

# Stochastic Processes of Risk Theory and Storage Theory in Physics and Biology



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By

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## PREFACE

Stochastic processes are ubiquitous in nature. They play a decisive role in the description of both physical phenomena and biological systems. The practical significance of these processes is very great. For example, they enable us to describe complex environments that we cannot fully control and whose description is not only a rich field for theoretical research but is also of decisive importance for the correct processing of practical applications. Stochastic processes are widely and effectively used in various physical and biological problems [1, 2]. This book contains articles that are directly related to this important topic.

The collection consists of four sections. The first section considers the issues of the application of boundary functionals in risk theory and the connection of the theory of stochastic processes with the thermodynamics of trajectories. Boundary functionals of random processes, i.e., functionals from selective trajectories associated with the exit of these trajectories beyond certain boundaries, describe numerous phenomena occurring in nature. Risk theory, applied mainly in economics, is of a general mathematical nature. It is closely related to such disciplines as queue theory and stochastic storage processes. In all these mathematical subsections of the theory of stochastic processes, some elements (particles) enter the system in accordance with certain patterns and leave the system according to other patterns. A similar description is made by kinetic equations in physics. The thermodynamics of trajectories is described in the articles included in the collection. The last article of the first section characterizes such boundary functionals as the time of the first reaching of the level and the value of overjump for random processes for general upper bounds on fluctuations of trajectory observables. The time of the first reaching of the level (target) is the most used boundary functional. Obtaining explicit expressions for the time of the first reaching for a random process for general upper bounds on fluctuations of trajectory observables requires using the results of risk theory, which was done.

The second section contains articles devoted to the influence of entropy changes on the behavior of the first reaching time. General relationships with numerous examples are considered, as well as the application of the

obtained expressions to nuclear reactors, electron transfer between quantum islands, and problems of tribology and thermodynamics of trajectories. Entropy changes accompanying nonequilibrium processes, including the processes of the first reaching of the level, also affect the moments of the process of the first reaching of the level and the average value of this quantity. Entropy changes include intrasystem changes and entropy exchange with the environment. External influences that form the entropy exchange with the environment can influence the speed and intensity of the processes of the first reaching of the level.

The third section contains articles on statistical distribution containing the time of first reaching as a thermodynamic parameter. On the basis of such a distribution, one can construct nonequilibrium thermodynamics, the results of which are compared in [3] with the results of statistical thermodynamics based on the method of nonequilibrium statistical operator and with the results of extended nonequilibrium thermodynamics. In trajectory thermodynamics, the cumulants of the statistical sum of the statistical distribution are written as the maximum eigenvalues of the kinetic equation. Other boundary functionals of random processes can also be thermodynamic variables and appear in the statistical distribution, as well as the time of first reaching the level. The articles from the third section consider in more detail the connections of the introduced distribution containing the time of first reaching the level with the method of nonequilibrium statistical operator and the superstatistics approach [4].

The articles in the fourth section consider various applications of stochastic storage processes, which, as noted, are closely related to risk processes. Noise-induced phase transitions, stochastic modeling of neutron processes in a nuclear reactor (in this case, a generalization of stochastic models of storage is carried out) and other applications are considered, in particular, to the behavior of aerosol systems, to radiation damages and durability of materials under radiation exposure, to a model of domain kinetics in biological membranes, and to stochastic eco-modelling: evolution of plants. The diversity of applications is explained by the general mathematical nature of stochastic storage processes.

The main results of this collection: a). Different approaches to physical and biological problems can be equivalent in a narrow range of parameters of stochastic processes; b). Such mathematical disciplines as risk theory, queueing theory, and stochastic storage theory can be very effective for solving various physical and biological problems, although their application is not very widespread; c). The articles in the first section note that many

results of the theory of stochastic processes can be effective in studying issues of statistical physics; d). Stochastic storage theory provides great opportunities for studying physical and biological systems.

