. Now let us consider the solution of the differential equation (1.5) - (1.6) for one layer. Then instead of $\sigma_{xx}^{(j)}$, $\sigma_{yy}^{(j)}$ и $\sigma_{xy}^{(j)}$ we substitute the following expressions:

$$\sigma_{xx}^{(j)} = \tilde{\lambda}_j \theta_j + 2\tilde{\mu}_j \frac{\partial u_j}{\partial x};$$
$$\sigma_{xy}^{(j)} = \tilde{\mu}_j \left(\frac{\partial u_j}{\partial \mathrm{y}} + \frac{\partial \vartheta_j}{\partial x} \right), \sigma_{yy}^{(j)} = \lambda_j \theta_j + 2\tilde{\mu}_j \frac{\partial \vartheta_j}{\partial y}. \qquad (1.7)$$

Where $\tilde{\lambda}_j$ and $\tilde{\mu}_j$ ($j = 1,2 \ldots$N) are operator elasticity moduli[13], which is determined by (1.2);

$\theta_j = \dfrac{\partial u_j}{\partial x} + \dfrac{\partial \vartheta_j}{\partial y}$ - volumetric expansion.

The equation of motion in stresses is reduced to the following form:

$$\bar{\mu}_n \left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial u^2{}_n}{\partial y^2} \right) + (\bar{\lambda}_n + \bar{\mu}_n) \frac{\partial}{\partial x} \left(\frac{\partial\ u_n}{\partial x} + \frac{\partial \vartheta_n}{\partial y} \right) - \rho_n \frac{\partial^2 u_n}{\partial t^2} = 0;$$
$$\bar{\mu}_n \left(\frac{\partial^2 \vartheta_n}{\partial x^2} + \frac{\partial \vartheta^2{}_n}{\partial y^2} \right) + (\bar{\lambda}_n + \bar{\mu}_n) \frac{\partial}{\partial y} \left(\frac{\partial\ u_n}{\partial x} + \frac{\partial \vartheta_n}{\partial y} \right) - \rho_n \frac{\partial^2 \vartheta_n}{\partial t^2} = 0; \qquad (1.8)$$

Where ρ_n - density of the material. We find the solution to the problem in the form:

$$u_n = U_n(y)\, e^{k(x-ct)}\ ;\ \vartheta_n = V_n(y) e^{k(x-ct)}\ ;\ n = 1,2,\ \ldots\ldots,N \qquad (1.9)$$

where $U_n(y)$ and $V_n(y)$ are the amplitude complex vector function; *k* is the wave number; $C = C_R + iC_i$ is the complex phase velocity; and ω is the complex frequency.

To clarify their physical meaning, let us consider two cases:

1) $k = k_R$; $C = C_R + iC_i$, then the solution (1.9) has the form of a sinusoid in x, the amplitude of which decays over time;

2) $k = k_R + ik_I$; $C = C_R$, then at each point x the oscillations are steady, but they decay along x.

In both cases, the imaginary parts k_I or C_I characterize the intensity of dissipative processes. Substituting (1.9) in (1.8), we obtain:

$$\overline{\mu}_n\left(U_n'' - k^2 U_n\right) + \left(\overline{\lambda}_n + \overline{\mu}_n\right) ik\left(ikU_n + V_n'\right) + \rho_n k^2 C^2 U_n = 0; \quad (1.10)$$

Thus, we have equations (1.10) of the second order for two regions each. We solve the problem directly, without reducing the equations to a fourth-order equation. All reasoning is given for a layer.

