Modeling Problem Solving in Physics with Simulated Experiments

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Engaging Lab Activities

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TABLE OF CONTENTS

CHAPTER 1

STRATEGIC APPROACHES TO PROBLEM SOLVING AND REASONING IN PHYSICS

1.1 Physics Knowledge as a Catalyst for Problem-Solving Skills

Problem-solving plays a significant role in physics instruction and learning. Students solve problems by illustrating concepts and principles, demonstrating procedures, and addressing points of potential confusion. Coursework typically includes homework problems to help students explore ideas and practice skills. Physics course exams consist of problems assessing students' understanding and ability to apply knowledge in new situations.

Research shows that problem-solving in physics, as employed, does not achieve its intended aims (Leak et al., 2017). Realizing the reasons requires a brief detour to present a contemporary perspective on physics expertise, learning, and critical thinking skills grounded in cognition and research in physics education.

Research on becoming experts in problem-solving in physics helps us understand the skills and knowledge we want students to gain. Modern educational theory is based on the epistemology of constructivism: in essence, it is based on the recognition that knowledge, as opposed to information, cannot be transmitted or observed but must be assembled as the result of cognitive processes within the human mind (von Glasersfeld, 1998). According to constructivism, learning has the following assumptions:

- The construction of knowledge requires purposeful and effortful activity.
- Prior knowledge must be linked with new ideas.
- An initial understanding of a concept must enable its application.

Following these assumptions, a successful analytical person must possess a solid physics knowledge foundation developed by discovering innovative ideas in a learning environment that supports the knowledge construction process. This process must account for several factors. Students often enter physics classrooms with pre-existing worldviews, frequently shaped by misconceptions. These existing perspectives will interact with new learning prompts derived from observations and experiences. Such prompts should allow students to revise and restructure their worldviews, enabling the practical application of knowledge.

Thus, the goal of developing students' robust critical thinking skills is to have students engage in cognitive activities that offer opportunities to construct knowledge while mastering skills. Following Meijer et al. (2006), students acquire skills to solve problems if they are provided with opportunities to proceed through the following four phases:

- Conceptual analysis (orienting, exploring)
- Strategic analysis (planning, choosing)
- Quantitative analysis (executing, determining, answering)
- Meta-analysis (reflecting, checking, challenging, relating).

Experts in problem-solving physics possess these strategies; however, most physics students do not have these inherent strategies; instead, they focus on identifying a formula and concluding the answer (Miller, 2023). However, according to Meijer et al. (2006), solving problems in physics is not about obtaining an eventual answer. Problem-solving should allow students to hypothesize, observe, explore, and reconstruct the problems' scenarios using various representations that would enable their solutions. Students must be provided with opportunities to acquire these skills. The depth and understanding of new knowledge affect students' engagement in problem-solving or lab activities, which transcend the quality of skills they gain from these activities.

Problem-solving and lab activities should intertwine in a coherent learning experience that enriches the knowledge construction and skills of applying the knowledge to solve problems. If problem-solving is considered a skill, it necessitates a well-defined strategy.

This book is to address these recommendations. The emphasis will be placed on developing students' skills to make them applicable to understanding natural phenomena using qualitative and quantitative reasoning. This will be perceived as the first step toward physics knowledge construction. The dual reasoning will have its merit:

- Qualitative reasoning will be developed by having students hypothesize the outcomes of experiments and providing opportunities to verify their predictions.
- Quantitative reasoning will be developed by enhancing covariational reasoning and by viewing formulas as dynamic functions that describe experimental processes and phases of change.

Students will apply covariational reasoning intertwined with exploring phenomena behaviors. In the following section, the underpinnings of covariational reasoning are discussed.

1.2 Understanding Covariations Explored in the Laboratory Activities

Covariational reasoning is a foundational way to think mathematically. The behaviors of various algebraic functions support covariations of two quantities. In the proposed lab activities, students will identify covariate parameters and explore their behaviors while rediscovering physics laws and principles. Covariational reasoning will be employed by:

- Identifying covariate parameters during experiments and formulating their mathematical relationships.
- Examining the structures of physics formulas and categorizing their algebraic forms by analyzing how variable parameters covary.

These enterprises will offer students opportunities to transfer their math knowledge into tangible experiences in the sciences.

Physics formulas are built on algebraic rules that can be categorized. The proposed categorization of covariations primarily reflects on the underlying algebraic structures and scientific interpretations (Sokolowski, 2021). Five categories of covariate reasoning will be explored during the lab activities, with the potential for further enrichment based on more detailed research.

