Notes of Seminar of Advanced Functional Analysis and Integral Transforms

Notes of Seminar of Advanced Functional Analysis and Integral Transforms:

Acta Matematica

Edited by

Francisco Bulnes

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PREFACE

The present book volume I is a collection of notes on expositions, talks and lectures on advanced functional analysis and integral transforms along the year of all sessions of the seminar celebrated all days Fridays of each week in my auditory-laboratory located in the Tecnológico de Estudios Superiores de Chalco, in Chalco, State of Mexico, Mexico.

In the book are presented advances on operator theory focused to the non-bounded and infinite dimensional operators, integral transforms constructed to star of its extension on functional integrals and whose domains are extended on meromorphic domains considering Banach and Hilbert structures, new integral transforms that born of science materials research and inverse problems in differential equations. Certain fine aspects of mathematical analysis on numerical fields and function spaces. Also notes and selected themes on harmonic and non-harmonic analysis, some directions from a point of view of orbits and their minimal types to the construction of certain characters corresponding to the generalized modules that can be the infinite dimension representations required to a space being a non-compact group and not necessarily compact. Some concrete applications of integral transforms to evaluation of electrical devices, and Riemannian supermanifolds in the functional study of expansion of the Universe considering torsion. All mentioned are some contents and items discussed and exposed through specialized research chapters that conform the book.

Yours, Prof. Dr. Francisco Bulnes
IINAMEI Director
Head of Research Department in Mathematics and Engineering, TESCHA
CDUV Rector
Selinus University, Associate Professor

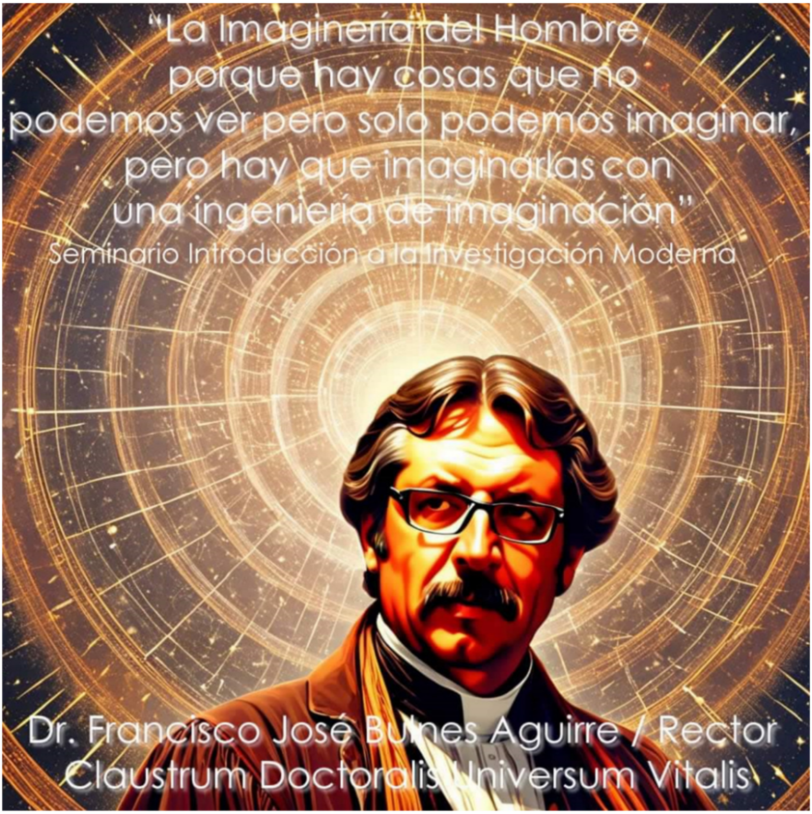

Fig. 1 Lithography of Dr. Francisco Bulnes, Seminar Director: Advanced Functional Analysis and Integral Transforms. The thought given in the top part, can be read "*The imagery of the man, because there are things that we cannot see, however only we can imagine but is necessary imagine them with an engineering of the imagination*"

CHAPTER 1

TORSION EVIDENCE AND EFFECTS IN SPINOR FRAMEWORK

ELISA VARANI

4.1 Introduction

The study of spinor in general relativity leads to important results about torsional effects. We can see from Hamiltonian formulation how spinors give rise to rotational terms associated with torsion.

In general relativity the affine connection is required to be symmetric, so torsion is zero, we want to verify if a spinor field can be considered a torsion source. The broader reference context is Einstein Cartan's gravitational theory, which is a generalization of general relativity; in this theory in addition to curvature there is torsion associated with intrinsic angular momentum density. The affine connection is not restricted to being symmetric as required in general relativity, torsion is included due to the antisymmetric part of the affine connection. In Einstein Cartan's theory torsion is connected to the spin tensor as expressed by the Cartan equations. These equations are obtained through the variation of the total action with respect to the torsion; the total action is intended as the sum of the action for the gravitational field and the action for the fermionic field, $s = s_G + s_f$, according to the minimal action principle $\delta S = 0$.

We consider these important hints about torsion and spin tensor to revisit general relativity with spinor fields, we focus on the requirement of symmetric affine connection and develop the calculation of the spin coefficients.

In order to include fermions in general relativity we introduce a local reference frame and define a tetrad of basis vectors. We refer to the Hamiltonian formulation and calculate the canonical momenta associated with the temporal variation of the tetrads, we find a fermionic rotational term. This term is connected to torsion as suggested by Cartan's equations.

Starting from a torsion-less theory we get a rotational current that would generate a torsion contribution.

Once the torsion presence in general relativity has been ascertained, the next step concerns the study of the tensor energy momentum for spinors.

The spin connection leads to important additional terms, the way to perceive these inputs is a scheme of discrete gravity, we calculate different contributions in the process of increasing perturbations starting from flat spacetime.

The relation between the energy and the metric leads to other conjectures involving gravitational waves and the reinterepretation of the cosmological constant.

4.2 Spinors in general relativity

In general relativity, physical laws are required to maintain the same form under a general coordinate transformation *(Diff* M^4), according to the general covariance principle.

Mathematically this concept is expressed by Weyl's theorem which implies the choice of a symmetric metric connection for the description of the gravitational field.

Fermionic fields are described by spinors, spinors are a representation of the Lorentz group, there are no analogous objects for a general coordinate transformation.

Conforming to the equivalence principle it is possible to identify a system of inertial coordinates so that the effects of the gravitational field are cancelled.

We consider a locally inertial reference frame, the tetrad field e_a^μ will be better defined later.

In order to construct the Dirac equation or more generally the action for the fermionic field we replace the ordinary derivative with a covariant derivative. The covariant derivative of a spinor must transform as a vector with respect to coordinate transformations and as a spinor with respect to a Lorentz transformation of the tetrad basis. Lorentz transformations rotate the vectors of the tetrad without changing the space-time coordinates.

After these clarifications we write the generally covariant Dirac action and calculate canonical momenta in consonance with the Hamiltonian formulation.

This calculation leads to a fermionic rotational current. This work moves between general relativity which is a torsion-free theory and the gravitational theory of Einstein- Cartan.

Torsion is a tensor defined from the affine connection $\Gamma_{\beta\gamma}^\alpha$:

$$\mathrm{T}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta} \tag{1}$$

In general relativity the affine connection is symmetric, so torsion is zero.