We find a particular solution of system (1.10) in the form

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = \begin{pmatrix} A_n \\ B_n \end{pmatrix} e^{r_n y},$$

where r_n is a constant. A homogeneous algebraic system concerning A_n and B_n has nontrivial solutions if its determinant is zero.

$$\begin{vmatrix} \bar{C}_{\mathrm{Tn}}^2({r^2}_n - k^2) - (\bar{C}_{\mathrm{Ln}}^2 - \bar{C}_{\mathrm{Tn}}^2)k^2 + \bar{C}^2 k^2 i(\bar{C}_{\mathrm{Ln}}^2 - \bar{C}_{\mathrm{Tn}}^2)kr_n \\ i(\bar{C}_{\mathrm{Ln}}^2 - \bar{C}_{\mathrm{TLn}}^2)kr_n \ \ \bar{C}_{\mathrm{TLn}}^2({r^2}_n - k^2) + (\bar{C}_{\mathrm{Ln}}^2 - \bar{C}_{\mathrm{Tn}}^2){r^2}_n + C^2k^2 \end{vmatrix} = 0, \quad (1.11)$$

$\overline{C}_{Ln}^2 = (\overline{\lambda}_n + 2\overline{\mu}_n)/\rho_n$. $\overline{C}_{Tn}^2 = \overline{\mu}_n/\rho_n$. at η = 0 the quantities $\overline{C}_{Ln}^2$ *u* $\overline{C}^2{}_{Tn}$ are the speed of compression and shear waves in an elastic medium, respectively [4]. Equation (1.11) can have four roots $(r_n)_{1,3} = \pm k\sqrt{1 - C^2/\overline{C}_{Ln}^2}$; $(r_n)_{3,4} = \pm k\sqrt{1 - C^2/\overline{C}_{Tn}^2}; n = 0,1; \quad i = 1......,4.$

As a result, we find four particular solutions of the form

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = \sum_{i=1}^{4} C_{ni} \begin{pmatrix} A_{ni} \\ B_{ni} \end{pmatrix} e^{(r_n)_i y}, \text{ (n = 0.1).} \quad (1.12)$$

Substituting the values of $(r_n)_i$ into (1.12) we find A_{ni}, B_{ni} when $r_n = (r_n)_i$.

The expressions for the offset are as follows:

$$\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = C_{11}\begin{pmatrix} ik \\ -k\overline{q}_1 \end{pmatrix} e^{-k\overline{q}_1 y} + C_{12}\begin{pmatrix} ik \\ k\overline{q}_1 \end{pmatrix} e^{k\overline{q}_1 y} +$$

$$+C_{13}\begin{pmatrix} -k\overline{S}_1 \\ -ik \end{pmatrix} e^{-ks_{11}y} + C_{14}\begin{pmatrix} -k\overline{S}_1 \\ -ik \end{pmatrix} e^{+ks_{11}y};$$

$$\begin{pmatrix} U \\ V \end{pmatrix} = C_{22}\begin{pmatrix} ik \\ -k\overline{q}_1 \end{pmatrix} e^{k\overline{q}_1 y} + C_{24}\begin{pmatrix} k\overline{s} \\ -ik \end{pmatrix} e^{\ k\overline{s}y}.$$

Therefore, for both rigid and sliding contacts, we obtain a set of six boundary conditions, which lead to six homogeneous equations with six unknowns C_{11}, C_{12}, C_{13}, C_{14}, C_{22}, C_{24}. For such a system of equations to have nontrivial solutions, the determinant of the coefficients must be equal to zero. The last equation gives the dispersion equation for dissipative systems, where

$$\overline{S}_n = (1 - C^2/\overline{C}^2{}_n)^{1/2};\ \ \overline{q}_n = (1 - C^2/\overline{C}^2{}_{Ln}), n = 0{,}1.$$

As an example, let us consider the problem of propagation of natural waves in a viscoelastic layer on a half-space.

