Numerical Methods in Structural Engineering and Selected Topics

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By

Mohammed Bin Salem

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PREFACE

Numerical engineering methods are mathematical techniques used to approximate solutions to complex engineering problems that are difficult or impossible to solve analytically. These methods are widely applied in structural analysis, fluid dynamics, thermodynamics, and various other engineering disciplines. Numerical methods allow engineers to solve real-world problems involving nonlinear equations, large datasets, and intricate geometries. They facilitate simulations, optimizations, and analyses that guide the design and testing of engineering systems. Common numerical methods in engineering are:

1- Finite Difference Method (FDM)

FDM is used to approximate derivatives by replacing them with finite difference equations. It is commonly applied in solving differential equations related to heat conduction, wave propagation, and structural analysis.

2- Finite Element Method (FEM)

FEM divides a complex structure into smaller elements and applies numerical techniques to solve governing equations. It is extensively used in mechanical and civil engineering for stress analysis, dynamic analysis, and thermal simulations.

3- Finite Volume Method (FVM)

FVM is commonly used in fluid mechanics and computational fluid dynamics (CFD). It conserves fluxes across control volumes and is suitable for solving partial differential equations governing fluid flow and heat transfer.

4- Newton-Raphson Method

This iterative method is used to solve nonlinear equations by approximating roots. It is widely applied in circuit analysis, power flow studies, and optimization problems.

5- Runge-Kutta Methods

These methods are used for solving ordinary differential equations (ODEs). They offer improved accuracy compared to basic numerical integration techniques such as Euler's method.

6- Applications of Numerical Methods

- **Structural Analysis:** Predicting stress, strain, and deformation in materials.
- **Fluid Dynamics:** Simulating airflow over aircraft wings, pipe flows, and turbulence modeling.
- **Thermal Analysis:** Evaluating heat transfer in mechanical and electronic systems.
- **Control Systems:** Designing and analyzing controllers in automation and robotics.
- **Optimization Problems:** Finding optimal design parameters for engineering projects.

7- Advantages and Limitations

Advantages:

- Enables solutions to complex engineering problems.
- Allows simulations before physical prototyping, reducing costs.
- Provides high accuracy when implemented correctly.
- Can handle nonlinear and dynamic systems.

Limitations:

- Computationally intensive and requires high processing power.
- Accuracy depends on step size and discretization.
- Errors can propagate if not properly managed.
- Requires expertise to implement and interpret results.

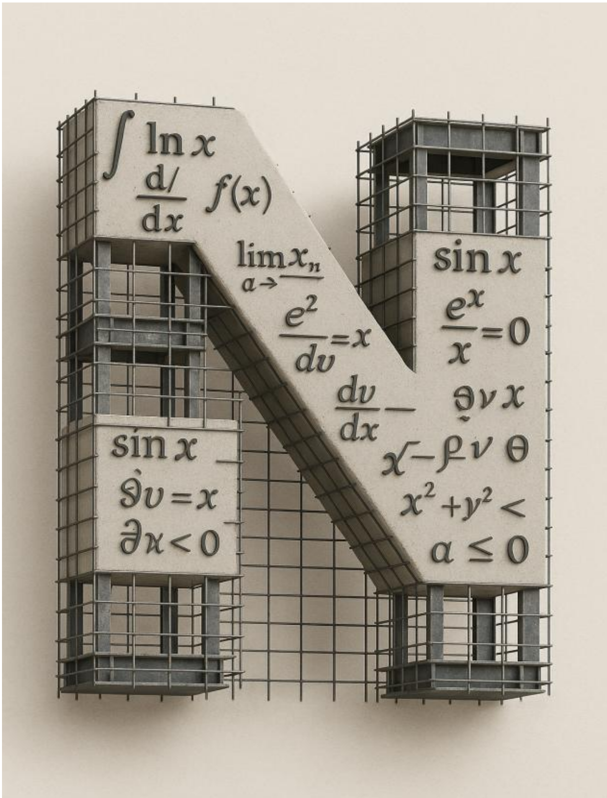
Numerical engineering methods are indispensable in modern engineering practice. Their ability to approximate solutions to complex problems makes them essential in research, design, and manufacturing processes. With advancements in computational power and algorithms, numerical methods will continue to evolve, providing even more accurate and efficient solutions for engineering challenges.

