A Journey into Quantum Wavefunction Optimization Algorithm

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Exploring the Path to the Optimal Solution

By

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I humbly dedicate this book at the feet of Maa Saraswati, the Goddess of Knowledge.

"Everything that comes to us that belongs to us if we create the capacity to receive it." By Rabindernath Tagore

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Preface

A Journey into Quantum Wavefunction Optimization Algorithm: Exploring the Path to the Optimal Solution offers an insightful and forward-looking exploration of quantum-inspired optimization methodologies and their integration with modern intelligent systems. This book captures the confluence of quantum mechanics, fuzzy logic, bio-inspired computing, and time series analysis, all directed toward solving complex optimization and forecasting problems.

The journey begins with a novel hybrid time series forecasting model that integrates neutrosophic theory with Particle Swarm Optimization (PSO), addressing the challenges posed by uncertainty, indeterminacy, and noise in real-world data. This foundation paves the way for deeper explorations into fuzzy time series forecasting, enriched through granular computing and biologically-inspired algorithms, revealing enhanced strategies for modeling nonlinear temporal behavior.

Building upon these foundations, the book introduces a fuzzy-quantum framework for time series forecasting. This model elegantly unifies fuzzy logic with quantum computing concepts, demonstrating how quantum probability amplitudes and wavefunctions can provide new insights into uncertain and dynamic data environments.

At the heart of this book lies the Fast Forward Quantum Optimization Algorithm, a novel technique designed for rapid convergence and high-efficiency optimization in unconstrained problem settings. Through analytical rigor and practical demonstration, the algorithm's performance and adaptability are thoroughly examined.

The application horizon broadens further with the Quantum Wavefunction Optimization Algorithm, particularly in solving the classic Traveling Salesman Problem. By simulating quantum wave behaviors and probabilistic transitions, the algorithm exemplifies the potential of quantum computation in tackling combinatorial challenges.

The final chapter applies the fast forward quantum optimization strategy to a multi-objective deep learning task, optimizing convolutional neural networks for digital image classification. This chapter demonstrates how quantum optimization can elevate performance in complex, high-dimensional AI systems, where traditional optimization techniques may fall short.

Throughout, *A Journey into Quantum Wavefunction Optimization Algorithm: Exploring the Path to the Optimal Solution* blends theory with practice, offering a compelling synthesis for researchers, engineers, and advanced learners. With its unique focus on quantum-computational perspectives applied to forecasting and optimization, the book not only advances the frontier of intelligent computing but also inspires new directions for future exploration.

The present book has been accomplished at National Taipei University of Technology (Taiwan), Jagiellonian University (Poland), and Central University of Rajasthan (India). All experiments were conducted at National Taipei University of Technology (Taiwan), Jagiellonian University (Poland), and Central University of Rajasthan (India). The empirical results presented in the book were published by esteemed journals.

Rajasthan, January, 2025

Pritpal Singh

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Lastly, I remain forever grateful to the Almighty for providing me the opportunity, strength, and wisdom to embark upon and complete this scholarly journey.

Chapter 1

A hybrid time series forecasting model using neutrosophic-PSO

"Where data dwells in shades of certainty, it is through indeterminacy and intelligent swarms that true patterns begin to emerge."

Abstract

This chapter introduces a novel time series forecasting model integrating neutrosophic set (NS) theory with the particle swarm optimization (PSO) algorithm. The model begins by transforming the time series data into a neutrosophic time series (NTS), characterized by three memberships: truth, indeterminacy, and falsity. Forecasting accuracy is shown to depend significantly on the optimal selection of the universe of discourse, a challenge addressed through PSO. The model was evaluated on three datasets: Alabama university enrollments, TAIFEX index, and TSEC weighted index. Experimental results demonstrate that the proposed model outperforms existing benchmarks, achieving average forecasting error rates of 0.80%, 0.015%, and 0.90%, respectively.

Keywords Neutrosophic set, PSO, time series forecasting, entropy.