Hard contact. The dispersion equation has the following form

$$\begin{vmatrix} (1+\overline{S}_1^2)\, e^{-\overline{\xi}q_1} & (1+\overline{S}_1^2)\, e^{\overline{\xi}q_1} & -2e^{\overline{\xi}q_1} & 2e^{\overline{\xi}q_1} & 0 & 0 \\ -2\overline{q}_1^{\ \xi\dot{q}_{11}} & 2\overline{q}_1^{\ \xi\dot{q}_{11}} & \left(\overline{s}_1 + \frac{1}{\overline{s}_1}\right)e^{-\overline{\xi}q_1} & \left(\overline{s}_1 + \frac{1}{\overline{s}_1}\right)e^{\overline{\xi}q_1} & 0 & 0 \\ (1+S_1^2)\, e^{\overline{\xi}q_1} & (1+\overline{S}_1^2)\, e^{-\overline{\xi}q_1} & -2e^{\xi\overline{S}_1} & 2e^{-\overline{\xi}q_1} & (1+s^2)\gamma_1 & -2/\gamma_1 \\ -2\overline{q}_1^{\ \xi\dot{q}_{11}} & 2\overline{q}_1^{\ \xi\dot{q}_{11}} & \left(\overline{s}_1 + \frac{1}{\overline{s}_1}\right)e^{\overline{\xi}q_1} & \left(\overline{s}_1 + \frac{1}{\overline{s}_1}\right)e^{-\overline{\xi}q_1} & \frac{2\overline{q}_1}{\gamma_{1\ 1}} & \left(\overline{s} + \frac{1}{\overline{s}}\right)/\gamma \\ e^{\xi\overline{q}_1} & e^{\xi\overline{q}_1} & e^{\xi\overline{q}_1} & e^{\xi\overline{q}_1} & -1 & -1 \\ \overline{q}e^{\xi\overline{q}_1} & \overline{q}_{1\ 1}e^{-\xi\overline{q}_1} & -\frac{1}{\overline{s}}e^{\xi\ \overline{s}} & -\frac{1}{\overline{s}}e^{\xi\ \overline{s}} & -\overline{q} & \frac{1}{s} \end{vmatrix} = 0 \qquad (1.13)$$

where ζ is the dimensionless wave number; $\gamma_1 = \overline{\mu}_l / \overline{\mu}$ or $\lambda+2\mu == 2(1-\nu)/(1-2\nu)$. As the relaxation kernel of the viscoelastic material, we will take the three-parameter kernel $R(t)=\frac{Ae^{-\beta t}}{t^{1-\alpha}}$ Rzhanitsyn-Koltunov [61], having a weak singularity, where A, α, β -material parameters [61]. We will accept the following parameters: $A = 0{,}048;\quad \beta = 0{,}05;\quad \alpha = 0{,}1$, using the complex representation for the elastic modulus described earlier. The roots of the frequency equation are solved by the Muller method, at each iteration of the Muller method the Gauss method is applied with the selection of the main element. Thus, the solution of equation (1.13) does not require the disclosure of the determinant. As an initial approximation, we choose the phase velocities of the waves of the elastic system. For free waves at η =0 phase velocities and wave number are real quantities. In the calculations, we take the following values of the parameters:

$\theta = \rho_1/\rho_2 =$ 0, 75; $E_{min} = 6.9 \cdot 10^6$ N/m^2; $E_{max} = 6.9 \cdot 105$ N/m^2;$\beta = 10^{-4}$; n = 1.

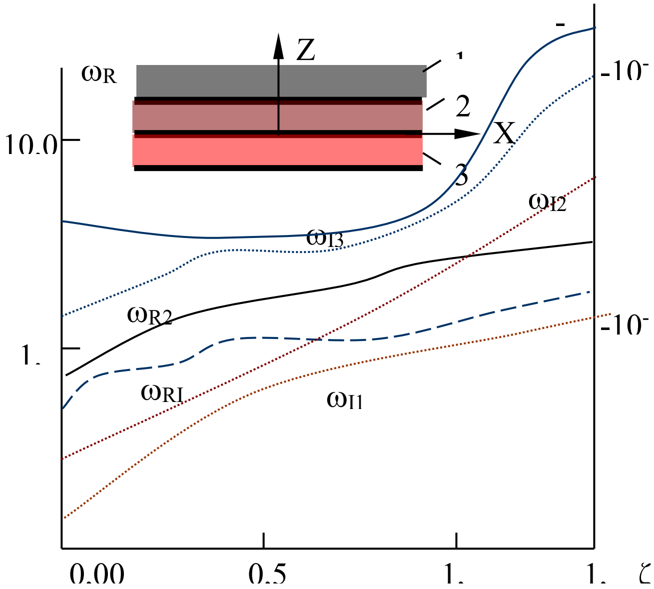

Fig. 1.3,a. Change in natural frequencies from wave number (dissipative homogeneous system)