The choice of the affine connection $\Gamma^{\alpha}_{\beta\gamma}$ is a consequence of two requirements:

- symmetry $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$
- metricity of the covariant derivative $\nabla_{\alpha} g_{\mu\nu} = 0.$ Ref[1]

These conditions are fulfilled if the affine connection coincides with the metric connection defined by the Christoffel symbols

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} = \frac{1}{2} g^{\alpha\lambda}\left(\partial_{\beta} g_{\lambda\gamma} + \partial_{\gamma} g_{\lambda\beta} - \partial_{\lambda} g_{\beta\gamma}\right) \tag{2}$$

4.3 Spinor action

We have already mentioned tetrads as a local inertial frame (Cartan's repère mobile) [2].

The following formulas summarize the relations between tetrads and the metric:

$$g_{\mu\nu} = \eta_{ab} e^{a}_{\mu} e^{b}_{\nu} \quad e^{a}_{\mu} e^{\nu}_{a} = \delta^{\nu}_{\mu} \quad e^{\mu}_{a} e^{b}_{\mu} = \delta^{b}_{a} \tag{3}$$

Both Greek and Latin indexes run from 0 to 3 and transform respectively under general coordinates and under local Lorentz transformations.

Within this system, we consider fermionic fields.
The covariant derivative of a Dirac spinor $D_{\mu}\psi$ acts according to the following definitions:

$$D_{\mu}\psi = \partial_{\mu}\psi - \frac{\mathrm{i}}{4}\omega_{\mu ab}\sigma^{ab}\psi \tag{4}$$

$$\overline{D_{\mu}\psi} = \partial_{u}\bar{\psi} + \frac{\mathrm{i}}{4}\bar{\psi}\omega_{\mu ab}\sigma^{ab} \tag{5}$$

Where $\sigma^{ab} = \frac{i}{2}[\gamma^{a}, \gamma^{b}]$, γ^{a} is usual (flat-space) γ-matrix and $\omega_{\mu ab}$ is the spin connection.

The spin coefficients are defined as follows:

$$\omega_{\mu ab} = e_{a\rho} \nabla_{\mu} e^{\rho}_{b} \tag{6}$$

∇_μ is the tensorial covariant derivative satisfying the metricity condition $\nabla_\mu g_{\nu\rho} = 0$

$$\omega_{\mu ab} = e_{a\rho}\nabla_\mu e_b^\rho = e_{a\rho}\left(\partial_\mu e_b^\rho + \Gamma_{\mu\lambda}^\rho e_b^\lambda\right) \tag{7}$$

The spin coefficients $\omega_{\mu ab}$ are antisymmetric in a and b,

$$\omega_{\mu ab} = -\omega_{\mu ba} \tag{8}$$

In the following calculations we use explicit antisymmetric equations (23)-(24).

We write the action for the fermionic field:

$$S\left(e_\mu^a, \psi, \bar\psi\right) = \frac{\mathrm{i}}{2}\int d^4\mathrm{x}\sqrt{g}\left(\bar\Psi\gamma^\mu D_\mu - \overline{D_\mu\Psi}\gamma^\mu + 2im\bar\Psi\right) \tag{9}$$

$\gamma^\mu = e_a^\mu\gamma^a$, and $\mathrm{d}^4x\sqrt{g}$ is the volume element.

We make explicit the covariant derivative and write the action as follows:

$$S_f = \int \mathrm{d}^4x\sqrt{g} \times \begin{bmatrix} \frac{\mathrm{i}}{2}\bar\psi\gamma^\mu\partial_\mu\psi - \frac{\mathrm{i}}{2}\partial_u\bar\psi\gamma^\mu\psi + \frac{1}{8}\bar\psi\gamma^\mu\omega_{\mu ab}\sigma^{ab}\psi + \\ \frac{1}{8}\bar\psi\sigma^{ab}\omega_{\mu ab}\gamma^\mu\psi - m\bar\psi\psi \end{bmatrix} \tag{10}$$

The terms of interaction with the spin connection are

$$\frac{1}{8}\bar\psi\gamma^\mu\omega_{\mu ab}\sigma^{ab}\psi + \frac{1}{8}\bar\psi\sigma^{ab}\omega_{\mu ab}\gamma^\mu\psi = \frac{1}{8}\bar\psi e_c^\mu\{\gamma^c, \sigma^{ab}\}\omega_{\mu ab}\psi \tag{11}$$

We define

$$T^{cab} = \{\gamma^c, \sigma^{ab}\} \tag{12}$$

The only nonvanishing elements of T^{cab} are the following,

$$T^{0ij} = \{\gamma^0, \sigma^{ij}\} = 2\varepsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} \tag{13}$$

$$T^{i0j} = \{\gamma^i, \sigma^{0j}\} = -T^{0ij} \tag{14}$$

$$T^{nij} = \{\gamma^n, \sigma^{ij}\} = 2\varepsilon_{ijn}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2\varepsilon_{ijn}\gamma^0\gamma^5 \tag{15}$$

The indices are flat indices: 0 timelike; n, i, j spacelike

4.4 Canonical momenta

We consider a global time function according to the 3+1 splitting of spacetime. Spacetime $\left(M^4, g_{\mu\nu}\right)$ is assumed to be globally hyperbolic. Topologically this means $M^4 \approx R \times \Sigma$, M^4 admits regular foliation with non-intersecting three-dimensional space-like hypersurfaces Σ_τ, τ is global time like function identifying the elements of the foliations (simultaneity surfaces).

We calculate the canonical moments according to the Hamiltonian formalism.

$$\pi_a^\alpha = \frac{\delta S_f}{\delta \partial_\tau e_a^\alpha} ; \pi_0^\alpha = \frac{\delta S_f}{\delta \partial_\tau e_0^\alpha} \tag{16}$$

For further explanations, the reader could refer to [3].

4.5 Spin rotational current

The only terms depending on the time derivative of the tetrad $\boldsymbol{\partial_\tau e_a^\alpha}$ are related to the spin connection, we isolate and analyze only this part of the action (10).

$$\frac{1}{8}\bar{\psi} e_c^\mu T^{cab} \omega_{\mu ab} \psi \tag{17}$$

Since the only non-vanishing elements of $\boldsymbol{T^{cab}}$ are $\boldsymbol{T^{0ij}}$, $\boldsymbol{T^{nij}}$, $\boldsymbol{T^{i0j}}$ we only consider contributions arising from the terms

$$\boldsymbol{\frac{1}{8}\bar{\psi} e_0^\mu \omega_{\mu ij} T^{0ij} \psi} \tag{18}$$

$$\boldsymbol{\frac{1}{8}\bar{\psi} e_i^\mu \omega_{\mu 0j} T^{i0j} \psi} \tag{19}$$

$$\boldsymbol{\frac{1}{8}\bar{\psi} e_n^\mu \omega_{\mu ij} T^{nij} \psi} \tag{20}$$

Spin coefficients are depending on tetrads, we make explicit calculations

$$\omega_{\mu ij} = e_{i\rho} \nabla_\mu e_j^\rho = e_{i\rho}\left(\partial_\mu e_j^\rho + \Gamma_{\mu\lambda}^\rho e_j^\lambda\right) \tag{21}$$

$$\boldsymbol{\omega_{\mu 0j} = e_{0\rho}\nabla_{\mu}e_{j}^{\rho} = e_{0\rho}\left(\partial_{\mu}e_{j}^{\rho} + \Gamma_{\mu\lambda}^{\rho}e_{j}^{\lambda}\right)} \quad (22)$$

The same we do for Christoffel symbols,

$$\Gamma_{\beta\gamma}^{\alpha} = \left\{ {\alpha \atop \beta\gamma} \right\} = \frac{1}{2}g^{\alpha\lambda}\left(\partial_{\beta}g_{\lambda\gamma} + \partial_{\gamma}g_{\lambda\beta} - \partial_{\lambda}g_{\beta\gamma}\right)$$

according to the equation $\boldsymbol{g_{uv} = e_{\mu}^{a}e_{va}}$ we replace the metric tensor with tetrads and look for $\boldsymbol{\partial_{\tau}e_{a}^{\alpha}}$.