This book is a collection of the author thoughts, ideas, lecture notes, exams, teaching, consulting work performed for design and construction and supervising experience for 31 years. The goals of this book are to know how to use numerical methods for the preliminary design of a safe and economical structure, providing numerical examples in SI units and US customary units for students and engineers to understand in clear and easy way the numerical preliminary design procedure of the structure's members such beams, columns, slabs and footings. In addition to dynamic loads. This book is divided into 4 chapters:

- ❖ Chapter 1 Introduction to Numerical Methods.
- ❖ Chapter 2 Numerical Structural Analysis.
- ❖ Chapter 3 Numerical Structural Design.
- ❖ Chapter 4 Numerical Selected Topics.
- ❖ Appendices

CHAPTER 1

INTRODUCTION



1.0 Introduction

Numerical analysis skills include the ability to formulate, analyze, and implement numerical algorithms that solve engineering problems. In numerical analysis, a numerical method is a mathematical tool designed to solve numerical problems. The implementation of a numerical method with an appropriate convergence check in a programming language is called a numerical algorithm:

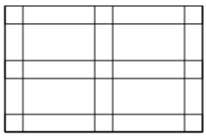
1- **Algorithm**s are procedures that were developed to solve mathematical problems in a finite number of steps.
2- **Numerical Method** is an algorithm that is used to obtain numerical solution of a mathematical problem.
3- **We need numerical** for:
   a- Analytical solution is not available
   b- Analytical solution is difficult to obtain or not practical.
4- **Mathematical model** is the formulation or equation that expresses the essential features of a physical system or process in mathematical terms. As an example:

   $F = ma$, F is the force acting on the body, m is the mass of the object and a is acceleration.

5- **Engineering problem solving process**:
   a- Problem definition
   b- Mathematical model
   c- Numerical solution:
      1- Numerical results
      2- Graphic results
   d- Implementation

   As an Example (Civil Engineering):

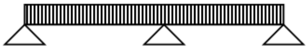
   a- Problem definition➔ Design Continuous beam B2



B2

Figure 1-1: Continuous Beam B2

b- Structural model



c- Graphic and Numerical results

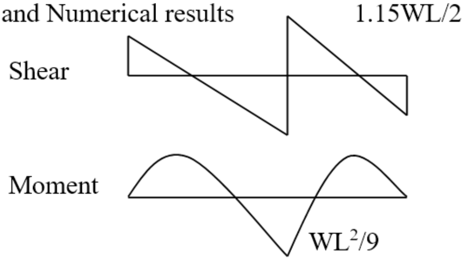


Figure 1-2: Shear and Moment Diagram

d- Implementation ➔ Design the beam using the values of shear and moment.

6- ERRORS

Error = $\frac{Exact\ Value - Approximate\ Value}{Exact\ Value} \times 100\%$

Or

Error = $\frac{Present\ Approximation - Previous\ Approximation}{Present\ Approximation} \times 100\%$

Stopping criterion ➔ $|Error\ | < \epsilon s$

Types of errors:

1- Round off errors (π,e,$\sqrt{7}, \dots \dots .\, etc.$)
2- Truncation errors result from using an approximation in place of exact mathematical procedure.

$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta v} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

3- Total Error = Round off errors + Truncation errors

Accuracy is how close the computed value to the true value.

Precision is how close the computed value to the previously computed value.

Inaccuracy is a systematic deviation from the actual value, Divergence.

If the results get closer and closer to the actual value that is **Convergence**.

1.1 Roots of Algebraic Equations

Roots are values of the variable x that satisfy f(x) = 0 for a given function.

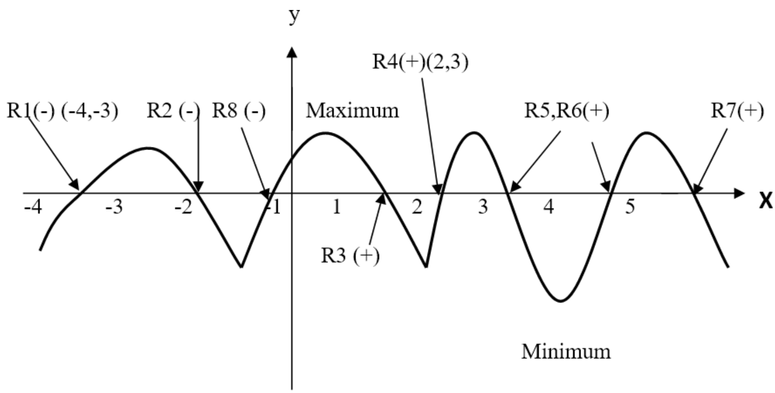


Figure 1-3: Roots Of the function f(x)

