# A Critical Examination of Lotteries

# A Critical Examination of Lotteries:

## *Prize Structures and Probabilities*

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**Cambridge Scholars**
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# INTRODUCTION

When you select fifteen numbers in a keno game, the chance of matching all fifteen out of the twenty numbers drawn is 1 in 428,010,179,098. That's a staggering 1 in 428 billion! Such a number is difficult to comprehend, highlighting just how improbable a perfect match truly is. Keno, like other lottery games, operates under the immutable laws of probability.

Lotteries are played in many countries around the world, attracting millions of hopeful participants who regularly or occasionally purchase tickets in pursuit of life-changing prizes. But how many of these dreamers have a real understanding of the mathematical principles governing their chances? This book is designed to provide essential knowledge about probability and its application to lottery games, empowering readers to comprehend the odds behind winning a prize.

This book is written for a diverse audience—from students eager to delve into the mathematics of chance to casual players curious about how lotteries operate. It offers a foundational introduction to the principles of probability and illustrates how these concepts apply to various lottery systems used globally. While this is not a comprehensive treatise on probability theory, it provides practical insights and simple mathematical rules that readers can easily grasp and use.

In an era where lottery advertisements, online gambling platforms, and betting games constantly entice us, understanding the risks associated with gambling is crucial. Education is a powerful tool to combat compulsive gambling tendencies, and it begins with a fundamental grasp of probability theory and its role in evaluating the likelihood of winning a lottery prize.

This book does not claim to reveal any guaranteed system or secret to winning the lottery. If such a foolproof strategy existed, the authors of countless books on the topic would already be millionaires and unlikely to share their insights for a modest fee. Instead, this work serves as a guide to understanding the reality of lottery odds and fostering informed decision-making.

## Chapter Overview

### Chapter 1: Lotteries, Odds, and Probabilities

The opening chapter introduces the essential concepts required to understand the mathematics of chance, setting the stage for a deeper exploration of lottery systems.

### Chapter 2: Probability of Simple and Compound Events

This chapter explores the probability of simple and compound events through practical examples, explaining the two fundamental principles of addition and multiplication.

### Chapter 3: Methods of Counting

Permutations and combinations, essential techniques for counting outcomes in probability calculations, are examined in this chapter. These methods are particularly useful when dealing with large numbers of possible outcomes.

### Chapter 4: Draws with Replacement

Building on earlier definitions, this chapter classifies lotteries based on the probability principle involved. It explains the concept of draws with replacement—also known as draws with repetition—using illustrative examples.

### Chapter 5: Draws without Replacement—One Draw Machine

This chapter focuses on lotteries without repetition, where draws are made from a single draw machine. Detailed examples help clarify the calculations involved.

### Chapter 6: Draws without Replacement—Two Draw Machines

Extending the previous discussion, this chapter considers lotteries involving two draw machines without repetition. Examples are provided to explain the associated calculations.

## Chapter 7: Selected Topics

The final chapter presents intriguing topics for reflection, challenging readers to consider aspects of lotteries they may never have contemplated before.

By the end of this book, readers will possess a clearer understanding of the principles of probability as they relate to lottery games and will be better equipped to evaluate their chances of winning. More importantly, they will gain a deeper appreciation for the mathematics behind these popular games of chance and the importance of making informed decisions when participating in them.

# CHAPTER 1

# LOTTERIES, ODDS, AND PROBABILITIES

**"O Fortuna,**
**velut luna**
**statu variabilis,**
**semper crescis**
**aut decrescis…"**

"Oh Fortune,
like the moon,
you are changeable,
ever waxing
and waning…"

**Carmina Burana**
(Profane Songs)

## Lotteries

A lottery is a form of gambling based on chance, rather than skill, in which participants pay money for a chance to win one or more prizes. The distribution of prizes is determined by a predefined set of rules outlined in the prize structure. Typically, lotteries offer a range of cash prizes that correspond to different combinations or permutations of the winning numbers. State governments and organizations commonly sponsor lotteries as a means of generating funds. The methods for determining the winning numbers have many variations, but essentially most of them involve matching a combination of numbers drawn by a ball drawing machine, or a number randomly generated by a computer (a random number generator).