1.2.1 Covariations with Constant Rates of Change

Quantities with a constant rate of change are modeled by linear functions, expressed as $f(x) = mx + b$, where m represents the rate of change of the quantity and b is the dependent quantity's initial value. Physics provides numerous opportunities to apply linear functions in practical contexts. For example:

- $F = mg$; the force of gravity acting on an object.
- $v = v_1 + at$; velocity of an object moving with a constant acceleration.
- $F = -kx$; the force exerted by a stretched spring. If the equation is given as $F = kx$, it describes only the magnitude of the force.

The formulas can be expressed using the function notation to highlight the quantity that changes. For example:

- $F(m) = mg$: The force of gravity expressed as a function of the object's mass, where g is the intensity of the gravitational field.
- $v(t) = v_1 + at$: The speed of an object moving with a constant acceleration expressed as a function of time.
- $F(x) = kx$: The magnitude of the spring force expressed as a function of the spring stretch, where k is the spring constant, and x is the spring stretch. Note that x can also represent the spring compression.

In all these formulas, the dependent parameters are placed on the left side. The independent parameters, enclosed in parentheses and denoted using function notation next to the dependent parameters, highlight their variable nature and distinguish them from constant parameters.

An interesting discussion arises when exploring the effects of other parameters embedded within each formula.

From a mathematical standpoint, in the equation $F(m) = mg$, the parameter g, representing the gravitational field constant, can be called a proportionality constant or the slope of the function $F(m) = mg$. From a scientific perspective, g serves as a mediating parameter, providing valuable insight into the relationship between mass and gravitational force. The gravitational field constant g is independent of the object's mass; however, it influences the object's weight or the force of gravity acting upon it. Notably, the object's weight depends on g, not vice versa.

Viewed through the lens of the proposed categorization, the constant quantities in $F(m) = mg$ and $F(x) = kx$, namely, g, and k—can be classified as mediating parameters.

- The gravitational constant g describes the external gravitational field affecting the force of gravity on an object within that field, and it is independent of the object's mass.
- Spring constant k defines the property of the spring (the medium). The same spring extension requires different force magnitudes de-

pending on the value of the spring constant, demonstrating its role as a mediating parameter.

In each case, these mediating parameters act as *proportionality constants* influencing the system's behavior.

Furthermore, in algebra, the general form of a linear function is typically expressed as $f(x) = mx + b,$ where m represents the slope. However, in physics, there is no strict convention for the order of inserting the factors in the equation. For example, in $F = mg$, the slope g appears as the second factor, whereas in $F(x) = kx$, the slope$-k$ appears first. While this lack of consistency does not significantly hinder interpretations of these formulas, scientific justification is often required to classify them correctly.

Formulas can involve more than one parameter, as in $F_B = \sigma V g$. A general interpretation of such formulas considers one parameter as the mediating parameter—in this case, g—while the density of the fluid σ and the displaced volume of the fluid V are treated as independent parameters. This classification depends on the specific experimental setup or the problem to be solved. Additionally, either σ or V can be considered the mediating parameter, with the other being treated as a variable. In such cases, the quantities of interest increase in magnitude as the independent parameters increase, and these changes are proportional to the mediating parameters. While the slope in each formula represents the mediating parameters, all these parameters exist independently of external actions. However, these external actions can be used to mathematically visualize and quantify the mediators.

Providing students with such analyses offers new insights into the underlying nature of physics formulas and their mathematical interpretations, enabling them to interpret physical phenomena more accurately.

1.2.2 Exploring Flexibility in Interpreting Covariations Built as Quotients

Many formulas in physics are assembled as quotients resembling rational expressions. Examples of such covariations include:

- $a = \frac{Gm}{d^2}$; the intensity of the gravitational field produced by any object.
- $R = \frac{\sigma l}{A}$; the resistance of a wire.

When expressed using function notation, these formulas take on clearer and more precise interpretations. For example, $a(d) = \frac{Gm}{d^2}$, or $R(A) = \frac{\sigma l}{A}$. If the independent parameter takes numerical values, the effects on the dependent parameter can be assessed by evaluating it for increasing values of the independent parameter and justifying their mutual behavior.

It is evident that in both formulas, the dependent parameters decrease as the independent variable increases. Can one claim that such formulas are always decreasing? Not necessarily.

It is important to recognize that formulas involving both a numerator and a denominator can represent different types of proportional relationships. For example:

- The function $a(m) = \frac{Gm}{d^2}$ is a linear because d and G are assumed constant parameters in this notation. More specifically $\frac{G}{d^2}$ represents the function's rate of change.
- The function $a(d) = \frac{Gm}{d^2}$ is a rational function. As d increases, a decreases.
- The function $a(G) = \frac{Gm}{d^2}$ is a linear function because it shows that G is the only variable in the expression, while m and d are treated as constants. This means that any hypothetical change in G would directly change the value of a. However, since G is a fundamental constant, $a(G) = \frac{Gm}{d^2}$ is a constant function with a slope of zero.