1.1 Introduction

In 1965, Zadeh (Zadeh, 1965) introduced the concept of representing the uncertainties that are found at different occasions. This theory is called as fuzzy set theory. It depends on the non-probabilistic portrayal of uncertainty of events, where the degree of membership of each event must lie between 0 and 1. Using fuzzy set theory, Song and Chissom (Song and Chissom, 1993) initially presented a model, which was termed as “fuzzy time series (FTS)”. In this modeling approach, each historical time series datum was characterized by the fuzzified linguistic variable (Singh, 2017). Later, various forecasting models were introduced by the researchers using the approach of Song and Chissom (Song and Chissom, 1993). For example, researchers were proposed FTS based model for forecasting the university enrollment dataset of Alabama (Chen, 1996; Lee and Chou, 2004; Cheng et al., 2006, 2008; Wong et al., 2010; Qiu et al., 2011; Chen and Tanuwijaya, 2011; Liu, 2007; Gangwar and Kumar, 2012; Singh and Borah, 2012). However, some researchers were additionally focused on forecasting the daily temperature of Taiwan (Lee et al., 2008; Wang and Chen, 2009; Singh and Borah, 2014a). In the recent years, many researchers were attracted toward the application of the FTS modeling approach in financial time series forecasting. In the literatures, numerous FTS forecasting models were proposed for forecasting the financial time series, which included the TAIEX (Yu and Huarng, 2008; Wang and Chen, 2009; Yu and Huarng, 2010; Wei et al., 2011; Huarng and Yu, 2012; Avazbeigi et al., 2010; Cheng et al., 2013) and the TIFEX (Bai et al., 2011; Kuo et al., 2010; Aladag et al., 2012; Avazbeigi et al., 2010).

Singh (Singh, 2017) explained that forecasting accuracy of the FTS modeling approach mainly depended on two factors, viz., determination of the effective length of intervals, and establishment of fuzzy logical relationships (i.e., called the decision rules). However, determination of effective decision rules mostly relied upon the appropriate selection of intervals. In FTS modeling approach, researchers utilized various optimization algorithms to resolve the issue of appropriate selection of intervals. In the literatures, several metaheuristics optimization algorithms were proposed (Cai et al., 2016; Zhang et al., 2018; Cao et al., 2019; Eshtay et al., 2019). But, in case of FTS modeling approach, researchers mainly used Genetic Algorithm (GA) (Chen and Chung, 2006), PSO (Kuo et al., 2010; Aladag et al., 2012), Simulated Annealing (SA) (Radmehr and Gharneh, 2012), Geese Movement Based Optimization Algorithm (GMBOA) (Singh and Dhiman, 2018) and Quantum Optimization Algorithm (QOA) (Singh et al., 2018) for the appropriate selection of intervals from the universe of discourse.

Real-time problems contain often inadequate or incomplete information. Atanassov (Atanassov, 1986) introduced the theory of Intuitionistic fuzzy sets (IFS) as an optional way to manage uncertainty in situations where available information isn't adequate for defining the uncertainty. In IFS, this loss of information is represented by using both membership and non-membership degree functions. In view of the IFS concept, Joshi and Kumar (Joshi and Kumar, 2012) introduced a new model for time series forecasting.

Above discussion indicated that fuzzy set theory (Zadeh, 1965) can represent the inherent uncertainties of time series data with truth-membership only; while the IFS can consider both truth-membership and false-membership to describe the uncertainties. To overcome these drawbacks of fuzzy set and IFS theories, Smarandache (Smarandache, 1999) introduced a new theory for the uncertainties representation based on the knowledge of neutral thought. This theory was termed as NS, which provided the clear distinction between fuzzy set theory and IFS. In NS, it is considered that an event can be characterized with three different degree of memberships, viz., truth-membership (T_f), indeterminacy-membership (I_f) and falsity-membership (F_f), where $\langle T_f, I_f, F_f \rangle \in [0, 1]$. In the literatures, several works have been reported showing the studies and applications of NS in various domains (Majumdar, 2015). However, no other work was reported in the literatures that could present a time series forecasting model using NS theory. By this motivation, this study was propagated toward the direction of designing time series forecasting model using NS theory and PSO algorithm. This study has identified following major problems while modeling time series dataset using NS theory and PSO algorithm as:

Problem 1: Identify a method for providing more effective representation of inherent uncertainties of time series dataset using NS theory.

Problem 2: Identify an approach to deal with three degree of memberships, viz., T_f, I_f and F_f together.

Problem 3: Identify a method to improve the forecasting accuracy.

In the following, solutions of these problems are discussed, which can be considered as the contributions of the study.

1) Contribution 1: For Problem 1, a mathematical method was presented to represent historical time series dataset in terms of NS theory. In this representation, each time series datum can be characterized in terms of T_f, I_f and F_f. This process of representation was termed as *neutrosophication*, and the transformed time series was termed as NTS. The model based on the NTS representation was termed as "NTS model".

2) Contribution 2: For Problem 2, this study found entropy (Richman and Moorman, 2000) as a suitable approach to deal with three degree of memberships, viz., T_f, I_f and F_f together. For combining these degree of memberships together, this study adopted the formula of entropy as proposed by Singh and Rabadiya (Singh and Rabadiya, 2018).

3) Contribution 3: While modeling with NTS, it was observed that forecasting accuracy depended on proper determination of the T_f, I_f and F_f. It was noticed that their appropriate determination depended on the optimal selection of the universe of discourse. Hence, for resolving Problem 3, this study adopted PSO (Eberhart and Kennedy, 1995) algorithm as it is computationally robust and convergent towards the global solution very fast.