Lottery draws are essentially random experiments, characterized by a well-defined set of rules, where the outcomes are entirely dependent on chance and cannot be accurately predicted. Similar games of chance include slot machines (known as poker machines in Australia), instant lotteries (commonly referred to as scratch-off or scratch-and-win), and Keno. Since all these games rely on randomness, they follow the principles outlined in the mathematical theory of probability.

## Chance

Uncertainty, or chance, is an inherent part of our daily lives. When we venture out of our homes in the early morning, we can never be certain of the events that may unfold. Surprisingly, even when we choose to stay at home, unexpected occurrences can still arise!

Chance represents the possibility or probability of various outcomes. It embodies the concept of uncertainty, standing in contrast to certainty. The study of chance falls within the realm of probability theory, a branch of mathematics that offers mathematical tools to understand what one can anticipate when events depend on possibilities.

Having a solid understanding of probability principles can be advantageous when participating in lotteries. Not because these principles can predict the winning numbers, but because they provide guidance on how to approach the game wisely and methodically. By applying probability principles, you can make informed decisions and enhance your overall lottery-playing strategy.

## Probability

Probability is the branch of mathematics dealing with how likely something is to happen. In other words, probability studies mathematical procedures to compute chance.

Probability deals with the likelihood of a given event. The chance that an event might happen is numerically expressed as a fraction and is defined as follows: The probability of an outcome to an event E, provided that all outcomes are equally likely, is the number of cases favourable to that outcome divided by the total number of cases possible.

$$\text{Probability of event } E = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

This definition is condensed in the formula

$$P(E) = \frac{n}{N}$$

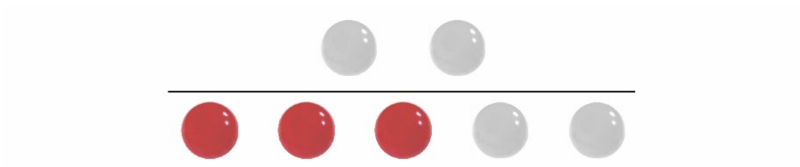
Where:
P(*E*) is the probability of the event E,
n is the number of favourable outcomes, and
N is the total number of possible outcomes.

**Example 1.1**

A box contains three red marbles and two white marbles. When drawing one marble at random, what is the likelihood of obtaining a white marble?

**Solution**

n = 2 (two white marbles)
N = 5 (two white marbles plus three red marbles)



$$P(E) = \frac{n}{N} = \frac{2}{5} = 0.4 = 40\%$$

**Example 1.2**

What is the probability of obtaining heads when tossing a coin?

**Solution**

n (number of favourable outcomes) = 1 (one head side)
N (number of possible outcomes) = 2 (two sides)

$$\mathrm{P}(E) = \frac{\mathrm{n}}{\mathrm{N}} = \frac{1}{2} = 0.5 = 50\%$$

Other examples of events are tossing coins, throwing dice, drawing cards from a standard deck of cards, and drawing marbles from a barrel.

As all outcomes are equally likely, the experiments performed must not be biased: coins must be fair (not bent or weighted) and tossed fairly, dice must be true, decks of cards must be shuffled, and any draw of a card or marble must be done at random.

## The Probability Scale

The probability of an event is expressed as a real number within the interval [0,1]. The lowest probability is zero, which represents an event that can never occur. This is the absolute impossibility of an event. An example of this is the probability of winning a prize (event *A*) in a particular lottery if you did not buy a ticket. It is not possible for you to win any prize.

$$\mathrm{P}(A) = \frac{0}{\mathrm{N}} = 0$$

P(*winning a prize*) = 0

The highest probability is 1, which represents an event that will certainly occur. This is the absolute certainty of an event. For example, if you bought so many tickets that you had in your possession the total number of possible combinations in a particular lottery, the probability that you will have the winning combination (event *B*) is one. You will certainly win.

$$P(B) = \frac{N}{N} = 1$$

P(*winning the jackpot*) = 1

It becomes axiomatic that any probability value must occur between zero and one. The lower the probability, the closer it is to zero. The higher the probability, the closer it is to one. In other words, events that are highly unlikely to occur have a probability close to zero, and events that are highly likely to occur have a probability close to one.

The probability scale is represented in figure 1.1 as an analogy to the mythical Tower of Babel: the probability that people in ancient times were able to build a tower reaching the troposphere (the lowest portion of Earth's atmosphere) was zero.