All three variations of the formula $a = \frac{Gm}{d^2}$ highlight different aspects of the gravitational field produced by a mass m. Can the universal gravitational constant $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$ be considered a mediator of the phenomenon? This is an open question. While G describes the field intensity and makes the formula universally applicable, it does not mediate the gravitational interaction. Instead, G serves as a proportionality constant in Newton's law of universal gravitation. To conceptualize the value of G, consider the force of attraction between two 1kg masses placed 1meter apart: the force is $6.67 \times 10^{-11} N$ and it illustrates the extremely weak nature of gravity compared to other fundamental forces.

The other formula $R = \frac{\sigma A}{l}$ describes the electrical resistance of a conductor in terms of its geometric parameters and material resistivity. Several interpretations of this formula can emerge:

- $R(l) = \frac{\sigma l}{A}$. This formula resembles a linear function, which indicates that the resistance increases directly as the length of the wire increases. The ratio $\frac{\sigma}{A}$ is the slope of the linear function or the proportionality constant.
- $R(A) = \frac{\sigma l}{A}$. This formula represents an inverse linear covariation because the resistance R decreases as the area increases. The product σl is the proportionality constant in this equation.
- $R(\sigma) = \frac{\sigma l}{A}$. This formula resembles a linear function, indicating that as the resistivity increases, the resistance of the wire also increases. The ratio $\frac{l}{A}$ is the slope of the linear function and its proportionality constant.

Discussing these formulas and their interpretations is a worthwhile endeavor, as mathematics traditionally defines rational expressions as those containing a variable in the denominator. The concept of rational functions allows for the exploration of how quantities behave in relation to one another, especially as one quantity becomes extremely large or small. The notions of linear and rational relationships in representing proportional quantities will be discussed in greater detail in Section 1.3.

1.2.3 The Relationship of Covariation in Parametric Functions

Linking the variation of two dependent parameters can also lead to new insights into a physical phenomenon. A good example is two-dimensional motion. While the causes of motion in the vertical direction do not affect the motion in the horizontal direction, linking, for example, vertical and horizontal positions, thus $x(t)$ and $y(t)$ reveals the object's path of motion. This connection is possible for two main reasons:

- The independent parameter, t, is shared by both functions $x(t)$ and $y(t)$, and
- Evaluation of both functions $x(t)$ and $y(t)$ produces the pair (x, y) and can lead to formulating $y(x)$ the path of motion.

It should be noted that in the final expression, the path of the object's motion does not contain the parameter *t*. This covariation can also involve finding the components of the velocity and acceleration functions, such as $v_x(t), v_y(t)$, $a_x(t)$, and $a_y(t)$.

Another example of such covariation is the investigation of temperature measured in two scales, $C(t)$ and $K(t)$, by formulating respective functions of $C(t)$ and $K(t)$. If the time parameter is considered the independent variable, the following conclusions emerge:

- The temperature is considered dependent on each function.
- The thermometer measures the temperature independently, producing different values due to different scales.
- By finding an algebraic function that represents the covariation between the temperatures, one can derive conversion equations $C(K)$ or $K(C)$.

When mediating parameters are included in these analyses, one can identify the specific heat capacity of the substance used in the experiment as influencing the rate of temperature change. However, the heat capacity does not affect the general conversion formulas $C(K)$ or $K(C)$, since temperature measurements on both scales will be altered at the same rates influenced by the substance's heat capacity value.

1.2.4 Multi-Parameter Covariation in Systems

This category involves systems composed of multiple objects, or one object simultaneously examined from two different perspectives (parameters). Algebraic structures representing multiple parameters are often called laws in physics, such as the law of conservation of momentum, the law of conservation of mechanical energy, or the gas law.

Since laws typically involve multiple objects and parameters, there is greater flexibility in how these parameters are classified. In this context, we will consider the law of conservation of mechanical energy, which involves both kinetic and potential energies.

Suppose that an object of mass $2kg$ moves in two dimensions and possesses kinetic and gravitational energies, then at any two selected time instants or positions, the total mechanical energy of the object remains constant:

$$2gh_1 + \frac{2v_1^2}{2} = 2gh_2 + \frac{2v_2^2}{2} = \text{constant value}$$

The law can be expressed more directly as a covariate relationship by assigning dependent and independent parameters.