By concluding the above discussion, this study presented a new time series forecasting model employing NTS and PSO. This new hybrid model was entitled as *NTS-PSO model.* Initially, performance of the NTS model was evaluated followed by the NTS-PSO model. Both these models were trained using the university enrollment dataset of Alabama (Chen, 1996). Finally, these models were validated by the TAIFEX index and TSEC weighted index datasets. Experimental results indicated that the NTS-PSO model outperformed various FTS- and non-FTS based models.

The remainder of this chapter is organized as follows. Section 1.2 provides the basics of NS and its application in time series forecasting. An overview of the PSO is given in Section 1.3. The proposed NTS model for time series forecasting is introduced in Section 1.4. The NTS-PSO based model for time series forecasting is presented in Section 1.5. Parameters for statistical analysis have been listed in Section 1.6. In Section 1.7, experimental results are discussed. Finally, conclusions and future directions are discussed in Section 1.8.

1.2 Background for the study

In this section, we provide the basics of the neutrosophic set (NS) followed by definitions of terminologies used throughout the article.

1.2.1 Neutrosophic set

Various theories of NS are presented as follows (Wang et al., 2005; Singh and Rabadiya, 2018):

Definition 1.2.1. (Neutrosophic Set (NS))(Wang et al., 2005). Assume that U be a universe of discourse. A neutrosophic set N for all $u \in U$ can be represented by a truth-membership function T_f, an indeterminacy-membership function I_f and a falsity-membership function F_f, where $T_f, I_f, F_f : U \to [0,1]$ and $\forall u \in U, u \equiv u(T_f(u), I_f(u), F_f(u)) \in N$.

Wang et al. (Wang et al., 2005) defined an instance of NS as a *single valued neutrosophic set (SVNS)*. The NS N can be represented as a SVNS on the universe of discourse $U = \{u_1, u_2, u_3, \dots, u_n\}$ as follows:

$$N = \sum_{i=1}^{n} \frac{u_i}{\langle T_f(u_i), I_f(u_i), F_f(u_i)\rangle} \tag{1.2.1.1}$$

Example 1.2.2. Let us consider that a time series dataset of daily temperature, whose universe of discourse U consists of n-different values as: $U = \{u_1, u_2, u_3, \dots, u_n\}$. Here, daily temperature conditions are considered as "Low", "Medium", "High", and so on. Now, these daily temperature conditions can be characterized using the three membership functions, viz., $T_f(u_i)$, $I_f(u_i)$ and $F_f(u_i)$. Now, a SVNS can be defined for the daily temperatu1axtre conditions on the U as:

$$N = \frac{u_1}{\langle T_f(u_1), I_f(u_1), F_f(u_1)\rangle} + \frac{u_2}{\langle T_f(u_2), I_f(u_2), F_f(u_2)\rangle} + \dots + \frac{u_n}{\langle T_f(u_n), I_f(u_n), F_f(u_n)\rangle} \tag{1.2.1.2}$$

In the following, the definition for *neutrosophic membership function* is provided as follows:

Definition 1.2.3. (Neutrosophic membership function)(Singh and Rabadiya, 2018). A *neutrosophic membership function* for any

$u \in U$ can be defined in terms of truth-membership function T_f, an indeterminacy-membership function I_f and a falsity-membership function F_f as follows:

$$T_f(u) = \frac{u - \min(U)}{\max(U) - \min(U)} \tag{1.2.1.3}$$

$$F_f(u) = 1 - T_f(u) \tag{1.2.1.4}$$

$$I_f(u) = \sqrt{T_f(u)^2 + F_f(u)^2} \tag{1.2.1.5}$$

In Eq. 1.2.1.3, "*min*" and *max* represent the minimum and maximum functions, which return the minimum and maximum values from the U, respectively.

Definition 1.2.4. (Complement of a SVNS)(Singh and Rabadiya, 2018). The complement of a SVNS N is denoted by N^c, and can be defined as: ${T_f}^c(u) = F_f(u)$, ${I_f}^c(u) = 1 - I_f(u)$ and ${F_f}^c(u) = T_f(u)$, such that $\forall u \in U$.

Definition 1.2.5. (Neutrosophication)(Singh and Rabadiya, 2018). The operation of *neutrosophication* transforms a crisp set into a neutrosophic set. Thus, a *neutrosifier* $\overline{N}$ is applied to a crisp subset i of the universe of discourse U yields a neutrosophic subset $\overline{N}(i : N)$, which can be expressed as:

$$\overline{N}(i : N) = \int_{u \in U} (T_f(u), I_f(u), F_f(u))N(u) \tag{1.2.1.6}$$

Here, $(T_f(u), I_f(u), F_f(u))N(u)$ represents the product of a scalar $(T_f(u), I_f(u), F_f(u))$ and NS $N(u)$; and $\int$ is the union of the family of NS $(T_f(u), I_f(u), F_f(u))N(u)$, $u \in U$.