Spin coefficients are antisymmetric, so we write

$$\omega_{\mu ij} = \frac{1}{2}\left(e_{i\rho}\partial_{\mu}e_{j}^{\rho} - e_{j\rho}\partial_{\mu}e_{i}^{\rho}\right) + \frac{1}{2}\Gamma_{\mu\lambda}^{\rho}\left(e_{i\rho}e_{j}^{\lambda} - e_{j\rho}e_{i}^{\lambda}\right) \quad (23)$$

$$\omega_{\mu 0j} = \frac{1}{2}\left(e_{0\rho}\partial_{\mu}e_{j}^{\rho} - e_{j\rho}\partial_{\mu}e_{0}^{\rho}\right) + \frac{1}{2}\Gamma_{\mu\lambda}^{\rho}\left(e_{0\rho}e_{j}^{\lambda} - e_{j\rho}e_{0}^{\lambda}\right)(24) \quad (24)$$

Temporal derivatives of the tetrad are manifest in these parts:

$$\omega_{\tau ij} = \frac{1}{2}\left(e_{i\rho}\partial_{\tau}e_{j}^{\rho} - e_{j\rho}\partial_{\tau}e_{i}^{\rho}\right) \quad (26)$$

$$\omega_{\tau 0j} = \frac{1}{2}\left(e_{0\rho}\partial_{\tau}e_{j}^{\rho} - e_{j\rho}\partial_{\tau}e_{0}^{\rho}\right) \quad (26)$$

The terms including time derivatives of the tetrad present in the Christoffel symbols are cancelled out, due to the antisymmetric part of the spin coefficients.

We only consider these contributions

$$\boldsymbol{\frac{1}{8}\bar{\psi}e_{0}^{\tau}\omega_{\tau ij}T^{0ij}\psi = \frac{1}{8}\bar{\psi}e_{0}^{\tau}[\frac{1}{2}\left(e_{i\rho}\partial_{\tau}e_{j}^{\rho} - e_{j\rho}\partial_{\tau}e_{i}^{\rho}\right)]T^{0ij}\psi} \quad (27)$$

$$\boldsymbol{\frac{1}{8}\bar{\psi}e_{i}^{\tau}\omega_{\tau 0j}T^{i0j}\psi = \frac{1}{8}\bar{\psi}e_{i}^{\tau}[\frac{1}{2}\left(e_{0\rho}\partial_{\tau}e_{j}^{\rho} - e_{j\rho}\partial_{\tau}e_{0}^{\rho}\right)]T^{i0j}\psi} \quad (28)$$

$$\boldsymbol{\frac{1}{8}\bar{\psi}e_{n}^{\tau}\omega_{\tau ij}T^{nij}\psi = \frac{1}{8}\bar{\psi}e_{n}^{\tau}\left[\frac{1}{2}\left(e_{i\rho}\partial_{\tau}e_{j}^{\rho} - e_{j\rho}\partial_{\tau}e_{i}^{\rho}\right)\right]T^{nij}\psi} \quad (29)$$

Finally, we obtain the conjugated momenta:

$$\boldsymbol{\pi_{a}^{\alpha} = \frac{\delta S_{f}}{\delta\partial_{\tau}e_{a}^{\alpha}} = \frac{1}{8}\bar{\psi}\left[e_{0}^{\tau}e_{i\alpha}T^{0ia} + \frac{1}{2}e_{i}^{\tau}e_{0\alpha}T^{i0a} + e_{n}^{\tau}e_{i\alpha}T^{nia}\right]\psi} \quad (30)$$

Index a is flat spacelike

$$\boldsymbol{\pi_0^\alpha = \frac{\delta S_f}{\delta \partial_\tau e_0^\alpha} = \frac{1}{8}\bar{\psi}\left[-\frac{1}{2}e_i^\tau e_{j\alpha}T^{i0j}\right]\psi} \tag{31}$$

Index 0 is flat timelike

The term (28) provides two contributions.

4.6 Torsion evidence and interactions terms

These considerations concern the analysis of interactions and the comparison with the results obtained in linearized general relativity.

We write the Dirac Lagrangian

$$L_f = \frac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{\mathrm{i}}{2}\partial_u\bar{\psi}\gamma^\mu\psi + \frac{1}{8}\bar{\psi}e_c^\mu\{\gamma^c,\sigma^{ab}\}\omega_{\mu ab}\psi - m\bar{\psi}\psi \tag{32}$$

since $\gamma^\mu = e_a^\mu\gamma^a$, (36) becomes

$$L_f = \frac{\mathrm{i}}{2}e_a^\mu\overline{(\psi}\gamma^a\partial_\mu\psi - \partial_u\bar{\psi}\gamma^a\psi) +$$

$$+\frac{1}{8}\bar{\psi}e_c^\mu\{\gamma^c,\sigma^{ab}\}\omega_{\mu ab}\psi - m\bar{\psi}\psi \tag{33}$$

We express the Dirac Lagrangian replacing equations of T_μ^a and s^{cab},
$T_\mu^a = \overline{(\psi}\gamma^a\partial_\mu\psi - \partial_u\bar{\psi}\gamma^a\psi)$ energy momentum tensor
$s^{cab} = -\frac{i}{4}\bar{\psi}\{\gamma^c,\sigma^{ab}\}\psi$ spin angular momentum tensor

$$\boldsymbol{L_f = \frac{\mathrm{i}}{2}e_a^\mu T_\mu^a + \frac{i}{2}e_c^\mu\omega_{\mu ab}s^{cab} - m\bar{\psi}\psi} \tag{34}$$

There are two terms of interaction between the gravitational field and the fermionic field.

The first interaction term $\boldsymbol{e_a^\mu T_\mu^a}$ consists of the product of the tetrad $\boldsymbol{e_a^\mu}$ with the energy-momentum tensor $\boldsymbol{T_\mu^a}$ for the Dirac field, the appearance of the tetrad field $\boldsymbol{e_a^\mu}$ is due to modified $\boldsymbol{\gamma}$-matrix being $\boldsymbol{\gamma^\mu = e_a^\mu\gamma^a}$. This interaction is also evident in linearized gravity.

The second term $\boldsymbol{e_c^\mu\omega_{\mu ab}s^{cab}}$ is related to the spin connection; in weak gravity the interaction part containing the spin connection vanishes (only to first order).

The linearized metric tensor is obtained by the flat metric plus a small perturbation $\boldsymbol{h_{\mu\nu}}$

$$\boldsymbol{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}} \tag{35}$$

In the linearized theory of general relativity, the interaction is represented by the product of the field $\boldsymbol{h_{\mu\nu}}$ with the Dirac energy- momentum tensor $\boldsymbol{T^{\mu\nu}}$.

In the non-relativistic limit this interaction has gravitoelectric and gravitomagnetic effects, both orbital and spin angular momenta couple in the same way to the gravitomagnetic field, the precession rate is universal for any angular systems [8].

Gravitomagnetism seems well explained with linearized gravity.

In this work we have analyzed in detail the spin coefficients.