## References

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# SECTION 1

## BOUNDARY FUNCTIONALS AND RISK THEORY IN STATISTICAL PHYSICS

### **1.1. Application of boundary functionals of random processes in statistical physics**

#### *Abstract*

The potential applications of boundary functionals of random processes, such as the extreme values of these processes, the moment of first reaching a fixed level, the value of the process at the moment of reaching the level, the moment of reaching extreme values, the time the process stays above a fixed level, and other functionals, are considered for the description of physical, chemical, and biological problems. Definitions of these functionals are provided, and characteristic functions are presented for the model with an exponential distribution of incoming demands. A generalization of these limitations is also considered. The potential uses of boundary functionals are demonstrated through examples such as a unicyclic network with affinity  $A$ , an asymmetric random walk, nonlinear diffusion, a two-level model, Brownian motion, and multiple diffusing particles with reversible target-binding kinetics.

#### *1. Introduction*

Random walks and processes with independent increments on finite Markov chains, as well as semi-Markov processes described by sums of random variables, generalize classical homogeneous processes with independent increments and are used to model queueing theory, risk theory, stochastic storage theory, and reliability theory. These processes are also widely used to solve various physical, chemical, and biological problems [1-8].

Boundary value problems for these processes are of considerable theoretical and practical interest, as they involve studying the distributions of functionals such as extreme values of the processes, the moment of first

reaching a fixed level, the value of the process at the moment of reaching that level, the moment of reaching extreme values, the time the process spends above a fixed level, and other related functionals. For example, key characteristics studied in risk theory are closely related to the distributions of boundary functionals for homogeneous processes with independent increments, such as extrema over finite intervals, absolute extrema, jump functionals, and others. Different aspects of the study of boundary functionals, using various mathematical approaches, are described in many works (see, e.g., Refs. [1-21, 66, 67]).

The most extensively studied boundary functional is the first-passage time (*FPT*) [22-58], which is used across various fields such as physics, chemistry, biology, and economics. The volume of work on *FPT* research exceeds several thousand papers. The list of existing and emerging areas of *FPT* research is also extensive. Many articles and books are also dedicated to the study of extreme values of random functions and processes. However, most of these applications pertain to mathematical statistics and related fields, while applications to physical or biological problems are relatively fewer, though still more numerous compared to other boundary functionals. Other boundary functionals are less frequently used compared to *FPT*, although their importance is arguably not less significant. In works [59-65], *FPT* was incorporated into statistical distributions as a thermodynamic parameter. It is possible to replace *FPT* with other boundary functionals in these distributions and apply them to study the behavior of a physical quantity chosen within the statistical distribution. For example, one could examine the effect of entropy change on the chosen functional, as was done for *FPT* in papers [60-62, 64-65]. A promising area of research could be the combined application of several boundary functionals to a given phenomenon.

In this article, we build on risk theory, whose primary application lies in the financial sector. However, risk theory is closely related to other fields such as renewal theory and applied mathematical disciplines like queueing theory, stochastic storage theory, and reliability theory. These disciplines often solve their problems by referring to risk theory, and conversely, risk theory draws on general mathematical principles and finds applications in these areas. Risk theory, as its name suggests, examines phenomena such as the presence of a risk process in survival and risk zones. This type of behavior is relevant to many physical, chemical, and biological processes and phenomena.

In this article, several boundary functionals are examined, with definitions provided and their behavior illustrated in relation to the conjugate thermodynamic parameter, which acts as a thermodynamic force. Their physical significance is explained. Some boundary functionals apparently have not been applied to physical problems before. Sections 4 and 5 are specifically devoted to examples of their potential applications in physical problems. The use of these functionals in risk theory, economics, queueing theory, and other applied mathematical disciplines has long been successfully practiced. Extreme value statistics are also applied in a wide range of fields, including physics.

The paper is organized as follows. Section 2 provides definitions of several boundary functionals. Section 3 presents the characteristic functions (*CF*) and moment-generating functions of these functionals for the exponential distribution model of incoming demands. Section 4 explores the application of these functionals to a unicyclic network with affinity  $A$  and to an asymmetric random walk. Section 5 applies the sojourn time functional above a given level to nonlinear diffusion and to multiple diffusing particles with reversible target-binding kinetics. Other tasks have been considered, particularly the two-level model and Brownian motion. The approximations made are discussed. Finally, Section 6 offers brief concluding remarks.