Definition 1.2.6. (Deneutrosophication)(Singh and Rabadiya, 2018). The operation of *deneutrosophication* transforms a NS into a crisp set.

A time series dataset can be converted into a NS, which is referred as a *neutrosophic time series (NTS)*. In the following, we provide the definition for the NTS as:

Definition 1.2.7. (Neutrosophic time series (NTS)). Let $F(u_i)(i = 1, 2, \ldots, n)$ be a subset of the universe of discourse U, on which truth-membership function $T_f(u_i)$, an indeterminacy-membership function $I_f(u_i)$ and a falsity-membership function $F_f(u_i)$

Table 1.1: The university enrollments dataset of Alabama (Chen, 1996).

Year	Actual Enrollments	Year	Actual Enrollments
1971	13055	1982	15433
1972	13563	1983	15497
1973	13867	1984	15145
1974	14696	1985	15163
1975	15460	1986	15984
1976	15311	1987	16859
1977	15603	1988	18150
1978	15861	1989	18970
1979	16807	1990	19328
1980	16919	1991	19337
1981	16388	1992	18876

are defined, and let $N(u_i)$ be a collection of $T_f(u_i)$, $I_f(u_i)$ and $F_f(u_i)$. Then, $N(u_i)$ is called a NTS on $F(u_i)$.

The SVNS can deal with the uncertainty associated with time series events. Entropy provides the measure of uncertainty represented by such set of events. It can be defined for the NTS value based on the following proposed formula as follows:

Definition 1.2.8. (Entropy of NTS). The entropy of a NTS $N(u_i)$ is denoted as a function $E(N(u_i))$, where $E(N(u_i)) : E(N(u_i)) \to [0, 1]$, which can be defined as follows:

$$E(N(u_i)) = 1 - \frac{1}{3}\sum_{i=1}^{n}\big(T_f(u_i) + I_f(u_i) + F_f(u_i)\big) \cdot E_1E_2E_3 \tag{1.2.1.7}$$

Here, $E_1 = \mid T_f(u_i) - {T_f}^c(u_i) \mid$, $E_2 = \mid I_f(u_i) - {I_f}^c(u_i) \mid$, and $E_3 = \mid F_f(u_i) - {F_f}^c(u_i) \mid$.

Definition 1.2.9. (Neutrosophic entropy relationship (NER)). Assume that $F(t_i - 1) = E(N(u_i))$ and $F(t_i) = E(N(u_j))$. The relationship between $F(t_i - 1)$ and $F(t_i)$ is referred as a NER, which can be represented as:

$$E(N(u_i)) \to E(N(u_j)), \tag{1.2.1.8}$$

where, $E(N(u_i))$ and $E(N(u_j))$ are referred as *previous state* and *current state* of the NER, respectively.

Definition 1.2.10. (High-order NER). If $F(u) = E_N(u_i)$ is caused by $F(u-1) = E_N(u_j), F(u-2) = E_N(u_k), \ldots$, and $F(u-n) = E_N(u_n)$ $(n > 0)$, then high-order NER can be defined as:

$$E_N(u_j), E_N(u_k), \ldots, E_N(u_n) \rightarrow E_N(u_i) \tag{1.2.1.9}$$

Definition 1.2.11. (Neutrosophic entropy relationship group (NERG)). Assume the following NERs:

$$\begin{gathered} E(N(u_i)) \rightarrow E(N(u_{k1})), \\ E(N(u_i)) \rightarrow E(N(u_{k2})), \\ \cdots \\ E(N(u_i)) \rightarrow E(N(u_{km})) \end{gathered}$$

The NERs having the same previous state can be grouped into a same neutrosophic entropy relationship group (NERG), which can be represented as follows:

$$E(N(u_i)) \rightarrow E(N(u_{k1})), E(N(u_{k2})), \ldots, E(N(u_{km})) \tag{1.2.1.10}$$

1.3 The PSO algorithm

Eberhart and Kennedy (Eberhart and Kennedy, 1995) proposed the PSO algorithm. It is a bio-inspired optimization algorithm, which mimics the social behavior of animals, such as bird flocking, fish schooling and swarm (Eberhart and Shi, 2001; Lin et al., 2010).

In time series forecasting model, the accuracy rate of forecasting depends upon the optimal selection of intervals from the universe of discourse (Singh, 2017). To resolve this issue, many researchers found PSO as an appropriate algorithm (Singh and Borah, 2014b). Forecasting accuracy of the model as proposed by Kuo et al. (Kuo et al., 2009) also indicated that the PSO algorithm was very efficient than GA. Therefore, to improve the forecasting accuracy of the proposed model, this study selected the PSO algorithm.