Note:

1- Chang of sign in the function(y) means that a Root exist within X_{LOWER} & X_{UPPER}

2- Brackets ➔ Root limits (X_L , X_U), for example R4 ➔ (X_L =2, X_U=3)

3- R1, R2, R8 are Negative roots, R3,R4,R5,R6,R7 are positive roots

4- Minimum➔ f'(x) =0, f''(x)>0

5- Maximum➔ f'(x)=0, f''(x)<0

Other types of roots:

f(x) = g(x) @ x=Root

Mathematical Model ➔
f(x)-g(x)=M(x)

M(x)=0 @ x=Root

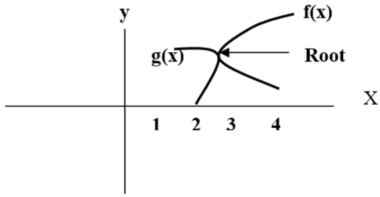


Figure 1-4: Root Of the functions f(x) and g(x)

Methods of determining the roots:

1- Incremental search method (ISM) or Direct search method.

2- Bisection Method or Bolzano or Half interval method. Slow but sure convergence.

3- False position method or Linear interpolation method. Slow convergence.

4- Secant Method. Fast convergence.

5- Newton Raphson Method. Fast convergence.

6- Newton Raphson Method 2nd order method.

7- Roots of Polynomial of order n.

8- Fixed Point Method.

9- Shooting Method.

Roots Example 1:

Determine the root of the function $f(x) = x^3 + 3x^2 - 9x + 1 \rightarrow [1,2]$ using:

- a- I.S.M. with Δx = 1 , 0.5 , 0.25
- b- Bisection, [1,2], Δx = 0.5
- c- False Position, [1,2], n=3
- d- Secant, [1,2], n=3
- e- Exact root for the interval [1,2] is 1.7687

Solution:

a- I. S. M. with Δx = 1 , n = number of iterations:

Table 1-1: I.S.M Iteration Results

n	X_n	f(x)	Remarks
0	1	-4	< 0
1	2	3	> 0
			R➔ { 1,2}

b- I.S. M. with Δx = 0.5

Table 1-2: I.S.M Iteration Results

n	X_n	f(x)	Remarks
0	1	-4	< 0
1	1.5	-2.375	< 0
2	2	3	> 0
			R➔ { 1.5,2}

c- I. S. M. with Δx = 0.25

Table 1-3: I.S.M Iteration Results

n	Xn	f(x)	Remarks
0	1	-4	< 0
1	1.25	-3.6	< 0
2	1.5	-2.375	< 0
3	1.75	-0.203	< 0
4	2	3	> 0
			R➔ { 1.75,2}

$$\boldsymbol{Root} = Xu - \left[\frac{\Delta x(f(xu)}{f(Xu) - f(Xu - \Delta x)}\right]$$

Table 1-4: I.S.M Iteration Results

$\Delta \boldsymbol{x}$	**Xu**	**Root**	$[\boldsymbol{X_U}, \boldsymbol{X_L}]$
1	2	1.57	[1,2]
0.5	2	1.721	[1.5,2]
0.25	2	1.766	[1.75,2]

d- <u>Bisection Method :</u>

$$\boldsymbol{Xr} = \frac{XU + XL}{2}$$

f(XL). f(Xr) < 0 ➔ XU = Xr $\qquad \epsilon_S = \left|\frac{\boldsymbol{Xr(n) - Xr(n-1)}}{\boldsymbol{Xr(n)}}\right|$

f(XL). f(Xr) > 0 ➔ XL = Xr

Table 1-5: Bisection Method Iteration Results

n	**XL**	**XU**	**Xr**	**f(XL)**	**f(Xr)**	**f(XL) . f(Xr)**
0	1	2	1.5	-4	-2.375	> 0
1	1.5	2	1.75	-2.375	-0.203	> 0
2	1.75	2	1.875	-0.203	1.26	< 0
3	1.75	1.875	1.8125	-0.203	0.497	< 0
4	1.75	1.8125	1.78125	-0.203	0.1389	< 0
5	1.75	1.78125	1.7656	-0.203	-0.034	> 0
6	1.7656	1.78125	1.7734	-0.034	0.05	< 0