### Notations and definitions

The article considers random processes with independent increments  $\{\xi(t), 0 \leq t \leq T\}$  defined on the probability space  $\{\Omega, F, P\}$  (works [9-10]) with real values in the phase space  $(R, \mathcal{B})$ ,  $R = (-\infty, \infty)$ ,  $\mathcal{B}$  is the  $\sigma$ -algebra of Borel sets on the line  $R$ . Here  $\Omega$  is the space of elementary events. For example,  $\Omega = \{\omega_1, \omega_2, \dots\}$ ,  $P(\{\omega_k\}) = p_k$  in the discrete case,  $\{\omega_k\}$  is a one-point set containing  $\omega_k$ ,  $F$  is the  $\sigma$ -algebra of subsets of  $\Omega$ ; elements of this  $\sigma$ -algebra are called (random) events;  $P$  is a probability - measure on  $\{\Omega, F\}$  such that  $P\{\Omega\} = 1$ . We assume that the process  $\{\xi(t), 0 \leq t \leq T\}$  is separable, and its sample functions belong to the space  $D_{[0,T]}$  of right-continuous functions without discontinuity of the second kind with finite left-side boundaries.

A process with independent increments is called homogeneous if the distribution of increments  $\xi(t) - \xi(s)$  depends only on  $t-s$  when  $s \leq t$ . If  $\xi(t)$  is a homogeneous process with independent increments, then we will assume that  $0 \leq t < \infty$ , and it is often assumed that  $\xi(0) = 0$ .

Let  $X$  be a random variable with cumulative distribution function  $F_X$ . The moment generating function (MGF) of  $X$  (or  $F_X$ ), denoted by  $M_X(t)$ , is:

$$M_X(t) = \mathbf{E}[e^{tX}] \quad (0.1)$$

provided this expectation exists for  $t$  in some open neighborhood of 0. That is, there is an  $h > 0$  such that for all  $t$  in  $-h < t < h$ ,  $\mathbf{E}[e^{tX}]$  exists. If the expectation does not exist in an open neighborhood of 0, we say that the moment-generating function does not exist.

If  $X$  is a continuous random variable, the following relation between its moment-generating function  $M_X(t)$  and the two-sided Laplace transform of its probability density function  $f_X(x)$  holds:  $M_X(t) = L\{f_X\}(-t)$ , since the probability density function two-sided Laplace transform is given as  $L\{f_X\}(s) = \int_{-\infty}^{\infty} e^{-sx} f_X(x) dx$  and the moment-generating function's definition expands (by the law of the unconscious statistician) to:

$$M_X(t) = \mathbf{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx. \quad (0.2)$$

The scaled cumulant generating function (SCGF) (see Eq. (7))  $k(r)$  of the process  $\zeta(t)$  Eq. (2) is:

$$k(r) = t^{-1} \ln \mathbf{E} e^{r\zeta(t)} = -cr + \lambda(\mathbf{E} e^{r\zeta_1} - 1). \quad (0.3)$$

If  $\xi(t)$  is a homogeneous stochastically continuous process with independent increments in  $R^m$ , then its characteristic function takes the form (a Lévy-Khinchin type representation):

$$\mathbf{E} e^{i(z, \xi(t))} = \exp\left\{t\left[i(z, a) - \frac{1}{2}(Bz, z) + \int_{|x| \leq 1} (e^{i(z, x)} - 1 - (z, x)) \Pi(dx) + \int_{|x| > 1} (e^{i(z, x)} - 1) \Pi(dx)\right]\right\}, \quad (0.4)$$

where  $a \in R^m$ ;  $B$  is a symmetric non-negative operator in  $R^m$ ;  $\Pi$  is a measure in  $R^m$ , for which  $\int \frac{(x, x)}{1+(x, x)} \Pi(dx) < \infty$ , and  $\Pi(\{0\}) = 0$ .