Momenta $\boldsymbol{\pi_a^\alpha}$ and $\boldsymbol{\pi_0^\alpha}$ have been calculated from (11), this term disappears if we consider the linearized theory for gravity, as already said.

Canonical momenta are related to the spin rotational current; we can compare the second member of the equations (30); (31) with the canonical spin angular momentum tensor.

$$\boldsymbol{s^{abc} = -\frac{i}{4}\bar{\psi}\{\gamma^a, \sigma^{bc}\}\psi} \tag{36}$$

The results we obtained confirm the interaction between the fermionic field and the gravitational field, moreover the spinor rotational current, usually associated with torsion, is highlighted through the calculation of canonical momenta.

In this way we can explain how torsion arises dynamically, due to the presence of spinors [5].

4.7 Energy momentum tensor for spinors in linear gravity

We want to analyze the energy-momentum tensor obtained for spinors in general relativity in the linear gravity approximation.

For this study we take as a reference the theory of spinors in general relativity, the formalism of the spin connection, covariant derivatives refer to that theory.

We consider linear gravity with $g_{uv} = \eta_{\mu v} + h_{\mu v}$, here we assume $h_{\mu v}$ as a first step of perturbation from flat spacetime, recursively we apply the same approximation till to n$h_{\mu v}$ contributions.

In this process we examine the energy momentum for a Dirac particle and calculate the tensor connected to each step of perturbation from flat to n$h_{\mu v}$.

The energy-momentum tensor has recursive behavior: in particular, at each step towards a little more curved space-time, a new contribution must be added to the energy-momentum tensor calculated in the former step.

In this approximation the tensor $T^{\mu v}$ linearly depends on the metric, this may then appear to be a clue that leads to a form of quantization of the energy if we assume the metric is quantized.

The analysis continues in more detail, we see in particular that there are intakes related to the spin connection so that the full energy momentum approximation is

$$T^{n\mu v} \cong \hat{T}^{\mu v} - \frac{n}{2}(h_a^\mu \hat{T}^{av} + h_a^v \hat{T}^{a\mu}) - n\frac{i}{4}(h_a^\mu h_{c|b}^v s^{abc} + h_a^v h_{c|b}^\mu s^{abc}) \quad (37)$$

Starting from additional contributions to the energy momentum tensor we try to attribute a new physical meaning to the cosmological constant. we consider the tensor as a source of the wave equation and study different solutions according to distinctive terms of the tensor.

4.8 Spinors and Quantum Gravity

Quantum gravity is an attempt to include general relativity and quantum mechanics principles.

Here we consider spinors as the encounter point of these two descriptions. Spinors can be described in general relativity thanks to the equivalence principle. We need a flat inertial reference frame where we can describe spinors as a representation of the Lorentz group: conforming to the equivalence principle we can set up a system of inertial coordinates so that the effects of the gravitational field are canceled out, we can find shelter from gravitation and set up spinors in locally flat spacetime.

Spinors are the quintessence of quantum mechanics.

Let's consider that for each point of curved spacetime we can introduce a flat reference system, we reset the memory of the process for a moment and then we return to realize that if we have arrived in the new reference frame, we owe it to a small jump $h_{\mu v}$ from the previous flat space time. During this path what's happened to the energy momentum tensor related to fermions? We see that this one is getting the same contributions at each step.

This result considers the spinor energy momentum tensor, as we can derive from the Dirac theory, in the general relativity background. Torsion is admitted in this theory even if the affine connection is symmetric [5] [6].

We find out that there is a relation between the changing in the metric tensor and the energy momentum tensor for the spinor field, the next question is how this combination is realized.

This mechanism involves gravitational waves.

Gravitational waves in linearized gravity are developed from the Einstein field equation, the perturbation $h_{\mu v}$ propagates as a wave according to the equation:

$$\square\, h_{\mu v} = -\frac{16\pi G}{c^2} T^{\mu v} \tag{38}$$

In the following analysis, we consider the terms of the energy-momentum tensor as sources of gravitational waves, we get non-homogeneous differential equations. At first order we get a differential equation typical for a standing wave if we choose appropriate boundary conditions.

A standing string has normal modes of vibration so that the energy is quantized, here the vibration does not concern a string but concerns the metric and the normal modes are referred to different perturbations of the flat spacetime.

If we consider the spin connection contributions to the energy momentum tensor, we obtain two other types of non-homogeneous equations that have a derivative term of the metric and these equations are related to forced and damped oscillations, in the next paragraphs we show in detail how these types of oscillations can generate opposite currents and the creation of whirlpools.

In the end, the curved space-time turns out to be obtained from the sum of many approximations of the size $h_{\mu v}$, the source for the metric perturbation is the energy momentum tensor associated with the metric tensor.

The curved space-time is generated by the normal modes of vibration of the metric having their source in the energy momentum tensor $T^{\mu v}$, this is a possible way to realize the coupling between the metric field and the spinor field; the peculiar behavior of the spinor field is achieved through the interaction with the gravitational field creating opposite contributions to the metric tensor.

Curvature measurements are obtained from detection of energy perturbation; we get an example of curvature energy spectra for hyperbolic paraboloid (see Fig 1)[7].

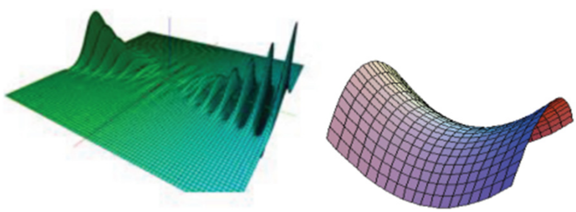

Fig. 1-1 Curvature energy spectra

4.9 Energy momentum tensor for the Dirac field in curved space time and in weak gravity

We start with the Lagrangian for the spinor field in the flat Minkowski space time.

$$L_f = \frac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{\mathrm{i}}{2}\partial_u\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi \tag{39}$$

For the calculation of the energy momentum tensor, we follow the Noether's theorem:

$$\hat{T}^{\mu v} = \frac{\partial L_f}{\partial(\partial_\mu\psi)}\partial_v\psi + \frac{\partial L_f}{\partial(\partial_u\bar{\psi})}\partial_v\bar{\psi} - \delta^\mu_v L_f \tag{40}$$

We get the energy momentum tensor in the explicit symmetric form:

$$\hat{T}^{\mu v} = \frac{1}{4}\left[\bar{\psi}\mathrm{i}\gamma^\mu\partial^\nu\psi - \overline{\partial^\nu\psi}\mathrm{i}\gamma^\mu\psi + \bar{\psi}\mathrm{i}\gamma^\nu\partial^\mu\psi - \overline{\partial^\mu\psi}i\gamma^\nu\psi\right] \tag{41}$$

Let's consider the energy-momentum tensor in curved spacetime with the covariant derivative.

$$D_v\psi \equiv \psi_{\|v} = \partial_v\psi + \Gamma_v\psi; \quad \overline{D^\nu\psi}=\partial_v\bar{\psi} - \Gamma_v\bar{\psi} \tag{42}$$

$$\Gamma_v = \frac{\mathrm{i}}{4}e_{b\mu}\nabla_v e^\mu_a\hat{\sigma}^{ab} \qquad \nabla_v e^\mu_a = \partial_v e^\mu_a + \Gamma^\mu_{v\lambda}e^\lambda_a \tag{43}$$

Christoffel symbols are so defined

$$\Gamma^\alpha_{\beta\gamma} = \left\{{\alpha \atop \beta\gamma}\right\} = \frac{1}{2}g^{\alpha\lambda}\left(\partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}\right) \tag{44}$$