Root = 1.7734 $\qquad \epsilon_S$ **= 0.441 < 0.5**

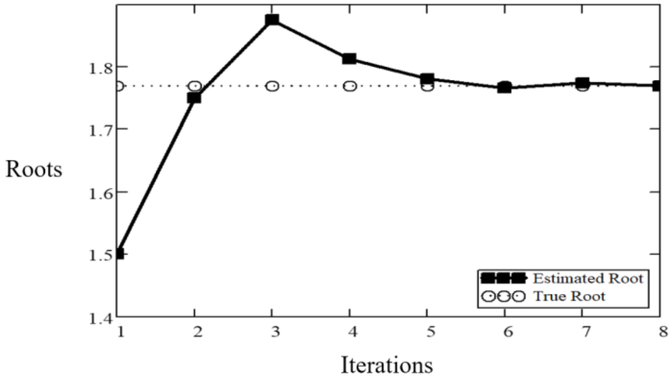


Figure 1-5: Roots Of the function f(x) - Bisection Method

e- <u>False Position Method:</u>

f(XL). f(Xr) < 0 ➔ XU = Xr **Xr = Xu –** $\frac{f(XU)(XL-XU)}{f(XL)-f(XU)}$

f(XL). f(Xr) > 0 ➔ XL = Xr

Table 1-6: False Position Method Iteration Results

n	XL	XU	Xr	f(XL)	f(XU)	F(Xr)	f(XL).f(Xr)
0	1	2	1.57	-4	3	-1.86	> 0
1	1.57	2	1.734	-1.86	3	-0.372	> 0
2	1.734	2	1.764	-.372	3	-0.052	> 0
3	1.764	2	1.768	-0.052	3	-0.008	> 0

R= 1.768

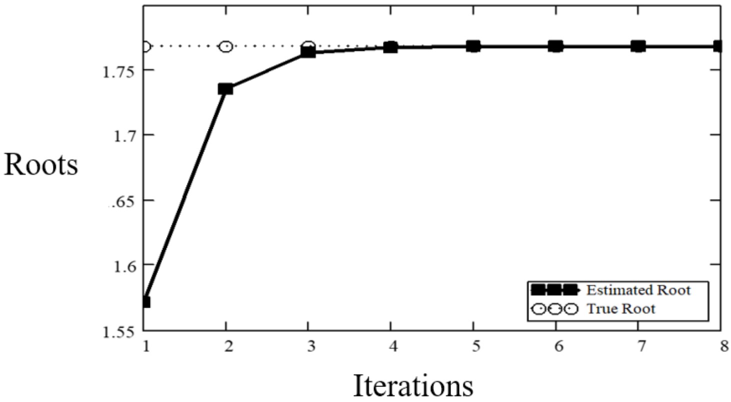


Figure 1-6: Roots Of the function f(x) - False Position Method

f- <u>Secant Method :</u>

$$Xr = X2 - \frac{f(X2)(X1-X2)}{f(X1)-f(X2)}$$

Table 1-7: Secant Method Iteration Results

n	X1	X2	Xr	f(X1)	f(X2)
0	1	2	1.57	-4	3
1	2	1.57	1.734	3	-1.86
2	1.57	1.734	1.775	-1.86	-.372
3	1.734	1.77	1.769	-.372	0.065

R= 1.769

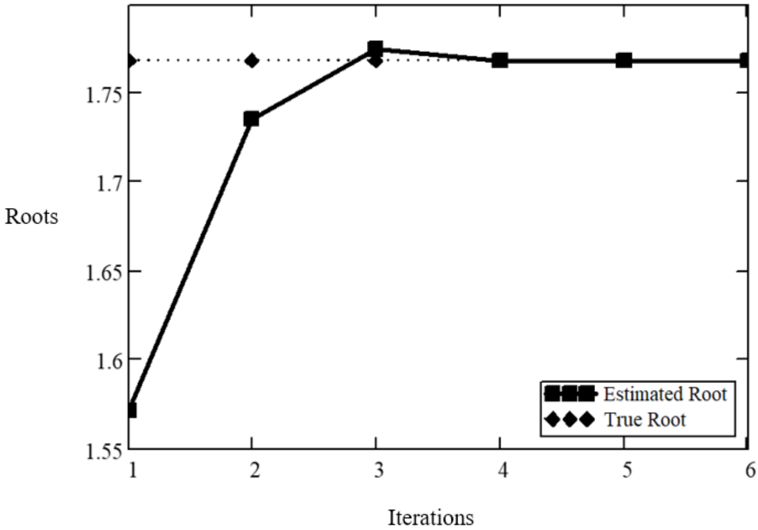


Figure 1-7: Roots Of the function f(x) - Secant Method

Roots Example 2:

Determine the root of the function $f(x) = \mathbf{2x^2 - e^x + 1}$ → [0,1] using:

1- Newton Raphson Method
2- Newton Raphson Method 2nd Order
3- Exact root is 0.741

Solution:

$$F(x) = 2x^2 - e^x + 1$$

$$F`(x) = 4x - e^x$$

a- Newton – Raphson Method, n = number of iterations:

Table 1-8: Newton-Raphson Method Iteration Results

n	**x**	**f(x)**	**f`(x)**	**Δx=** $\frac{f(x)}{f'(x)}$
0	1	0.282	1.282	0.22
1	1-0.22 = 0.78	0.036	0.939	0.038
2	0.78-0.038= 0.742	0		

Root = 0.742

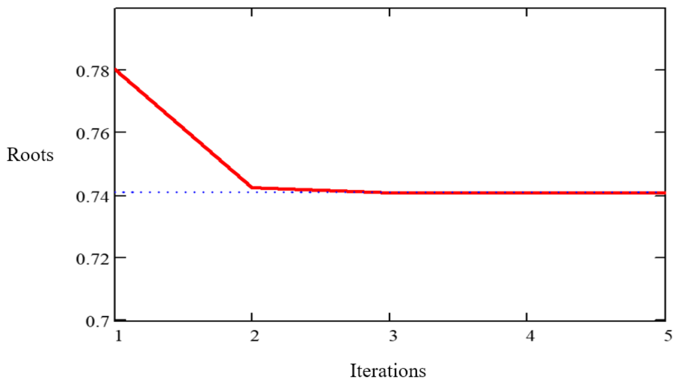


Figure 1-8: Roots Of the function f(x) - Newton – Raphson Method

b- Newton – Raphson Method , 2nd order Method:

$$F(x) = 2x^2 - e^x + 1$$

$$F'(x) = 4x - e^x$$

$$F''(x) = 4 - e^x$$

Table 1-9: Newton – Raphson 2nd order Method Iteration Results

n	x	*f(x)*	f' (x)	f''(x)	$\Delta x = \dfrac{f'(x)}{f'(x) - \dfrac{f''(x) \times f(x)}{2f'(x)}}$
0	2	1.611	0.611	-3.389	0.3172
1	1.683	1.283	1.351	-1.381	0.639
2	1.043	0.338	1.335	1.161	0.285
3	0.758	0.015	0.899	1.865	0.017
4	0.741	0	0.866	1.902	0

Root = 0.741

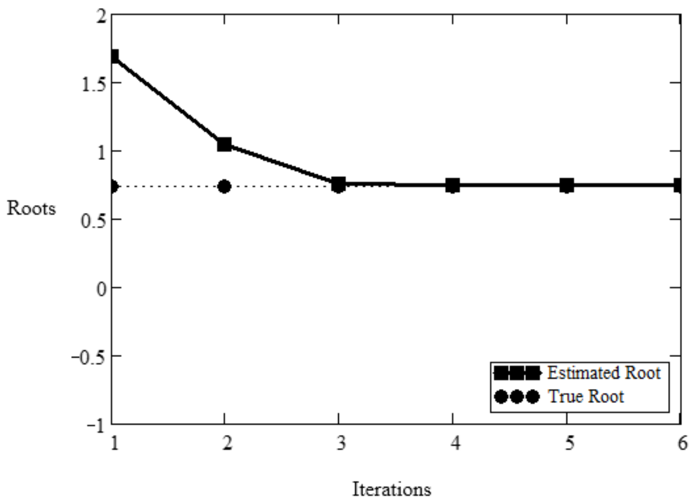


Figure 1-9: Roots Of the function f(x) - Newton – Raphson Method 2nd Order

1.2 Linear Algebraic Equations

A linear equation is an algebraic equation where each term has an exponent of 1 and when this equation is graphed, it always results in a straight line. Linear algebraic equation is written in the form ax + b = 0 or ax + by + c = 0, where a, b and c are real numbers and x and y are variables.