Suppose that the total initial mechanical energy of the object is 200 J, and that one is interested in attending to the object's velocity expressed in terms of the object's variable altitude, h.

Under this condition, the energy conservation equation is:

$$200 = 2gh_2 + v_2^2$$

Solving for v_2, we get:

$$v_2(h_2) = \sqrt{200 - 2gh_2}$$

This equation allows us to calculate the magnitude of the object's speed at any height along its path and to plot the velocity as a function of altitude.

The analysis would follow a similar approach but yield a different result if the dependent parameter were the object's altitude h_2, instead of velocity. Laws involving systems of objects offer greater opportunities to develop algebraic structures due to the diversity and complexity of the parameters involved.

1.2.5 Exploring Covariation in Composite Functions

Covariation resulting from composite functions is seldom emphasized in traditional school-level physics formulas. Its algebraic foundation lies in composing two functions, specifically, when the independent parameter of one function depends on another parameter. Algebraically, such covariations are expressed as follows.

$$y = f(g(x)).$$

In such notation, $g(x)$ is considered an inner function of y and $f(x)$ is the outer function.

Problem-solving in physics does not typically require the construction of formal composite functions, as the numerical output of the inner function can simply serve as the input for the outer function. However, formulating composite expressions can offer deeper insight into the structure of the relationships involved and can aid in graph sketching.

Consider the gravitational potential energy, U_G of an object that is traditionally calculated as

$$U_G(y) = mgy,$$

where y represents the position of the object above or below the established line of reference.

Suppose the object's position changes according to

$$y = h(t) = v_1 t + \frac{gt^2}{2}$$

Then the gravitational potential energy U_G can be expressed as a composite function by substituting y with $h(t)$:

$$U_G(h(t)) = mg(v_1 t + \tfrac{gt^2}{2}).$$

Although the units of $U_G(h(t))$ remain in *Joules*, the independent variable is no longer the object's position, but it is time, and the expression for the energy can be expressed as $U_G(t)$. Using the composite function and knowing the time, the gravitational potential energy can be calculated directly, and a graph of $U_G(t)$ can also be sketched, providing more insight into the energy behavior.

Similarly, the expression for rotational kinetic energy,

$$K = \tfrac{1}{2} I\omega^2$$

can also be reformulated through the concept of composite functions. If the angular velocity is given as

$$\omega(t) = \omega_0 + \alpha t,$$

then the kinetic energy can be described as a function of time:

$$K(t) = \tfrac{1}{2} I(\omega_0 + \alpha t)^2.$$

Students are introduced to techniques for finding composite functions in high school mathematics courses; therefore, applying these skills in physics should be both natural and feasible. The effort to classify physics formulas offers a fresh perspective on these expressions, broadening students' understanding of their algebraic flexibility and introducing dynamic approaches to problem-solving.

The categories outlined in this section are not exhaustive but serve multiple purposes. If they enhance students' perspectives on physics methods and support conceptual understanding, they fulfill their intended role. These ideas will be integrated into the proposed lab activities in Chapter 3.

1.3 Direct and Inverse Proportional Covariations in Math and Physics

This section will explore in greater depth how physics formulas can be categorized through the lens of proportional relationships. These categorizations will primarily reflect their underlying algebraic structures, which will later be examined within the context of scientific interpretation. An accompanying worksheet for independent student work is provided to support the development of these skills and their application in the problem-solving activities in Chapter 3.

We will refer to functional notation $f(x)$, as a concise way to designate independent and dependent variables, referred to as *physical quantities* in physics. We recall that in mathematics, the independent variable is traditionally denoted as x, and the corresponding values of the dependent variable are determined using a specified mathematical relationship known as the function.

Example 1 Given are $f(x) = 3x + 4$ and $g(x) = 3x^2 + 4x$.

Evaluate both functions at $\boldsymbol{x = 2}$:

- $\boldsymbol{f(2) = 3(2) + 4 = 10}$
- $\boldsymbol{g(2) = 3(2)^2 + 4(2) = 20}$

We observe that the values of these functions depend not only on the value of x but more significantly on the mathematical structure of the functions themselves.

Functional notation is not traditionally used to represent scientific formulas. However, it can be helpful to indicate quantities that vary within a given formula or experiment.

1.3.1 Linking Covariate Relationships in Mathematics with Physics

Given a relationship between a dependent and an independent variable, the underlying function can take various algebraic forms, such as linear, quadratic, rational, exponential, or logarithmic. Within these forms, specific behaviors can also be identified, most notably as *proportionality* (direct variation) and *inverse proportionality* (inverse variation).