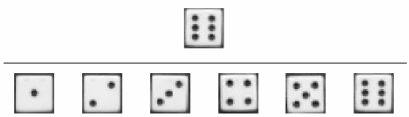
Probability is usually expressed as a fraction or a decimal, but it can be converted into a percentage. Thus, a probability of 2/5 is equivalent to 0.4 or 40 percent.

**Example 1.3**

What is the probability of throwing the number six when rolling a six-sided die?

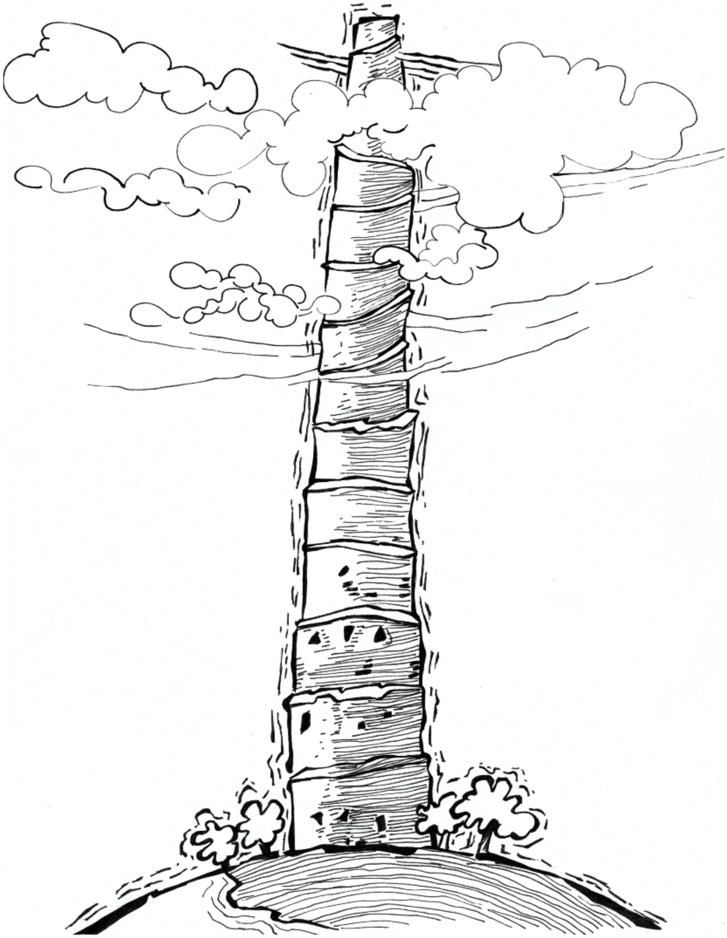
**Solution**

n (number of favourable outcomes) = 1 (one side)
N (number of possible outcomes) = 6 (six sides)



$$P(E) = \frac{n}{N} = \frac{1}{6} = 0.167 = 16.7\%$$

REALM OF DREAMS: WHERE PROBABILITIES MIGHT BE CLOSER TO ZERO



REALM OF REALITY: WHERE PROBABILITIES MIGHT BE CLOSER TO ONE

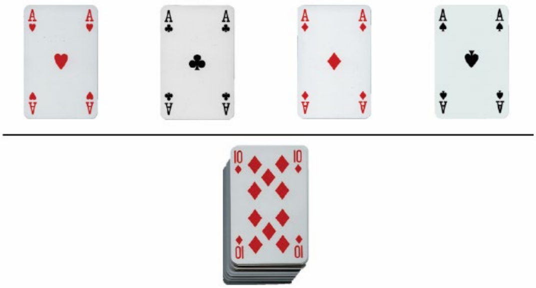
**Fig. 1-1** Probability's Tower of Babel

**Example 1.4**

What is the probability of drawing any ace when selecting one card at random from a full deck of cards?

**Solution**

n (number of favourable outcomes) = 4 (four aces)
N (number of possible outcomes) = 52 (fifty-two cards)



$$P(E) = \frac{n}{N} = \frac{4}{52} = \frac{1}{13} = 0.077 = 7.7\%$$

## Odds

In games of chance probabilities are also expressed in terms of odds. The odds of an event occurring is the ratio of the number of ways the event can occur (successes) to the number of ways the event cannot occur (failures). The odds of an event can also be defined as the ratio of the probability of the event happening to the probability of the event not happening.