Methods of solving Linear Algebraic Equations:

1- Gauss Elimination Method
2- L U Factorization Method
3- Cramer's Rule Method
4- Gauss Seidel Method

1.2.1 Gauss Elimination Method

Gauss elimination, in linear and multilinear algebra, a process for finding the solutions of a system of simultaneous linear equations by first solving one of the equations for one variable (in terms of all the others) and then substituting this expression into the remaining equations.

Gauss Elimination Method Example:

Use Gauss Elimination to solve the linear equations

$3x1 + 18x2 + 9x3 = 18$

$2x1 + 3x2 + 3x3 = 117$

$4x1 + x2 + 2x3 = 283$

Solution:

1- Construct the matrix Matrix [A]

$$[A] = \begin{bmatrix} 3 & 18 & 9 & 18 \\ 2 & 3 & 3 & 117 \\ 4 & 1 & 2 & 283 \end{bmatrix} \rightarrow \begin{bmatrix} a11 & a12 & a13 & b1 \\ a21 & a22 & a23 & b2 \\ a31 & a32 & a33 & b3 \end{bmatrix}$$

2- Eliminate a21

Divide 1st row by a11➔ $\frac{3 \quad 18 \quad 9 \quad 18}{3} = 1\ \ 6\ \ 3\ \ 6$

Multiply the results by -a21 ➔ -2 -12 -6 -12

Add the results to the 2nd row and have a new matrix

$$[A1]\begin{bmatrix} 3 & 18 & 9 & 18 \\ 0 & -9 & -3 & 105 \\ 4 & 1 & 2 & 283 \end{bmatrix}$$

3- Eliminate a31

Divide 1st row by a11➔ $\frac{3 \quad 18 \quad 9 \quad 18}{3} = 1\ \ 6\ \ 3\ \ 6$

Multiply the results by -a31 ➔ -4 -24 -12 -24

Add the results to the 3rd row and have a new matrix

$$[A2] = \begin{bmatrix} 3 & 18 & 9 & 18 \\ 0 & -9 & -3 & 105 \\ 0 & -23 & -10 & 259 \end{bmatrix}$$

4- Eliminate a32

Divide 2nd raw in matrix [A2] by a22➔ $\frac{0 \quad -9 \quad -3 \quad 105}{-9} = 0\ \ 1\ \ \frac{1}{3}\ \ \frac{-105}{9}$

Multiply the results by -a32 ➔ $0 \quad 23 \quad \frac{23}{3} \quad \frac{-2415}{9}$

Add the results to the 3rd row in [A2] and have a new matrix

$$[A3] = \begin{bmatrix} 3 & 18 & 9 & 18 \\ 0 & -9 & -3 & 105 \\ 0 & 0 & -\frac{7}{3} & -\frac{28}{3} \end{bmatrix}$$

5- Back Substitution

$$x3\left(-\frac{7}{3}\right) = -\frac{28}{3} \Rightarrow x3 = 4$$

$$x2(-9) - 3(\,x3 = 4\,) = 105 \;\Rightarrow x2 = -13$$

$$x1(3) + 18(x2 = -13) + 9(\,x3 = 4\,) = 18 \Rightarrow x1 = 72$$

Note that the Gauss Elimination procedure is to construct matrix A3 from matrix A:

$$[A] = \begin{bmatrix} 3 & 18 & 9 & 18 \\ 2 & 3 & 3 & 117 \\ 4 & 1 & 2 & 283 \end{bmatrix} \rightarrow [A3] = \begin{bmatrix} 3 & 18 & 9 & 18 \\ 0 & -9 & -3 & 105 \\ 0 & 0 & -\dfrac{7}{3} & -\dfrac{28}{3} \end{bmatrix}$$

1.2.2 L U Factorization Method

A factorization of a matrix A in the form A = LU, where L is unit lower triangular and U is upper triangular, is called an LU factorization of A.