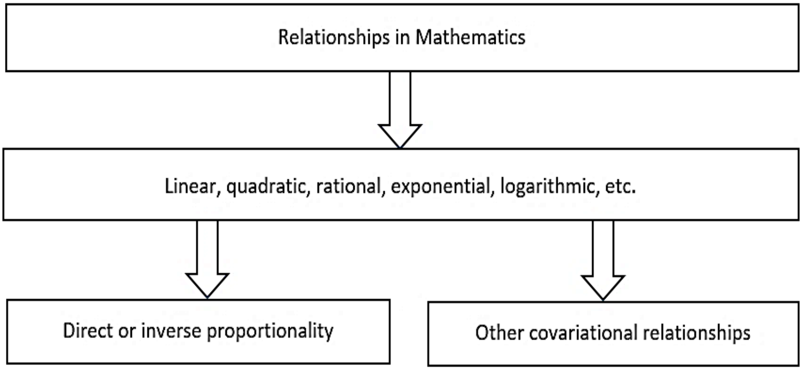


Fig. 1.1 Classification of Mathematical Relationships

How can we distinguish and correctly classify the types of covariates in physical formulas? Since various covariate relationships are constructed using mathematical principles, we will rely on mathematics to classify them.

1.3.1.1 Proportional and Direct Proportional Relationships

We begin by formally defining what constitutes a directly proportional relationship.

Definition: relationship is directly proportional if the value of $\frac{f(x)}{x}$ is constant for each pair of coordinates in the relation, excluding the point (0,0). This constant, resulting by evaluating $\frac{f(x)}{x}$, is called the *constant of proportionality*, denoted by k. Thus,

$$k = \frac{f(x)}{x}.$$

The equation of a directly proportional relationship can be written as

$$f(x) = kx.$$

Example 2. Does the function $f(x) = 3x + 4$ represent a directly proportional relationship?

Discussion. Preliminarily, we can classify the function as a linear covariate or, more simply, as a linear function. But does this function represent directly proportional variables? We could certainly follow the function form and decide; however, it is also beneficial to evaluate $k = \frac{f(x)}{x}$ and verify if the ratios satisfy the condition to be identical.

Let us generate and analyze the function's table with data.

x	0	1	2	3
$f(x) = 3x + 4$	4	7	10	13

Suppose we verify the values for each pair of coordinates using the ratio $\frac{f(x)}{x}$. When substituting respective values, the ratios are $\frac{7}{1}, \frac{10}{2}$ and $\frac{13}{3}$. The ratios are not constant; therefore, $f(x) = 3x + 4$ does not represent a proportional relation. Note that we excluded (0, 4) from this calculation, because it generates an undefined value.

Although this function has a constant rate of change of 3 (known as the gradient or slope), its values are not directly proportional to each other. This function has a constant rate of change of three, which is equal to its derivative, $f'(x) = 3$, but this value is not called the proportionality coefficient of this function.

How will this function behave if we reduce its form to $f(x) = 3x$?

x	0	1	2	3
$f(x) = 3x$	0	3	6	9

Following the rule, we check whether this function has a constant proportionality factor. It occurred that $k = \frac{3}{1} = \frac{6}{2} = \frac{9}{3} = 3,$ thus this function has a constant proportionality factor, representing a direct proportional relationship of the form $f(x) = kx$.

Generalizing this definition, can $f(x) = kx^n$ be called a proportional relation?

Yes, all functions of the form $f(x) = kx^n$ represent proportionality, but not all are directly proportional. Only the linear function (with n =1) of the form $f(x) = kx$ is directly proportional.

Below are graphs of functions representing proportional relationships with $n = 1, n = 2,$ and $n = 3$.

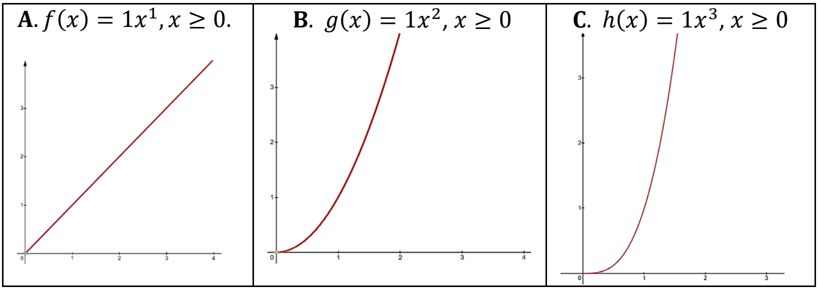


Fig. 1.2 Graphs of Directly Proportional and Proportional Relationships

We observe that these functions are polynomials of varying degrees. Furthermore, the degree of the polynomial determines the general shape of the graph and the nature of the relationship between the variables.