**Odds in Favour**

The odds in favour of the event *E* are expressed as the ratio of the number of favourable outcomes to the number of unfavourable outcomes. This is written as

$$O_{for}(E) = \frac{n}{(N-n)}$$

Where:

$O_{for}(E)$ = the odds in favour of event E,

N = total possible outcomes, and
n = favourable outcomes.

**Odds Against**

The odds against the event *E* are expressed as the ratio of the number of unfavourable outcomes to the number of favourable outcomes. This is written as

$$O_{against}(E) = \frac{(N-n)}{n}$$

Where:
$O_{against}(E)$ = the odds against event E,
N = total possible outcomes, and
n = favourable outcomes.

As it might be expected, the odds for the event happening are the reciprocal of those against. Thus, for instance, if event *E* has odds 10 to 1 against, it has odds 1 to 10 in favour.

$$O_{for}(E) \times O_{against}(E) = 1$$

**Example 1.5**

When tossing one coin, what are the odds of obtaining heads?

**Solution**

n (number of favourable outcomes) = 1
N (number of possible outcomes) = 2

$$O_{for}(\text{heads}) = \frac{n}{(N-n)} = \frac{1}{(2-1)} = \frac{1}{1}$$

Also written as 1:1 (read as “1 to 1”)

$$O_{against}(\text{heads}) = \frac{(N-n)}{n} = \frac{(2-1)}{1} = \frac{1}{1}$$

Also written as 1:1 (read as “1 to 1”)

**Example 1.6**

When rolling one die (six-sided), what are the odds of obtaining the number four?

**Solution**

n (number of favourable outcomes) = 1
N (number of possible outcomes) = 6

$$O_{for}(\textit{number four}) = \frac{n}{(N-n)} = \frac{1}{(6-1)} = \frac{1}{5}$$

Also written as 1:5 (read as "1 to 5")

$$O_{against}(\textit{number four}) = \frac{(N-n)}{n} = \frac{(6-1)}{1} = \frac{5}{1}$$

Also written as 5:1 (read as "5 to 1")


**Example 1.7**

One card is drawn at random from a full deck of cards. What are the odds of choosing an ace?

**Solution**

n (number of favourable outcomes) = 4
N (number of possible outcomes) = 52

$$O_{for}(\textit{ace}) = \frac{n}{(N-n)} = \frac{4}{(52-4)} = \frac{4}{48} = \frac{1}{12}$$

Also written as 1:12 (read as "1 to 12")

$$O_{against}(\textit{ace}) = \frac{(N-n)}{n} = \frac{(52-4)}{4} = \frac{48}{4} = \frac{12}{1}$$

Also written as 12:1 (read as "12 to 1")

## Difference Between Odds and Probability

$$\textit{Probability of event } E = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

Remember that the total number of possible outcomes is the sum of the favourable outcomes plus the unfavourable outcomes.

$$\textit{Odds in favour of event } E = \frac{\text{Number of favourable outcomes}}{\text{Number of unfavourable outcomes}}$$

It is a common practice in lottery websites to denote odds as

$$1 \text{ in } \frac{\text{Total number of possible outcomes}}{\text{Number of favourable outcomes}}$$

This expression is actually the probability. For a more detailed explanation see the references below.

## References

Fulton, Lawrence V et al (2012). Confusion between odds and probability, a pandemic? *Journal of Statistics Education*, volume 20, number 3.

Voas, David and Watt, Laura (2024). The odds are it's wrong: Correcting a common mistake in statistics, *Teaching Statistics*, 1–12, DOI 10.1111/test.12391.

# CHAPTER 2

# PROBABILITY OF SIMPLE AND COMPOUND EVENTS

## Simple Events

In statistics and probability theory, simple events are the most basic possible outcomes of an experiment or random process.

A simple event cannot be broken down into smaller parts and represents a single occurrence.

**Characteristics of simple events:**

1. Single outcome: Each simple event corresponds to exactly one specific outcome of the experiment.
2. Mutually exclusive: No two simple events can occur at the same time. If one occurs, it excludes all others.
3. Part of the sample space: Simple events collectively form the sample space, which is the set of all possible outcomes.

**Examples of simple events:**

- Coin toss: The simple events are "Heads" and "Tails".
- Dice roll: For a six-sided die, the simple events are rolling a 1, 2, 3, 4, 5, or 6.
- Card draw: Drawing a specific card, such as the Ace of Spades, is a simple event.