The characteristic function of a random variable  $\xi$  with a distribution function  $F(x) = P\{\xi < x\}$  is defined in expression (7).

For the characteristic function (CF) of the process  $\xi(\theta_s)$ , defined as:

$$\mathbf{E} e^{i\alpha \xi(\theta_s)} = s \int_{-\infty}^{\infty} \mathbf{E} e^{i\alpha \xi(t)} e^{-st} dt, \quad (0.5)$$

we consider  $\xi(\theta_s)$  as a randomly stopped process [68]. Randomly stopped processes include an exponentially distributed random variable  $\theta_s$  that is independent of the process  $\xi(t)$ ,  $P\{\theta_s > t\} = e^{-st}$ ,  $s > 0$ ,  $t > 0$ . The characteristic function of a randomly stopped process is the Laplace-Carson transform of the characteristic function given by equation (7). The dependence on time  $t$  is determined by applying the inverse Laplace-Carson transform  $L^{-1}$ . Since  $\mathbf{E}(\theta_s) = 1/s$ , the parameter  $s$  can be interpreted as the average inverse random time.

Some boundary functionals considered below describe physical quantities characterizing time. These quantities are positive, and their characteristic functions transform into the Laplace transform, which has a probabilistic meaning. For example, for the Laplace transform of the quantity  $\tau^+$ , the moment of the first exit for the level  $x > 0$ , Eqs. (5)-(6), the equality  $\mathbf{E}e^{-\gamma\tau^+} = P\{\tau^+ \leq \tau\}$ ,  $P\{\tau > t\} = e^{-\gamma t}$ , is true, where  $\tau$  is a random variable independent of  $\xi = \tau^+$  and having an exponential distribution with parameter  $\gamma$ . Matching against a randomly stopped process (0.5) results in a match of the form:  $s \leftrightarrow \gamma$ ,  $\theta_s \leftrightarrow \tau$ .

For the characteristic function (0.5) of  $\xi(\theta_s)$  the following relation is satisfied:  $\varphi(s, \alpha) := \mathbf{E}e^{i\alpha\xi(\theta_s)} = \frac{s}{s - \Psi(\alpha)}$ , and also, the fundamental factorization identity, which is also the identity of infinitely divisible factorization:  $\varphi(s, \alpha) = \varphi_+(s, \alpha)\varphi_-(s, \alpha)$ ,  $\varphi_{\pm}(s, \alpha) = \mathbf{E}e^{i\alpha\xi^{\pm}(\theta_s)}$ , where  $\zeta^{\pm}(t) = \sup_{0 \leq t' \leq t}(\inf) \zeta(t')$  represents the extremes of the process  $\zeta(t')$  on the interval  $[0, t]$ . These relationships are used in Section 5.5.

Some boundary functionals of the random process are defined in Section 2 by formulas (5)-(6), (9)-(11). Let us also define additions to the process extrema on the interval  $[0, t]$ :

$$\bar{\xi}(t) = \xi(t) - \xi^+(t), \quad \tilde{\xi}(t) = \xi(t) - \xi^-(t). \quad (0.6)$$

The moment of the first jump through the level  $x > 0$  and the values of the first undershoot and the jump that covers the level  $x > 0$  is defined in expression (5). The moment of the first achievement of the maximum is denoted by:

$$T^+(t) := \inf\{u \geq 0: \xi(u) = \xi^+(t)\}. \quad (0.7)$$

This functional is discussed in Section 5.2, formula (57).