This algorithm is initiated by considering a set of particles, where each of the particles are allowed to move randomly in the problem space in the searching of optimal solution. During the movement, velocity of each particle is computed using the following equation as:

$$\begin{aligned} V_{id,t} = \alpha \times V_{id,t} + m_1 \times R_f \times (PREV_{id} - CURR_{id,t}) \\ + m_2 \times R_f \times (GLOBAL_{best} - CURR_{id,t}) \end{aligned} \tag{1.3.1.1}$$

Finally, position of each of the particles are updated using the following equation:

$$CURR_{id,t} = CURR_{id,t} + V_{id,t} \tag{1.3.1.2}$$

The steps for the standard PSO is presented in Algorithm 1.

Algorithm 1: Standard PSO algorithm

1 i: denotes the ith particle.
2 d: denotes the dimension of the problem space.
3 α: indicates the inertia weight factor.
4 $CURR_{id,t}$: defines the current position of the particle i in iteration t.
5 $PREV_{id}$: defines the previous best position of the particle i that experiences the best fitness value ($PREV_{best}$) so far.
6 $GLOBAL_{best}$: defines the global best fitness value ($GLOBAL_{best}$) among all the particles.
7 R_f: denotes random function generating random value in the range of $[0, 1]$.
8 m_1 and m_2: indicate the self-confidence coefficient and the social coefficient, respectively.
9 $V_{id,t}$: defines the velocity of the particle i in iteration t, limited to $[-V_{max}, V_{max}]$.
10 Define n number of particles as: $P_1, P_2, \ldots, P_n$ in the problem space.
11 Randomly initialize the positions and velocities of all particles in the problem space.
12 Compute the fitness of each particle $P_1, P_2, \ldots, P_n$.
13 Identify the particle having the optimal fitness value, denoted as $PREV_{best}$, and consider this particle as P_i.
14 Compute the velocities of each particle using Eq. 1.3.1.1.
15 Update the positions of the particles with respect to $PREV_{best}$ of P_i using Eq. 1.3.1.2.
16 Compute the fitness of each particle $P_1, P_2, \ldots, P_n$ at their new positions.
17 Compare the fitness of each particle with its previous best. If current fitness is better than $GLOBAL_{best}$, update $GLOBAL_{best}$.
18 Repeat Steps 12 to 17 until a stopping criterion is met (e.g., a sufficiently good $GLOBAL_{best}$ is obtained).

1.4 The proposed NTS model

In this section, a new NTS model is introduced for time series forecasting. An algorithmic representation of the proposed NTS model is

Table 1.2: NTS representation of university enrollments.

Year	Actual Enrollments	NTS Representation	Entropy Value
1971	13055	$\frac{13055}{\langle 0.026,0.974,0.974\rangle}$	0.438
1972	13563	$\frac{13563}{\langle 0.035,0.965,0.966\rangle}$	0.472
1973	13867	$\frac{13867}{\langle 0.040,0.960,0.961\rangle}$	0.490
1974	14696	$\frac{14696}{\langle 0.054,0.946,0.948\rangle}$	0.537
1975	15460	$\frac{15460}{\langle 0.067,0.933,0.935\rangle}$	0.578
1976	15311	$\frac{15311}{\langle 0.065,0.935,0.937\rangle}$	0.572
1977	15603	$\frac{15603}{\langle 0.070,0.930,0.933\rangle}$	0.587
1978	15861	$\frac{15861}{\langle 0.074,0.926,0.929\rangle}$	0.599
1979	16807	$\frac{16807}{\langle 0.090,0.910,0.914\rangle}$	0.644
1980	16919	$\frac{16919}{\langle 0.092,0.908,0.913\rangle}$	0.649
...	...	...	...
1989	18970	$\frac{18970}{\langle 0.127,0.873,0.882\rangle}$	0.733
1990	19328	$\frac{19328}{\langle 0.133,0.867,0.877\rangle}$	0.745
1991	19337	$\frac{19337}{\langle 0.133,0.867,0.877\rangle}$	0.745
1992	18876	$\frac{18876}{\langle 0.126,0.874,0.883\rangle}$	0.730

shown in Algorithm 2. Initially, an experiment was carried on by employing the university enrollments dataset of Alabama (Chen, 1996). This dataset is shown in Table 1.1. Each phase of the proposed model is explained next.

Step 1.4.1. Define the universe discourse of the historical time series dataset.