**Probabilities of Simple Events:**

The probability of a simple event, denoted as P($E$), is a measure of the likelihood that the event will occur. For experiments where outcomes are equally likely, the probability of a simple event is:

$$P(E) = \frac{n}{N}$$

Where:
n = number of favourable outcomes
N = total number of possible outcomes

For example:

- In a fair six-sided die, the probability of any simple event (e.g., rolling a 4) is 1/6.

Simple events are foundational in probability, as more complex events are often described as combinations or unions of simple events.

**Example 2.1**

When tossing one coin, what is the probability of obtaining heads?

**Solution**

n (number of favourable outcomes) = 1 (one head side)
N (number of possible outcomes) = 2 (two sides)

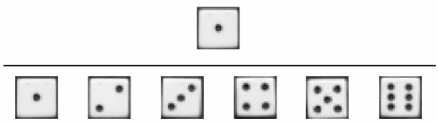

$$P(E) = \frac{n}{N} = \frac{1}{2} = 0.50 = 50\%$$

**Example 2.2**

When rolling one die (six-sided), what is the probability of obtaining the number one?

**Solution**

n (number of favourable outcomes) = 1 (one side)
N (number of possible outcomes) = 6 (six sides)



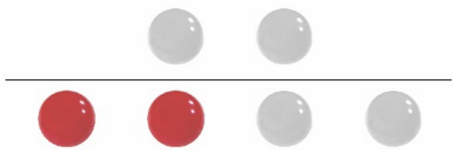
$$P(E) = \frac{n}{N} = \frac{1}{6} = 0.167 = 16.7\%$$

**Example 2.3**

When drawing one marble from a box containing two red marbles and two white marbles, what is the probability of drawing a white marble?

**Solution**

n (number of favourable outcomes) = 2 (two white marbles)
N (number of possible outcomes) = 4 (four total marbles)



$$P(E) = \frac{n}{N} = \frac{2}{4} = \frac{1}{2} = 0.5 = 50\%$$

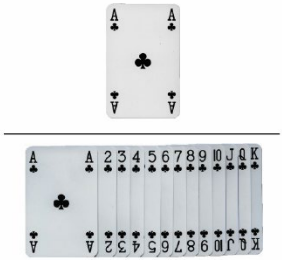
**Example 2.4**

One card is drawn at random from the suit of clubs. What is the probability that the card chosen will be the ace of clubs?

**Solution**

n (number of favourable outcomes) = 1 (the ace of clubs)

N (number of possible outcomes) = 13 (thirteen cards in a suit)



$$P(E) = \frac{n}{N} = \frac{1}{13} = 0.077 = 7.7\%$$

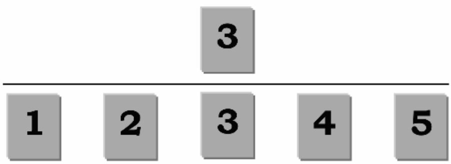

**Example 2.5**

The numbers one through five are each written on one of five cards, and the cards are thoroughly mixed in a bag. One card is drawn at random. What is the probability of drawing the card with the number three on it?

**Solution**

n (number of favourable outcomes) = 1 (one card with number 3)

N (number of possible outcomes) = 5 (five cards)




$$P(E) = \frac{n}{N} = \frac{1}{5} = 0.20 = 20\%$$

## Compound Events

The examples given in the previous section involve the probabilities when considering a single event. But it might happen that two or more events are to be considered when occurring either simultaneously or one after the other.

Then we have compound events, and the addition and multiplication principles are applied to work out the probabilities of such events.

In probability theory, compound events refer to events that are formed by combining two or more simple events. These events can involve multiple outcomes from a single experiment or from multiple experiments. The probability of a compound event depends on the nature of its constituent events and how they are related.

Two main types of compound events are commonly taken from ordinary events: considering two or more different outcomes when performing one experiment and considering two or more different outcomes when performing two or more experiments.

Examples of considering two or more different outcomes when performing *one experiment* include the probabilities of obtaining the following:

- ✓ heads or tails when tossing a coin once,
- ✓ the number one or the number two when throwing a die once,
- ✓ the ace of hearts or the ace of clubs when drawing a card from a suit,
- ✓ a green marble or a blue marble when drawing a marble from a box.