- All graphs originate from the coordinate system's origin (0,0) and exhibit increasing behavior, though their growth rates vary.
- Among them, only a linear function displays a constant rate of change.
- The quadratic function increases proportionally to the square of the variable, while the cubic function increases proportionally to its cube.
- Though the functions $g(x) = 1x^2$ and $h(x) = 1x^3$ have coefficients of 1; they do not possess constant proportionality coeffi-

cients, as the ratio $\frac{f(x)}{x}$ is not constant. Instead, their rate of change itself changes with x, indicating nonlinear relationships.

Exponential and logarithmic functions can also be considered proportional dependencies when they satisfy the condition of proportional growth or change. This proportionality is defined by the specific mathematical structure that governs their rates of change, which can be defined according to their mathematical structures.

Let us return to physics. This discipline offers numerous opportunities to apply the concept of proportionality in practical contexts.

Example 3. Let us discuss the formulas presented below, which are taken from a typical physics textbook: $F = mg, F = kx,$ and $d = \frac{at^2}{2}$.

Discussion

We cannot clearly define the nature of proportional relationships in physics formulas without first identifying which quantities are variables and which are constants. Revisiting function notation helps clarify these classifications by explicitly designating dependent and independent variables.

Consider the formula:

$$F = mg$$

Suppose we examine how gravitational force depends on the mass of the body. In this case, we will write this relationship as

$$F(m) = mg$$

where g is the strength of the gravitational field, which can be called the constant of proportionality. Following the properties of directly proportional relationships, we say that the force of gravity is directly proportional to the strength of the gravitational field. We will suggest the graph Figure 1.1 A, to visualize the relationship with the force F labeled on the vertical axis and the mass, m on the horizontal axis.

Suppose now that the gravitational force is expressed as a function of the gravitational field intensity. In that case, the constant of proportionality is not the strength of the gravitational field but the mass of the body; thus, we use $F(g) = mg$. The graph of $F(g)$ will be similar, except that the field strength g, will be marked on the horizontal axis.

Consider the formula

$$F(x) = kx.$$

This relationship is described as the spring force, expressed as a function of the spring's stretch, where the spring constant k is the proportionality constant of this function, and its spring extension x is an independent variable. The graph of the function will be like Figure 1.1 A.

Would it be practical to analyse $F(k) = xk$? Perhaps not, but this covariation can still offer insights into how the function behaves in these circumstances, which is also useful for problem-solving.

Consider the formula

$$d(t) = \frac{at^2}{2}.$$

To clarify the meaning of the parameters in this formula, we will state that the body's position is proportional to the square of time. If the time is doubled, the position will quadruple. The expression $\frac{a}{2}$ is not called the proportionality constant. The expression affects the shape of the parabola, representing the object's position. The graph of this function will be like Figure 1.2 B. Analysing $d(a) = \frac{at^2}{2}$ would rather be impractical.

In all these examples, we observe that the initial values of the functions are zero, reflecting their expressions as proportional relationships. The independent quantities are emphasized by placing their symbols in parentheses next to the dependent quantities.

1.3.1.2 Inversely Proportional Relationships

In the following section, inversely proportional quantities will be discussed. Mathematical analysis of these forms is simplified due to their similarity to proportional relationships. Our focus will primarily be on applying these covariational relationships within physics.

We will begin by providing a formal definition of an inversely proportional relationship:

> *Definition:* Two variable quantities x and $f(x)$ are *inversely proportional* if, for each pair of the coordinates (excluding the point (0,0), the product $f(x) \cdot x$ is constant. This constant is called the constant of proportionality, denoted by k.
>
> $$k = f(x) \cdot x.$$
>
> If this condition is met, the relationship can be expressed as:
>
> $$f(x) = \frac{k}{x}.$$

Example 4. Consider $g(x) = \frac{4}{x} + 2$.

Discussion. The function can be described as a covariate or a rational function, but it does not necessarily represent inverse proportionality due to the added constant 2. When constructing a table of values for some values of x and calculating the proportionality constant for each pair of resulting coordinates, it would turn out that $k = g(x) \cdot x$ is not constant.

However, if we reject 2, then the function

$$g(x) = \frac{4}{x}$$

represents a true inverse linear proportionality, where the proportionality constant is $k = 4$.

Extending this definition, rational functions of the form

$$f(x) = \frac{k}{x^n}$$

where $n > 0$ and $k > 0$ represent diverse types of inverse proportionalities. For example,

- $f(x) = \frac{10}{x^2}$ is called inverse quadratic proportionality.
- $f(x) = \frac{4}{x^3}$ is called an inverse cubic proportionality.