[Explanation:] Assume that E_{min} and E_{max} be the minimum and maximum enrollments of the historical time series dataset. Based on E_{min} and E_{max}, the universe of discourse U can be defined as: $U = [E_{min} - A_N, E_{max} + A_P]$, where A_N and A_P be the two adjustment factors. From the historical time series dataset as shown in Table 1.1, we have E_{min} =13055 and E_{max} =19337. Hence, initially, we can assume that A_N =1555 and A_P =50663. Therefore, the universe of discourse U is obtained as: $U = [11500, 70000]$, where, $min(U)$=11500 and $max(U)$=70000.

Step 1.4.2. Apply the neutrosophication process to represent the historical time series dataset into NS.

[Explanation:] Each value of the historical time series dataset is represented using NS as:

$$N_{t_i} \quad = \quad \frac{t_i}{\langle T_f(t_i), I_f(t_i), F_f(t_i)\rangle}; i = 1, 2, \ldots, n \qquad (1.4.1.1)$$

In Eq. 1.4.1.1, T_f, I_f and F_f represent the truth-membership, indeterminacy-membership and falsity-membership functions for the historical time series value t_i, where $T_f, I_f, F_f : U \rightarrow [0,1]$ and $\forall t_i \in U, t_i \equiv t_i(T_f(t_i), I_f(t_i), F_f(t_i)) \in N_{t_i}$. Here, the values of $T_f(t_i)$, $I_f(t_i)$ and $F_f(t_i)$ can be computed using Eqs. 1.2.1.3-1.2.1.5, respectively.

For example, the enrollment value for the year 1971 is 13055. We can represent the enrollment value 13055 as a NS using Eq. 1.4.1.1 as:

$$N_{13055} = \frac{13055}{\langle T_f(13055), I_f(13055), F_f(13055)\rangle} \qquad (1.4.1.2)$$

Here, $T_f(13055)$, $I_f(13055)$ and $F_f(13055)$ are obtained (Eqs. 1.2.1.3-1.2.1.5) as:

$$T_f(13055) = \frac{13055 - min(U)}{max(U) - min(U)} = \frac{13055 - 11500}{70000 - 11500} = 0.026$$

$$F_f(13055) = 1 - T_f(13055) = 1 - 0.026 = 0.974$$

$$I_f(13055) = \sqrt{T_f(13055)^2 + F_f(13055)^2} = \sqrt{(0.026)^2 + (0.974)^2} = 0.974$$

Hence, Eq. 1.4.1.2 can be represented in terms of NS for the enrollment value 13055 as:

$$N_{13055} = \frac{13055}{\langle 0.026, 0.974, 0.974\rangle} \qquad (1.4.1.3)$$

In this way, the historical time series dataset as shown in Table 1.1, is represented in terms of NS using the neutrosophication process. This representation is called as the NTS (refer to Definition 1.2.7).

The NTS representation of university enrollments dataset of Alabama is shown in Table 1.2.

Algorithm 2: The proposed NTS model

```
Input: t_i ∈ U.
Define the universe discourse of the historical time series
  dataset.
for t_i ∈ U do
  | Apply the neutrosophication process to represent the
  |   historical time series dataset into NS.
  | Compute entropy for the NTS dataset.
  | Establish the NERs among the entropy values.
  | Create the NERGs.
  | Obtain the forecasted values from the NTS dataset.
Output: Forecasted t_i.
```

Step 1.4.3. Compute entropy for the NTS dataset.

[**Explanation:**] If one year's enrollment value is t_i, then its entropy can be obtained using Eq. 1.2.1.7 as:

$$E(N(t_i)) = 1 - \frac{1}{3}\sum_{i=1}^{n}(T_f(t_i) + I_f(t_i) + F_f(t_i)) \times E_1E_2E_3 \quad (1.4.1.4)$$

Here, $E_1 =\mid T_f(t_i) - {T_f}^c(t_i) \mid$, $E_2 =\mid I_f(t_i) - {I_f}^c(t_i) \mid$, and $E_3 =\mid F_f(t_i) - {F_f}^c(t_i) \mid$.

For example, the enrollment value of year 1971 is 13055, whose NS representation is given in Eq. 1.4.1.3. Hence, by following Eq. 1.4.1.4, entropy for the enrollment value 13055, can be obtained as:

$$\begin{aligned} E(N(13055)) &= 1 - \frac{1}{3}(0.026 + 0.974 + 0.974) \times 0.948 \times 0.948 \times 0.948 \\ &= 0.438 \end{aligned}$$

Here, $E_1 =\mid T_f(13055) - {T_f}^c(13055) \mid=\mid 0.026 - 0.974 \mid= 0.948$, $E_2 =\mid I_f(13055) - {I_f}^c(13055) \mid=\mid 0.974 - 0.026 \mid= 0.948$, and $E_3 =\mid F_f(13055) - {F_f}^c(13055) \mid=\mid 0.974 - 0.026 \mid= 0.948$.