Examples of situations considering two or more different outcomes when performing *two or more experiments,* include the following:

- ✓ tossing a coin twice or more,
- ✓ tossing two or more separate coins simultaneously or one after the other,
- ✓ rolling a die twice or more,
- ✓ rolling two or more separate dice simultaneously or one after the other,
- ✓ drawing two or more cards from a group of cards,
- ✓ drawing two or more marbles from a box.

# The Addition Principle

## Mutually Exclusive Events

A box contains three red marbles and one black marble. When drawing one marble at random, what is the probability that it will be either red or black?

Because it has been stated that the marbles are either red or black, one marble cannot possibly be red and black at the same time. Therefore, when drawing one marble, the occurrence of one event (drawing a red marble) excludes the possibility of the other (drawing a black marble) happening, and vice versa.

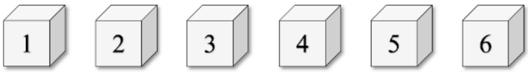
These are mutually exclusive events.

*Two events are mutually exclusive if they cannot occur simultaneously*

The probability of such events is worked out by means of the addition principle, which states that the probability that one or another of several mutually exclusive events happens is the sum of the probabilities of the individual events.

If we have the events *A* and *B*, then

$$P(A \text{ or } B) = P(A) + P(B)$$

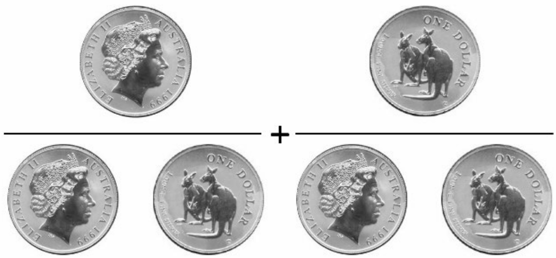


**Example 2.6**

When tossing one coin, what is the probability of throwing either heads or tails?

**Solution**

A = Heads is obtained.
B = Tails is obtained.
E = Either heads or tails is obtained.




$$P(E) = P(A) + P(B) = \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1 = 100\%$$

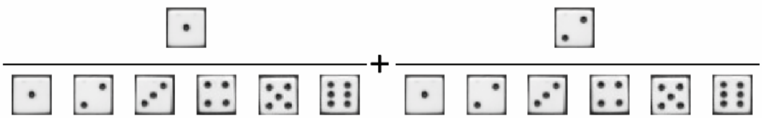
In other words, throwing either heads or tails is a certain event.

**Example 2.7**

When rolling one die (six-sided), what is the probability of getting either one or two?

**Solution**

A = The number one is obtained.
B = The number two is obtained.
E = Either the number one or the number two is obtained.



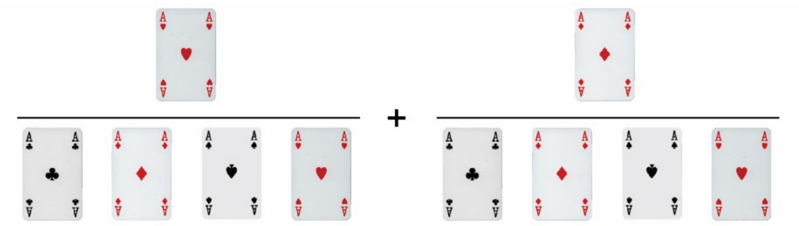

$$P(E) = P(A) + P(B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} = 0.33 = 33\%$$

**Example 2.8**

One card is drawn at random from a set of cards comprising the four aces. What is the probability that the card chosen will be either the ace of hearts or the ace of diamonds?

**Solution**

A = The ace of hearts is obtained.
B = The ace of diamonds is obtained.
E = Either the ace of hearts or the ace of diamonds is obtained.



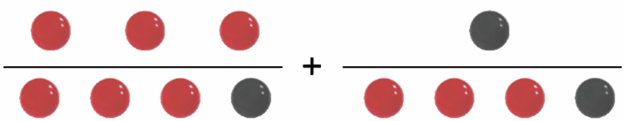
$$P(E) = P(A) + P(B) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2} = 0.50 = 50\%$$


**Example 2.9**

One marble is drawn at random from a box containing three green marbles and one blue marble. What is the probability that it will be either green or blue?

**Solution**

A = A green marble is obtained.
B = A blue marble is obtained.
E = Either a green marble or a blue marble is obtained.



$$P(E) = P(A) + P(B) = \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1 = 100\%$$