A. Graph $f(x) = \frac{1}{x^1}$, $x > 0$.	B. Graph $f(x) = \frac{1}{x^2}$, $x > 0$.	C. Graph $f(x) = \frac{1}{x^3}$, $x > 0$.

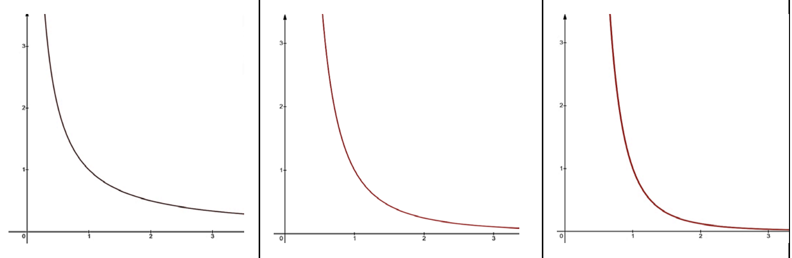


Fig. 1.3 Graphs of Inversely Proportional Quantities

All these graphs in Figure 1.3 are decreasing: as the values of x increase, the corresponding values of $f(x)$ decrease. Additionally, for all these rational functions, the following limits hold:

$$\lim_{x\to\infty} \frac{k}{x^n} = 0,$$

which means that the function values approach zero as x becomes very large and

$$\lim_{x\to 0^+} \frac{k}{x^n} = \infty,$$

indicating that the function values grow without bound as x approaches zero from the positive side. While the domain of all these functions was $x > 0$, these functions could also be sketched for $x < 0$.

Observing the curves of the graphs, we note that the rate at which the functions decrease varies depending on the exponent n in the denominator of $f(x) = \frac{k}{x^n}$. Specifically, the rate of decrease becomes more pronounced as the exponent n increases. This behavior can also be justified by examining the second derivative of these functions, which reflects the degree of concavity**.** The greater the exponent n, the stronger the concavity, illustrating how sharply the function curves downward.

Root, exponential, and logarithmic functions can also represent inverse proportionality, depending on how their variables relate.

How do we apply inverse proportionality to interpret relationships in physics formulas?

Let us consider the formula for centripetal acceleration:

$$a_c = \frac{v^2}{R}.$$

Similar to proportional relationships, we first need to decide which quantity on the right side of the equation is treated as a variable.

- Suppose $a_c(R) = \frac{v^2}{R}$.
  This function represents an inverse linear proportionality for which the proportionality constant is v^2. Its graph resembles Figure 1.3 A, showing how a_c decreases as R increases.
- Alternatively, suppose we write $a_c(v) = \frac{v^2}{R}$.
  This notation informs that the centripetal acceleration is proportional to the square of the velocity. Its graph is similar to Figure 1.2 B, demonstrating the quadratic increase of a_c with increasing v.

Let us consider the gas law

$$PV = nRT.$$

We observe that the law is not expressed in an explicit functional form. The values of all these quantities—pressure P, volume V, and amount of gas $n -$ can vary, except for the gas constant R, while maintaining the equality between the left and right sides of the equation.

If we transform the gas law into a functional form

$$P(T) = \frac{nRT}{V},$$

then the pressure P is proportional to the temperature T, with the proportionality constant

$$k = \frac{nR}{V}.$$

Its graph resembles Figure 1.2 A, showing a linear increase of pressure with temperature.

Alternatively, if we write

$$P(V) = \frac{nRT}{V},$$

then the pressure P is inversely proportional to the volume V, with a proportionality constant

$$k = nRT.$$

The graph of P(V) resembles resembles Figure 1.3 A, depicting a decreasing curve as volume increases.

A classic example of proportionality is Newton's law of universal gravitation:

$$F = \frac{GmM}{d^2}.$$

If we consider the force as a function of distance,

$$F(d) = \frac{GmM}{d^2},$$

we see that the force decreases with the square of the distance, and its graph will be similar to Figure 1.3 B (an inverse square relationship).

If instead we consider the force as a function of mass

$$F(m) = \frac{GmM}{d^2},$$

the force increases linearly with the mass m, where the proportionality constant is

$$k = \frac{GM}{d^2}.$$

This relationship corresponds to a graph similar to Figure 1.2 A (a linear increase).

An interesting example to include in this discussion is the equation for wave velocity:

$$v = f\lambda.$$

Is it practical to analyze this equation as a linear function and apply directly proportional relationships? Probably not—especially when we assume the medium's properties remain constant. In this case, the wave speed $v = f\lambda$ is a constant function with zero rate of change, because as the frequency f increases, the wavelength λ decreases proportionally.