According to the Noether theorem now we obtain:

$$T^{\mu v} = \frac{1}{4}[\bar{\psi}\mathrm{i}\gamma^\mu D^\nu\psi - \overline{D^\nu\psi}\mathrm{i}\gamma^\mu\psi + \bar{\psi}\mathrm{i}\gamma^\nu D^\mu\psi - \overline{D^\mu\psi}i\gamma^\nu\psi] \tag{45}$$

The tensor has the same form but covariant derivatives instead of the ordinary ones

$$\hat{T}^{\mu v} = \frac{1}{4}\left[\bar{\psi}i\gamma^\mu\partial^\nu\psi - \overline{\partial^\nu\psi}i\gamma^\mu\psi + \bar{\psi}i\gamma^\nu\partial^\mu\psi - \overline{\partial^\mu\psi}i\gamma^\nu\psi\right] \tag{46}$$

Now we consider the tensor in weak gravity approximation,

$$g_{uv} = \eta_{\mu\nu} + h_{\mu\nu} \tag{47}$$

$$\gamma^{\mu} = \hat{\gamma}^{\mu} - \tfrac{1}{2} h^{\mu}_{a} \hat{\gamma}^{a} \tag{48}$$

We make explicit calculation and we get

$$T^{\mu\nu} = \hat{T}^{\mu\nu} + \tfrac{1}{4}\Big[\bar{\psi} i \left(-\tfrac{1}{2} h^{\mu}_{a}\hat{\gamma}^{a}\right) D^{\nu}\psi - \overline{D^{\nu}\psi} i \left(-\tfrac{1}{2} h^{\mu}_{a}\hat{\gamma}^{a}\right)\psi + \bar{\psi} i(-\tfrac{1}{2} h^{\nu}_{a}\hat{\gamma}^{a}) D^{\mu}\psi - \overline{D^{\mu}\psi} i(-\tfrac{1}{2} h^{\nu}_{a}\hat{\gamma}^{a})\psi\Big] \tag{49}$$

$$T^{\mu\nu} = \hat{T}^{\mu\nu} + \tfrac{1}{4}\{-\tfrac{1}{2} h^{\mu}_{a}[\bar{\psi}\mathrm{i}(\hat{\gamma}^{a})\partial^{\nu}\psi - \overline{\partial^{\nu}\psi}\mathrm{i}(\hat{\gamma}^{a})\psi] - \tfrac{1}{2} h^{\nu}_{a}[\bar{\psi}\mathrm{i}(\hat{\gamma}^{a})\partial^{\mu}\psi - \overline{\partial^{\mu}\psi} i(\hat{\gamma}^{a})\psi]\} + \ldots . \tag{50}$$

$$T^{\mu\nu} = \hat{T}^{\mu\nu} - \tfrac{1}{2} h^{\mu}_{a}\hat{T}^{a\nu} - \tfrac{1}{2} h^{\nu}_{a}\hat{T}^{a\mu} - \tfrac{1}{2} h^{\mu}_{a}\tfrac{i}{4}[\bar{\psi}(\hat{\gamma}^{a})\Gamma^{\nu}\psi + \bar{\psi}\Gamma^{\nu}(\hat{\gamma}^{a})\psi] - \tfrac{1}{2} h^{\nu}_{a}\tfrac{i}{4}[\bar{\psi}(\hat{\gamma}^{a})\Gamma^{\mu}\psi + \bar{\psi}\Gamma^{\mu}(\hat{\gamma}^{a})\psi] \tag{51}$$

$$T^{\mu\nu} = \hat{T}^{\mu\nu} - \tfrac{1}{2} h^{\mu}_{a}\hat{T}^{a\nu} - \tfrac{1}{2} h^{\nu}_{a}\hat{T}^{a\mu} - \tfrac{1}{2} h^{\mu}_{a}\tfrac{i}{2}[\bar{\psi}\{\hat{\gamma}^{a},\Gamma^{\nu}\}\psi] - \tfrac{1}{2} h^{\nu}_{a}\tfrac{i}{2}[\bar{\psi}\{\hat{\gamma}^{a},\Gamma^{\mu}\}\psi] \tag{52}$$

$$\Gamma^{\nu} \cong \tfrac{\mathrm{i}}{4} h^{\nu}_{c|b}\hat{\sigma}^{bc} \quad ; \Gamma^{\mu} \cong \tfrac{\mathrm{i}}{4} h^{\mu}_{c|b}\hat{\sigma}^{bc} \tag{53}$$

$$T^{\mu\nu} = \hat{T}^{\mu\nu} - \tfrac{1}{2} h^{\mu}_{a}\hat{T}^{a\nu} - \tfrac{1}{2} h^{\nu}_{a}\hat{T}^{a\mu} + \tfrac{i}{4} h^{\mu}_{a} h^{\nu}_{c|b}\tfrac{i}{4}[\bar{\psi}\{\hat{\gamma}^{a},\hat{\sigma}^{bc}\}\psi] + \tfrac{i}{4} h^{\nu}_{a} h^{\mu}_{c|b}\tfrac{i}{4}[\bar{\psi}\{\hat{\gamma}^{a},\hat{\sigma}^{bc}\}\psi] \tag{54}$$

$$s^{abc} = -\tfrac{i}{4}[\bar{\psi}\{\hat{\gamma}^{a},\hat{\sigma}^{bc}\}\psi] \tag{55}$$

spin angular momentum tensor

$$T^{\mu\nu} \cong \hat{T}^{\mu\nu} - \tfrac{1}{2} h^{\mu}_{a}\hat{T}^{a\nu} - \tfrac{1}{2} h^{\nu}_{a}\hat{T}^{a\mu} - \tfrac{i}{4} h^{\mu}_{a} h^{\nu}_{c|b} s^{abc} - \tfrac{i}{4} h^{\nu}_{a} h^{\mu}_{c|b} s^{abc} \tag{56}$$

If we ignore the spin contribution, we can write the energy momentum tensor approximated as

$$T^{\mu\nu} \cong \hat{T}^{\mu\nu} - \tfrac{1}{2} h^{\mu}_{a}\hat{T}^{a\nu} - \tfrac{1}{2} h^{\nu}_{a}\hat{T}^{a\mu} \tag{57}$$

So, if we start with a flat space-time and then we introduce a little deformation in the metric the energy momentum tensor is changing of a factor $-N^{\mu v}$ each time.

$$N^{\mu v} = \frac{1}{2}(h^{\mu}_{a}\hat{T}^{av} + h^{v}_{a}\hat{T}^{a\mu}) \tag{58}$$

$N^{\mu v}$ is symmetric for μv

$$T^{\mu v} \cong \hat{T}^{\mu v} - N^{\mu v} \tag{59}$$

We approximate the metric, each time putting ourselves in a new reference frame, starting from the flat we proceed with discrete path of the size $h_{\mu v}$ to a little more curved spacetime. Fig 2.