In this way, entropy values for each of the NSs are computed using Eq. 1.2.1.7. Entropy value corresponding to each of the enrollment value is shown in Table 1.2.

Step 1.4.4. Establish the NERs among the entropy values.

Table 1.3: NERs for university enrollments.

NERs	NERs	NERs	NERs
$0.438 \rightarrow 0.472$	$0.572 \rightarrow 0.587$	$0.605 \rightarrow 0.646$	$0.733 \rightarrow 0.745$
$0.472 \rightarrow 0.490$	$0.578 \rightarrow 0.572$	$0.625 \rightarrow 0.578$	$0.745 \rightarrow 0.745$
$0.490 \rightarrow 0.537$	$0.578 \rightarrow 0.581$	$0.644 \rightarrow 0.649$	$0.745 \rightarrow 0.730$
$0.537 \rightarrow 0.578$	$0.581 \rightarrow 0.563$	$0.646 \rightarrow 0.701$	-
$0.563 \rightarrow 0.563$	$0.587 \rightarrow 0.599$	$0.649 \rightarrow 0.625$	-
$0.563 \rightarrow 0.605$	$0.599 \rightarrow 0.644$	$0.701 \rightarrow 0.733$	-

Table 1.4: NERGs for university enrollments.

NERGs	NERGs
$0.438 \rightarrow 0.472$	$0.599 \rightarrow 0.644$
$0.472 \rightarrow 0.490$	$0.605 \rightarrow 0.646$
$0.490 \rightarrow 0.537$	$0.625 \rightarrow 0.578$
$0.537 \rightarrow 0.578$	$0.644 \rightarrow 0.649$
$0.563 \rightarrow 0.563, 0.605$	$0.646 \rightarrow 0.701$
$0.572 \rightarrow 0.587$	$0.649 \rightarrow 0.625$
$0.578 \rightarrow 0.572, 0.581$	$0.701 \rightarrow 0.733$
$0.581 \rightarrow 0.563$	$0.733 \rightarrow 0.745$
$0.587 \rightarrow 0.599$	$0.745 \rightarrow 0.745, 0.730$

[**Explanation:**] Based on Definition 1.2.9, we can establish NERs between two consecutive entropy values of enrollments dataset. For example, in Table 1.2, the entropy values for Years 1973 and 1974 are 0.490 and 0.537, respectively. So, we can establish a NER between 0.490 and 0.537 as: $0.490 \rightarrow 0.537$. In this way, we have obtained the NERs for the enrollments dataset, which are presented in Table 1.3.

Step 1.4.5. Create the NERGs.

[**Explanation:**] Based on Definition 1.2.11, the NERs can be grouped into a NERG. For example, in Table 1.3, there are NERs with same previous state as: $0.578 \rightarrow 0.572$, $0.578 \rightarrow 0.581$. Therefore, these NERs can be grouped into the NERG as: $0.578 \rightarrow 0.572, 0.581$. In this way, the NERGs for the remaining NERs are created. These NERGs for the enrollments dataset, are shown in Table 1.4.

Step 1.4.6. Obtain the forecasted values from the NTS dataset.

[**Explanation:**] Deneutrosophication process is applied to obtain the forecasted values from the NTS representation of the historical time series dataset. In the following, a deneutrosophication process is proposed, which includes the following process as:

1. For a particular forecasting day/year t_{i+1}, obtain the corresponding entropy value as: $E(N(t_i))$.

2. Find the NERG for the corresponding $E(N(t_i))$, which can be represented in the following form as:

$$E(N(t_i)) \rightarrow E(N(t_{k1})), E(N(t_{k2})), \ldots, E(N(t_{kn})) \quad (1.4.1.5)$$

   Here, $E(N(t_i))$ is called the previous state of the neutrosophic time series value for the day/year t_i; whereas $E(N(t_{k1})), E(N(t_{k2})), \ldots, E(N(t_{kn}))$ are called the current state of the NTS values for the days/years $t_{k1}, t_{k2}, t_{k3}, \ldots, t_{kn}$, respectively.

3. Obtain the average entropy available in the current state of the NERG as:

$$E(N)_{average} = \frac{1}{n} \sum_{i=1}^{n} E(N(t_{kn})) \quad (1.4.1.6)$$

4. Obtain the actual time series values associated with the previous state's of the NERG as: t_i.

5. Apply the following deneutrosophication formula to calculate the forecasted value for the day/year t_{i+1} as:

$$Forecast(t_{i+1}) = \frac{t_i \times E(N)_{average}}{E(N(t_i))} \quad (1.4.1.7)$$

   Here, the value of $E(N)_{average}$ can be obtained from Eq.1.4.1.6.