Considering instead

$$\lambda = \frac{v}{f},$$

or more specifically

$$\lambda(f) = \frac{v}{f},$$

makes the equation more interesting to classify and interpret, as it represents an inverse proportionality between wavelength and frequency.

Following this discussion, students can engage in independent work to deepen their understanding.

1.3.1.3 Proportional Reasoning in Practice

The main goal of this assignment is to assess how well students can independently classify selected physics formulas based on their algebraic and functional characteristics. Students are not required to generate tables of values to sketch graphs for these relationships.

If students have prior familiarity with the physics concepts behind these formulas, additional questions can be included to encourage deeper reflection on the physical phenomena or scenarios that the formulas describe.

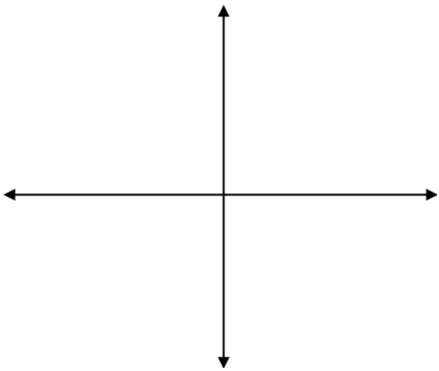
Assignment

The following formulas illustrate various physical relationships, with constant and variable quantities expressed using functional notation. For each formula:

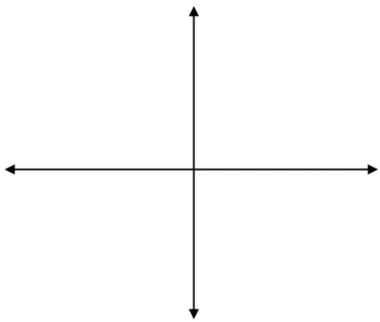
- Identify the dependent variable, the independent variable(s), and the constant(s).
- Specify the type of proportionality involved (write *no proportionality* if the formula does not represent a proportional relationship).

- State the proportionality constant.
- Sketch the graph corresponding to the underlying function.
- Label the vertical and horizontal axes on the graph, including their physical quantities and units.

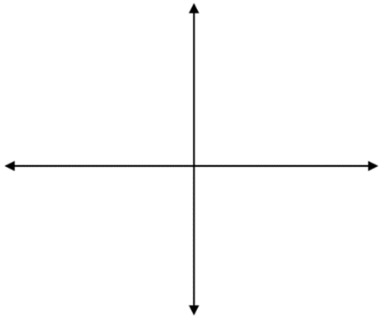
1. $p(v) = mv$



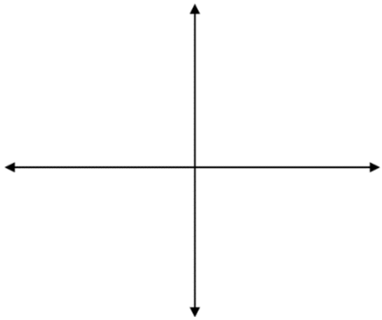
2. $K(m) = \frac{mv^2}{2}$



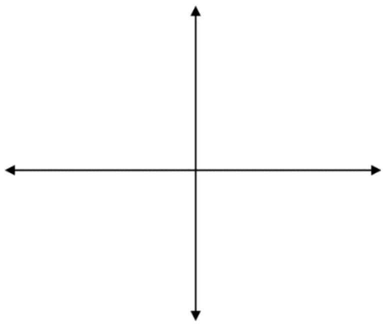
3. $F(M) = \frac{GMm}{d^2}$



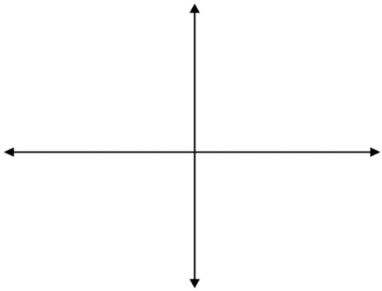
4. $x(t) = v_o t + \frac{at^2}{2}$



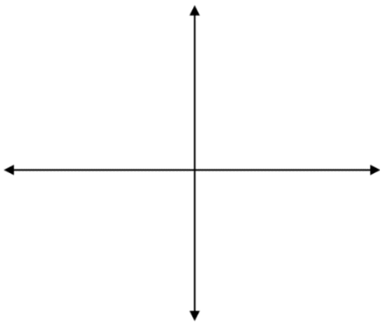
5. $P(V) = \frac{nRT}{V}$



6. $T(m) = 2\pi\sqrt{\frac{k}{m}}$



7. $U_G(y) = mgy$



8. $V(p) = \frac{nRT}{p}$