The result at the first step is the following equation:

$$T'^{\mu v} \cong \hat{T}^{\mu v} - \frac{1}{2}h^{\mu}_{a}\hat{T}^{av} - \frac{1}{2}h^{v}_{a}\hat{T}^{a\mu} \tag{60}$$

Second step

$$T''^{\mu v} \cong T'^{\mu v} - \frac{1}{2}h^{\mu}_{a}T'^{av} - \frac{1}{2}h^{v}_{a}T'^{a\mu} \cong \hat{T}^{\mu v} - h^{\mu}_{a}\hat{T}^{av} - h^{v}_{a}\hat{T}^{a\mu} \tag{61}$$

Third step

$$T'''^{\mu v} \cong \hat{T}^{\mu v} - \frac{3}{2}h^{\mu}_{a}\hat{T}^{av} - \frac{3}{2}h^{v}_{a}\hat{T}^{a\mu} \tag{62}$$

And so on for n- times,

$$T^{n\mu v} \cong \hat{T}^{\mu v} - \frac{n}{2}h^{\mu}_{a}\hat{T}^{av} - \frac{n}{2}h^{v}_{a}\hat{T}^{a\mu} \tag{63}$$

here n is not a world index, it only indicates the n-steps of the path.

At each step we sum up the effect of the metric approximation with the factor

$-N^{\mu v} = -\frac{1}{2}h^{\mu}_{a}\hat{T}^{av} - \frac{1}{2}h^{v}_{a}\hat{T}^{a\mu}$ inherited from the previous energy-momentum tensor.

We have completely ignored second order contributions.

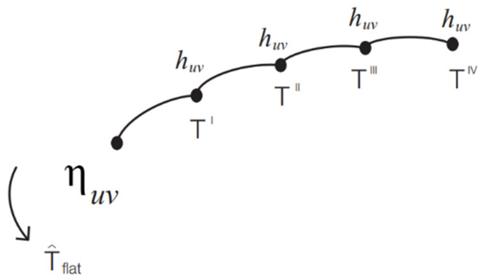

Fig. 1-2 Iteration sequence to approximate the energy-momentum tensor through discrete paths of size $h_{\mu v}$ up to slightly more curved space-time.

Now if we include the spinor angular momentum, we have to consider the tensor

$$T^{n\mu v} \cong \hat{T}^{\mu v} - \frac{n}{2}(h_a^\mu \hat{T}^{av} + h_a^v \hat{T}^{a\mu}) + n\frac{i}{4}h_a^\mu h_{c|b}^\nu \frac{i}{4}[\bar{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] + n\frac{i}{4}h_a^\nu h_{c|b}^\mu \frac{i}{4}[\bar{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] \tag{64}$$

$$T^{n\mu v} \cong \hat{T}^{\mu v} - \frac{n}{2}(h_a^\mu \hat{T}^{av} + h_a^v \hat{T}^{a\mu}) - n\frac{i}{4}(h_a^\mu h_{c|b}^\nu s^{abc} + h_a^\nu h_{c|b}^\mu s^{abc}) \tag{65}$$

First step with spinor:

$$T'^{\mu v} = \hat{T}^{\mu v} - \frac{1}{2}h_a^\mu \hat{T}^{av} - \frac{1}{2}h_a^v \hat{T}^{a\mu} + \frac{i}{4}h_a^\mu h_{c|b}^\nu \frac{i}{4}[\bar{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] + \frac{i}{4}h_a^\nu h_{c|b}^\mu \frac{i}{4}[\bar{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] \tag{66}$$

$$T''^{\mu v} = T'^{\mu v} - \frac{1}{2}h_a^\mu T'^{av} - \frac{1}{2}h_a^v T'^{a\mu} + \frac{i}{4}h_a^\mu h_{c|b}^\nu \frac{i}{4}[\bar{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] + \frac{i}{4}h_a^\nu h_{c|b}^\mu \frac{i}{4}[\bar{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] \tag{67}$$

$$T''^{\mu v} = \hat{T}^{\mu v} - \frac{1}{2}h_a^\mu \hat{T}^{av} - \frac{1}{2}h_a^v \hat{T}^{a\mu} + \frac{i}{4}h_a^\mu h_{c|b}^\nu \frac{i}{4}[\bar{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] + \frac{i}{4}h_a^\nu h_{c|b}^\mu \frac{i}{4}[\bar{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] \tag{68}$$

$$-\frac{1}{2}h_a^\mu\left(\hat{T}^{av} - \frac{1}{2}h_b^a \hat{T}^{bv} - \frac{1}{2}h_b^v \hat{T}^{ba} + \frac{i}{4}h_k^a h_{c|b}^\nu \frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi] + \frac{i}{4}h_k^\nu h_{c|b}^a \frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi]\right)$$

$-\frac{1}{2}h_a^\nu\left(\hat{T}^{a\mu}-\frac{1}{2}h_b^\mu\hat{T}^{b\mu}-\frac{1}{2}h_b^\mu\hat{T}^{ba}+\frac{i}{4}h_k^a h_{c|b}^\mu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^k,\hat{\sigma}^{bc}\}\psi]+\right.$
$\left.\frac{i}{4}h_k^\mu h_{c|b}^a\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^k,\hat{\sigma}^{bc}\}\psi]\right)$

$+\frac{i}{4}h_a^\mu h_{c|b}^\nu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}\psi]+\frac{i}{4}h_a^\nu h_{c|b}^\mu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}\psi]$

$T'^{a\nu}=\hat{T}^{a\nu}-\frac{1}{2}h_b^a\hat{T}^{b\nu}-\frac{1}{2}h_b^\nu\hat{T}^{ba}+\frac{i}{4}h_k^a h_{c|b}^\nu\frac{i}{4}[\bar{\psi}\{\hat{\gamma}^k,\hat{\sigma}^{bc}\}\psi]+$
$\frac{i}{4}h_k^\nu h_{c|b}^a\frac{i}{4}[\bar{\psi}\{\hat{\gamma}^k,\hat{\sigma}^{bc}\}\psi]$ (69)

$T'^{a\mu}=\hat{T}^{a\mu}-\frac{1}{2}h_b^\mu\hat{T}^{b\mu}-\frac{1}{2}h_b^\mu\hat{T}^{ba}+\frac{i}{4}h_k^a h_{c|b}^\mu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^k,\hat{\sigma}^{bc}\}\psi]+$
$\frac{i}{4}h_k^\mu h_{c|b}^a\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^k,\hat{\sigma}^{bc}\}\psi]$ (70)

We don't consider second order contribution like $h_a^\nu h_b^\mu$ but include $h_a^\mu h_{c|b}^\nu$

$$71a.\,T''^{\mu\nu}\cong\hat{T}^{\mu\nu}-\frac{1}{2}h_a^\mu\hat{T}^{a\nu}-\frac{1}{2}h_a^\nu\hat{T}^{a\mu}+\frac{i}{4}h_a^\mu h_{c|b}^\nu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}\psi]+\frac{i}{4}h_a^\nu h_{c|b}^\mu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}\psi]$$

$$-\frac{1}{2}h_a^\mu\left(\hat{T}^{a\nu}+\ldots\right)-\frac{1}{2}h_a^\nu\left(\hat{T}^{a\mu}+\ldots\right)+\frac{i}{4}h_a^\mu h_{c|b}^\nu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}\psi]+\frac{i}{4}h_a^\nu h_{c|b}^\mu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}\psi]$$

$$71b.\,T''^{\mu\nu}\cong\hat{T}^{\mu\nu}-h_a^\mu\hat{T}^{a\nu}-h_a^\nu\hat{T}^{a\mu}+2\frac{i}{4}h_a^\mu h_{c|b}^\nu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}\psi]+2\frac{i}{4}h_a^\nu h_{c|b}^\mu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}\psi]$$