Matrix [A] = $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$

Decompose [A] into [U] and [L] using factorization methods:

1- Upper triangular matrix [U] = $\begin{bmatrix} u11 & u12 & u13 \\ 0 & u22 & u23 \\ 0 & 0 & u33 \end{bmatrix}$

2- Lower triangular matrix [L] = $\begin{bmatrix} l11 & 0 & 0 \\ l21 & l22 & 0 \\ l31 & l32 & l33 \end{bmatrix}$

3- [L] [U] = [A]

4- System of linear equations: [A] [X] = [b]
- a- [L] [d] = [b]
- b- [U] [X] = [d]
- c- Back Substitution ➔ [X]

L U Factorization Method Example:

6x1 + 15x2 + 55x3 = 12

15x1 + 55x2 + 225x3 = 80

55x1 + 225x2+ 979x3 = 160

Solve the system using [U] and [L] matrices, CHOLESKY Method.

Solution:

$$[A] = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \qquad b = \begin{bmatrix} 12 \\ 80 \\ 160 \end{bmatrix}$$

1- Upper triangular matrix [U]:

a- First Row (i=1) $u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ii}{}^2} \quad = \quad 10.15$

$$u_{11} = \sqrt{a_{11}} = \sqrt{6} = 2.44949$$

$$u_{12} = \frac{a12}{u11} = \frac{15}{2.44949} = 6.123724$$

$$u_{13} = \frac{a13}{u11} = \frac{55}{2.44949} = 22.45366$$

b- Second Raw (i=2)

$$u_{22} = \sqrt{a22 - (u12)^2} = \sqrt{55 - (6.123724)^2} = 4.1833$$

$$u23 = \frac{a23 - u12\,u13}{u22} = \frac{225 - 6.123724(22.45366)}{4.1833} = 20.9165$$

c- Third Row (i=3):

$$u_{33} = \sqrt{a33 - (u13)^2 - (u23)^2}$$
$$= \sqrt{979 - (22.45366)^2 - (20.9165)^2} = 6.110101$$

THUS, The Cholesky method yields

$$[U] = \begin{bmatrix} 2.44949 & 6.123724 & 22.45366 \\ 0 & 4.1833 & 20.9165 \\ 0 & 0 & 6.110101 \end{bmatrix}$$

2- Lower triangular matrix [L] :

a- First Row (i=1) : $l11 = \sqrt{a_{11}} = \sqrt{6} = 2.4494$

b- Second Row (i=2) : $l_{21} = \frac{a21}{l11} = \frac{15}{2.44949} = 6.123724$

$$l_{22} = \sqrt{a22 - (u21)^2} = \sqrt{55 - (6.123724)^2} = 4.1833$$

c- Third Row (i=3) : $l_{31} = \frac{a31}{l11} = \frac{55}{2.44949} = 22.45366$

$$l32 = \frac{a32 - l21\, l31}{l22} = \frac{225 - 6.123724(22.45366)}{4.1833} = 20.9165$$

$$l_{33} = \sqrt{a33 - (l31)^2 - (l32)^2} = \sqrt{979 - (22.45366)^2 - (20.9165)^2} = 6.110101$$

THUS, The Cholesky method yields

$$[L] = \begin{bmatrix} 2.44949 & 0 & 0 \\ 6.123724 & 4.1833 & 0 \\ 22.45366 & 20.9165 & 6.110101 \end{bmatrix}$$

3- The [d] :

$$d1 = \frac{b1}{l11} = \frac{12}{2.4494} = 4.89916$$

$$d2 = \frac{b2 - l21\ d1}{l22} = \frac{80 - 6.123724 * 4.89916}{4.1833} = 11.95202$$

$$\begin{aligned} d3 = \frac{b3 - l31\ d1 - l32\ d2}{l33} \\ &= \frac{160 - 22.45366 * 4.89916 - 20.9165 * 11.95202}{6.110101} \\ &= -32.732437 \end{aligned}$$

[U] [X] = [d]

$$\left[\begin{bmatrix} 2.44949 & 6.123724 & 22.45366 \\ 0 & 4.1833 & 20.9165 \\ 0 & 0 & 6.110101 \end{bmatrix}\right] \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 4.89916 \\ 11.95202 \\ -32.732437 \end{bmatrix}$$

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} -22.9996 \\ 29.64259 \\ -5.3571 \end{bmatrix}$$

1.2.3 Cramer's rule Method

Cramer's rule is a specific formula used for solving a system of linear equations containing as many equations as unknowns, efficient whenever the system of equations has a unique solution.

Cramer's rule Method Example:

0.3x1 + 0.52x2 + 1x3 = -0.01

0.5x1 + 1x2 + 1.9x3 = 0.67

0.1x1 + 0.3x2+ 0.5x3 = -0.44

Solve the system using Cramer's rule Method.