In the following, an example is illustrated to obtain the forecasted value of the university enrollments dataset as follows:

Example 1.4.7. Suppose, we want to forecast the enrollment for the year $F(1976)$. For this, find the entropy value for the previous year, i.e., $F(1975)$. From Table 1.2, the entropy value for the year $F(1975)$ is 0.578. Now, obtain the NERG, whose previous state is 0.578. From Table 1.4, the NERG having previous state 0.578 is as: $0.578 \rightarrow 0.572, 0.581$. In this NERG, right hand side entropy values, i.e., "$0.572, 0.581$", are called the current state of the NERG. Now, by using Eq. 1.4.1.6, compute the average entropy available in current state of the NERG as:

$$E(N)_{average} \quad = \quad \frac{(0.572 + 0.581)}{2} \approx 0.577$$

Now, based on Eq. 1.4.1.7, the forecasted enrollment for the year $F(1976)$, can be computed as:

$$Forecast(1976) = \frac{15460 \times 0.577}{0.578} = 15433.25$$

In this way, we have obtained the forecasted values for the enrollments dataset using the proposed model.

In this study, the performance of the proposed model is evaluated with the help of average forecasting error rate ($AFER$). It can be defined as follows:

$$AFER = \frac{\sum_{i=1}^{n} |F_i - A_i|/A_i}{N} \times 100\% \tag{1.4.1.8}$$

In Eq. 1.4.1.8, each F_i and A_i represent the forecasted and actual value of year/day i, respectively; and N is the total number of years/days to be forecasted. A smaller value of $AFER$ indicates a good forecasting accuracy.

1.5 The proposed NTS-PSO model

This section introduced NTS-PSO model for time series forecasting. This model was designed by integrating the PSO algorithm with the NTS model. The main function of the PSO algorithm was to select the optimal universe of discourse by minimizing the error rate. An algorithmic representation this hybrid model is presented in Algorithm 3. The detailed description of the hybrid model is explained next by employing the university enrollments dataset of Alabama.

Step 1.5.1. Select the objective function.

[**Explanation:**] The main objective is to improve the forecasting accuracy of the NTS model, which is possible by selecting the optimal intervals from the universe of discourse U. For this purpose, performance of the model is required to evaluate by adopting suitable evaluation parameter. This parameter can also be used as an objective function. In the literatures, the average forecasting error rate (AFER) was chosen by most of the researchers (Singh, 2015). It is obvious that if the proposed model selects the optimal universe of discourse, then it will definitely minimize the AFER. Now, based on the AFER (as defined in Eq. 1.4.1.8), we can formulate the objective function as:

$$\text{Minimize } AFER = \frac{\sum_{i=1}^{k} \mid A_i - F_i \mid / A_i}{K} \tag{1.5.1.1}$$

subject to the constraints

$$\begin{aligned} (E_{min} - A_N) &\leq \min(U) \\ (E_{max} + A_P) &\leq \max(U) \\ (E_{min} - A_N) &< (E_{max} + A_P) \end{aligned} \tag{1.5.1.2}$$

Here, $(E_{min} - A_N)$ and $(E_{max} + A_P)$ be the lower and upper bounds of the universe of discourse U, where A_N and A_P be the two adjustment factors. Here, the min and max represent the minimum and maximum functions, respectively. Here, K is the total number of days/years to be forecasted.

Step 1.5.2. Define n-number of universe of discourses for the historical time series dataset.

[**Explanation:**] Initially, the universe of discourses are randomly selected that can be represented as: $U_1 = [E_{min} - A_N, E_{max} + A_P]$, $U_2 = [E_{min} - A_N, E_{max} + A_P]$, ..., $U_n = [E_{min} - A_N, E_{max} + A_P]$. Here, the E_{min} and E_{max} are the minimum and maximum enrollments of the university enrollments dataset of Alabama; and the A_N and A_P are the two adjustment factors. For the university enrollments dataset of Alabama, we have performed the experiments on five randomly selected universe of discourses, which can be represented as: $U_1, U_2, \ldots, U_5$.

Step 1.5.3. Define particle for each of the individual universe of discourses $U_1, U_2, \ldots, U_n$.

[**Explanation:**] Assume that $P_1, P_2, \ldots, P_n$ be the number of particles, where each particle is assigned to the individual universe of discourse, i.e., $U_1 \in P_1$, $U_2 \in P_2$, ..., $U_n \in P_n$. Hence, for the university enrollments dataset of Alabama, five different particles are defined for the universe of discourses as: $U_1, U_2, \ldots, U_5$, where $U_1 \in P_1$, $U_2 \in P_2$, ..., $U_5 \in P_5$.

Step 1.5.4. Apply the neutrosophication process to represent the historical time series dataset into NS.