At the moment we don't consider these kinds of contribution

$$-\frac{1}{2}h_a^\mu(-\frac{1}{2}h_b^\mu\hat{T}^{b\nu}+\frac{i}{4}h_k^a h_{c|b}^\nu\frac{\mathrm{i}}{4}[\bar{\psi}\{\hat{\gamma}^k,\hat{\sigma}^{bc}\}\psi]) \quad (72)$$

because they are second order or more, so we write:

$$71c.\,T''^{\mu\nu}\cong\hat{T}^{\mu\nu}-(h_a^\mu\hat{T}^{a\nu}+h_a^\nu\hat{T}^{a\mu})-2\frac{i}{4}(h_a^\mu h_{c|b}^\nu s^{abc}+h_a^\nu h_{c|b}^\mu s^{abc})$$

We can see from equation (65) that the spinor contribution $\frac{i}{4}h_a^\mu h_{c|b}^\nu\frac{i}{4}[\bar{\psi}\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}\psi]$ is different from zero only if the derivative $h_{c|b}^\nu\neq$

0, that to say the coupling between the gravitational field h_a^μ and torsion is effective only if the deformation of the metric is not constant.
$h_{c|b}^{\nu} = h_{c|b}^{v}\partial_b h_c^\nu \neq 0$ (73)

4.10 Energy-momentum tensor depending on discrete steps in the metric

The main results we get is this equation describing the energy momentum tensor related to the changing in the metric and to torsion

$$T^{n\mu v} \cong \hat{T}^{\mu v} - \frac{n}{2}(h_a^\mu \hat{T}^{av} + h_a^v \hat{T}^{a\mu}) + n\frac{i}{4}(h_a^\mu h_{c|b}^\nu + h_a^\nu h_{c|b}^\mu)s^{abc} \quad (74)$$

We write the equation in a more compact way

$$T^{n\mu v} \cong \hat{T}^{\mu v} - nN^{\mu v} + n\ \frac{i}{4}B^{uv} \quad (75)$$

The first term $\hat{T}^{\mu v}$ is about Dirac particle in flat spacetime.

$$N^{\mu v} = \frac{1}{2}\left(h_a^\mu \hat{T}^{av} + h_a^v \hat{T}^{a\mu}\right) \quad (76)$$

This term represents the coupling between the deformed gravitational space and the Dirac particle, it's symmetric.

The contribution to the energy momentum tensor is negatively increasing at each step, as if spinors could lose energy in the coupling with the gravitational field.

$$B^{uv} = (h_a^\mu h_{c|b}^\nu s^{abc} + h_a^\nu h_{c|b}^\mu s^{abc}) \quad (77)$$

B^{uv} is symmetric for the world indices, but it contains the spinor angular momentum antisymmetric for the flat indices of the local reference frame. Here we have the coupling between the gravitational field and the spinor angular momentum through the derivative $h_{c|b}^\nu$.

It's apparently negative. Let's check $\frac{i}{4}s^{abc}$

Here we have to deal with $M^{abc} = \{\hat{\gamma}^a, \hat{\sigma}^{bc}\}$ and specify non vanishing terms.

We consider Dirac matrices so defined:

$$\hat{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};\ \hat{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad (78)$$

$$\hat{\sigma}^{ij}=\frac{i}{2}[\hat{\gamma}^i,\hat{\gamma}^j]=\varepsilon_{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \tag{79}$$

The only non-vanishing terms of $M^{abc}=\{\hat{\gamma}^a,\hat{\sigma}^{bc}\}$ are

$$M^{0bc}=2\varepsilon_{bck}\begin{pmatrix}\sigma^k & 0\\ 0 & -\sigma^k\end{pmatrix} \tag{80}$$

index 0 time like; bck flat spacelike

$$M^{b0c}=-M^{0bc}=-2\varepsilon_{bck}\begin{pmatrix}\sigma^k & 0\\ 0 & -\sigma^k\end{pmatrix} \tag{81}$$

$$M^{kbc}=2\varepsilon_{bck}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} \tag{82}$$

So, if we make explicit $-n\,\frac{i}{4}B^{uv}=-n\,\frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|b}+h^{\nu}_{a}h^{\mu}_{c|b})s^{abc}$ (83)

we get three kinds of contributions:

84.I) $-n\frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|b}+h^{\nu}_{a}h^{\mu}_{c|b})s^{abc}=-n\frac{i}{4}(h^{\mu}_{0}h^{\nu}_{c|b}+h^{\nu}_{0}h^{\mu}_{c|b})s^{0bc}=$

$+n\frac{i}{4}(h^{\mu}_{0}h^{\nu}_{c|b}+h^{\nu}_{0}h^{\mu}_{c|b})\frac{\mathrm{i}}{4}[\bar{\psi}M^{0bc}\psi]=$

$$=-\frac{n}{8}(h^{\mu}_{0}h^{\nu}_{c|b}+h^{\nu}_{0}h^{\mu}_{c|b})\varepsilon_{bc\kappa}\bar{\psi}\begin{pmatrix}\sigma^{\kappa} & 0\\ 0 & -\sigma^{\kappa}\end{pmatrix}\psi$$

84.II) $-n\frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|b}+h^{\nu}_{a}h^{\mu}_{c|b})s^{abc}=-n\frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|0}+$

$h^{\nu}_{a}h^{\mu}_{c|0})s^{a0c}=+n\frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|0}+h^{\nu}_{a}h^{\mu}_{c|0})\frac{\mathrm{i}}{4}[\bar{\psi}M^{a0c}\psi]=$

$$=+\frac{n}{8}(h^{\mu}_{a}h^{\nu}_{c|0}+h^{\nu}_{a}h^{\mu}_{c|0})\varepsilon_{ac\kappa}\bar{\psi}\begin{pmatrix}\sigma^{\kappa} & 0\\ 0 & -\sigma^{\kappa}\end{pmatrix}\psi$$

here index a is flat spacelike.

84.III) $-n\frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|b}+h^{\nu}_{a}h^{\mu}_{c|b})s^{abc}=-n\frac{i}{4}(h^{\mu}_{k}h^{\nu}_{c|b}+h^{\nu}_{k}h^{\mu}_{c|b})s^{kbc}=$

$=+n\frac{i}{4}(h^{\mu}_{k}h^{\nu}_{c|b}+h^{\nu}_{k}h^{\mu}_{c|b})\frac{\mathrm{i}}{4}[\bar{\psi}M^{kbc}\psi]=-\frac{n}{8}(h^{\mu}_{k}h^{\nu}_{c|b}+$
$h^{\nu}_{k}h^{\mu}_{c|b})\varepsilon_{bc\kappa}\bar{\psi}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\psi$

The energy momentum tensor has a recursive form and for each successive approximation we can detect a negative factor associated with the energy momentum tensor for the Dirac field in flat spacetime, this contribution is multiplied by n at each step.

In addition, we get positive and negative contributions from spinor angular momentum, we mean this could be associated to expansion and compression or growing and damping deformation of the metric.

Spinor as source of gravitational waves loses energy in creating curvature and torsion of the metric.

4.11 Cosmological constant reinterpreted

We suggest that the terms calculated explicitly in the energy momentum tensor could be a declaration about the cosmological constant and dark energy: spinor and torsion enter in the mechanism of expansion.

We consider the field equation with the cosmological constant

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^2}\hat{T}^{\mu\nu} \tag{85}$$

Now if we start with the field equation without cosmological constant and develop the energy momentum tensor we get

$$G^{\mu\nu} = -\frac{8\pi G}{c^2}T^{\mu\nu} \tag{86}$$

$$G^{\mu\nu} = -\frac{8\pi G}{c^2}(\hat{T}^{\mu\nu} - N^{\mu\nu} + \frac{i}{4}B^{u\nu}) \tag{87}$$

$$G^{\mu\nu} + \frac{8\pi G}{c^2}(-N^{\mu\nu} + \frac{i}{4}B^{u\nu}) = -\frac{8\pi G}{c^2}\hat{T}^{\mu\nu} \tag{88}$$

$$\Lambda g^{\mu\nu} = \frac{8\pi G}{c^2}(-N^{\mu\nu} + \frac{i}{4}B^{u\nu}) \tag{89}$$

$$\Lambda(\eta^{\mu\nu} + h^{\mu\nu}) = \frac{8\pi G}{c^2}[-\frac{1}{2}(h^{\mu}_{a}\hat{T}^{a\nu} + h^{\nu}_{a}\hat{T}^{a\mu}) + \frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|b} + h^{\nu}_{a}h^{\mu}_{c|b})s^{abc}] \tag{90}$$

The cosmological constant is positive and creates acceleration in the expansion of the universe, the value is associated to vacuum energy or to dark energy. There is a problem with the interpretation of the cosmological constant as vacuum energy: the value of the constant calculated from quantum field theory is too big and would produce acceleration too high compared with measurements [4].

In this frame the cosmological constant is associated with spinor coupling to the gravitational field and to torsion interaction.

Torsion is implicated in the expansion of the universe and could replace the role of dark energy to explain repulsive force contrasting gravity [9].

The cosmological constant could assume different value for each step of expansion

$$\Lambda g^{\mu\nu} = \frac{8\pi G}{c^2}(-nN^{\mu\nu} + n\ \frac{i}{4}B^{uv}) \tag{91}$$

here we have attractive decelerating term (minus sign) and a mixed term of torsion with positive (repulsive) and negative contribution, creating different acceleration of the expansion.

4.12 Gravitational waves sources

We consider different contributions as wave source and look forward to the solutions as a superpositions of the two effects.

$$\square h^{\mu\nu} = -\frac{16\pi G}{c^2}[\hat{T}^{\mu\nu} - \frac{1}{2}(h^{\mu}_{a}\hat{T}^{av} + h^{\nu}_{a}\hat{T}^{a\mu}) + \frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|b} + h^{\nu}_{a}h^{\mu}_{c|b})s^{abc}] \tag{92}$$

$$\square h^{\mu\nu} = \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{93}$$

$$\square h^{\mu\nu} = -\frac{16\pi G}{c^2}[-\frac{1}{2}\left(h^{\mu}_{a}\hat{T}^{av} + h^{\nu}_{a}\hat{T}^{a\mu}\right)] \tag{94}$$

This is a non-homogeneous wave equation like the elastic string wave equation- describing tension in a fixed string due to an external force:

$h^{\mu\nu}$ could be interpreted as the tension in the string and the non-homogeneous term $-\frac{1}{2}\left(h^{\mu}_{a}\hat{T}^{av} + h^{\nu}_{a}\hat{T}^{a\mu}\right)$ is assumed to be the force acting on the string forcing vibrations.

$$\square h^{\mu\nu} = -\frac{16\pi G}{c^2}[\frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|b} + h^{\nu}_{a}h^{\mu}_{c|b})s^{abc}] \tag{95}$$

Here we have three terms.

$$\square h^{\mu\nu} = -C\frac{1}{8}(h_0^{\mu}h_{c|b}^{\nu} + h_0^{\nu}h_{c|b}^{\mu})\varepsilon_{bc\kappa}\bar{\psi}\begin{pmatrix}\sigma^{\kappa} & 0\\ 0 & -\sigma^{\kappa}\end{pmatrix}\psi; \quad (96)$$

$$-\frac{16\pi G}{c^2} = C$$

cbk are flat spacelike indices

$$\square h^{\mu\nu} = +C\frac{1}{8}(h_a^{\mu}h_{c|0}^{\nu} + h_a^{\nu}h_{c|0}^{\mu})\varepsilon_{ac\kappa}\bar{\psi}\begin{pmatrix}\sigma^{\kappa} & 0\\ 0 & -\sigma^{\kappa}\end{pmatrix}\psi \quad 97)$$

0 is flat time like index

$$\square h^{\mu\nu} = -C\frac{n}{8}(h_k^{\mu}h_{c|b}^{\nu} + h_k^{\nu}h_{c|b}^{\mu})\varepsilon_{bc\kappa}\bar{\psi}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\psi \quad (98)$$

We notice that the nonhomogeneous terms contain first order partial derivatives, like for heat equation covered by vacuum wave equation. These first order terms are associated to damping or growth phenomena, depending on the plus or minus sign; there are opposite currents creating whirlpool, torsion from spinor becomes torsion of the metric.

So, the picture we can figure out from (94) is about background stationary waves due to spinor fields related to discrete expansion. In addition to this, equation (95) describes the torsion effects arising when spinors are present in curved spacetime. Stationary wave plus whirlpool are the effects of the deformation of the metric due to the presence of spinors.

4.13. Planck scale considerations and early universe

In this investigation we have assumed the approximation of linear gravity with progressive steps till to $nh_{\mu v}$ so that $g_{uv} = \eta_{\mu v} + nh_{\mu v}$.

For each perturbation of the size $h_{\mu v}$ the energy momentum tensor is so written

$$T^{n\mu v} \cong \hat{T}^{\mu v} + n\left[-\frac{1}{2}(h_a^{\mu}\hat{T}^{av} + h_a^{v}\hat{T}^{a\mu}) + \frac{i}{4}(h_a^{\mu}h_{c|b}^{\nu} + h_a^{\nu}h_{c|b}^{\mu})s^{abc}\right] \quad (61)$$

This is a model of discrete expansion driven through gravitational waves originating from spinors.

But how does this mechanism start? We could suppose vacuum fluctuation for the first deformation $h_{\mu v}$,spinors coupled to gravitational field assume new energy contributions and the presence of spinors deforms the metric.

How are the discrete levels of the metric connected? Gravitational wave originates from spinors having their source in the energy momentum tensor, the tensor changes modifying the gravitational field with the perturbation of the size $h_{\mu v}$.

In this model we are not fixing the size of $h_{\mu v}$, could it be at Planck scale? Let's calculate what happens with $\mathrm{l_p} = \sqrt{\frac{\hbar \mathrm{G}}{\mathrm{c}^3}}$, if we consider the size of the space deformation of $h_{\mu v}$ of the order of l_p, we are describing the universe at the Planck time $t_p \cong 10^{-43}s$, according with the cosmological picture of the early beginnings of the Universe.

What is the fuel to expand to different levels? The fuel is the energy tensor of the previous level, this is the source for the wave deforming the metric curvature.

If we consider the discrete steps used for this model of expansion, we can distinguish the contribution of two kinds of waves (56) and (57).

We get two different phenomenology according to the solutions of the differential equations: the former source contribution $h_a^\mu \hat{T}^{av} + h_a^v \hat{T}^{a\mu}$ allows gravitational waves creating a new size of the metric as expansion.

This torsional source $\frac{i}{4}(h_a^\mu h_{c|b}^v + h_a^v h_{c|b}^\mu)s^{abc}$, contributes to other metric changesets related to vortices creation through growing and damping metric perturbation.

The spiral shape of galaxies could be a good example of the effects of this mechanism on a huge scale.

We propose a different interpretation of the cosmological constant, expansion is due to the additional term of the energy-momentum tensor.

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