Functionals related to crossing or reaching the upper level are designated by the index "+". Let's call them upper functionals. Lower functionals related to crossing the lower-level  $x < 0$  is designated similarly:

$$\begin{aligned}\tau^-(x) &= \inf\{t: \xi(t) < x\}, \quad \gamma^-(x) = x - \xi(\tau^-(x)), \\ \gamma_-(x) &= \xi(\tau^-(x) - 0) - x, \quad \gamma_x^-(x) = \gamma^-(x) + \gamma_-(x), \\ T^-(t) &= \inf\{u \geq 0: \xi(u) = \xi^-(t)\}.\end{aligned}$$

## ***2. Definitions of some boundary functionals***

In book [12], alongside the First-Passage Time (*FPT*), several other boundary functionals are discussed: the distribution of extreme values of a process, the distribution of time spent above a given level, the return time of a function within an interval, and more. The significance of these distributions for risk theory is also defined in works [7, 16, 19-21, 67].

### **2.1. On the physical meaning of risk theory**

In general, risk theory applied to physical problems is associated with numerous issues explored in physical tasks involving the first-passage time of a certain level. However, the application of risk theory entails a more detailed examination and analysis of these issues.

In guides on risk theory, one can find discussions regarding life insurance models. As outlined in Reference [12], we derive from the overarching theory of stochastic processes, considering risk theory as a subset of this general framework. This article employs elements of risk theory that are related to the theory of stochastic processes and possess a common mathematical foundation.

The mathematical theory of risk, rooted in stochastic process theory, is predominantly utilized in problems related to insurance, bankruptcy assessments, and associated phenomena. Nevertheless, risk theory is a mathematical discipline closely linked to queue theory and stochastic storage theory. Due to the generality of the mathematical theory, it can also be applied in other scientific fields, such as physics, chemistry, and biology. Its research subjects may include systems at certain critical values. For instance, the shell of a nuclear reactor is designed for operation within a specific time interval even before it commences operation. The expiration of this period can be regarded as analogous to bankruptcy in the financial domain. The operational lifespan of the reactor shell may be extended based on experimental studies of the material composing the shell. Alternatively,

an annealing procedure may be conducted, wherein the shell is placed in a high-temperature environment that mitigates material defects. Various processes related to the end of the operational lifespan, its extension, and the restoration of functionality may serve as subjects for investigation within risk theory. The physical problems associated with risk theory are extensive and diverse. The first-passage time of a level, one of the most utilized boundaries functionals, is employed in risk theory and has numerous physical and other applications.

## 2.2. Risk process

The actuarial risk models serve the clarification of the relation between the premiums and the surpluses during the insurance business and provide the tools for the correct calculation of the risks, premiums, and reserves. We assume that the insurance company begins at time 0 with initial capital  $u \geq 0$  and takes a total premium up to time  $t$  equal to  $C(t)$ . We can see that  $C(t)$  represents an increasing stochastic process, and in its simplest version, when it increases with a constant premium rate  $c > 0$ , the function  $C(t)$  obtains the form of a deterministic linear function  $C(t) = ct$ .

Let  $N(t) = N_t$  denote the number of claims occurring during the interval  $[0, t]$  and  $\{T_k, k \geq 0\}$  denote the set of consecutive arrival times of the claims. Hence  $N(t) = \max\{k \geq 0 : T_k \leq t\} = \min\{k \geq 0 : T_{k+1} > t\}$ , where by convention  $T_0 = 0$ . The size of the  $k$  claim is denoted by  $\xi_k$ .

The primary quantity in risk theory is the classical risk process, often referred to as the reserve risk process, which takes the form:

$$R(t) = u + ct - S(t). \quad (1)$$

In expression (1)  $S(t) = \sum_{k=1}^{N_t} \xi_k$ ,  $P\{\xi_k \neq 0\} = 1$ , represent the amounts of payments (claims),  $\xi_k > 0$ ;  $(\xi_k)$  is a sequence of independent identically distributed random variables with some distribution function  $F(x) = P\{\xi_1 \leq x\}$ ,  $\mu = E\xi_1$  ( $F(0) = 0, \mu < \infty$ ),  $N = (N_t)_{t \geq 0}$  is the Poisson process,  $N_t = \sum_k I(T_k \leq t)$ , where  $I(T_k \leq t) = \begin{cases} 1 & T_k \leq t \\ 0 & T_k > t \end{cases}$ ;  $(T_{k+1} - T_k)_{k \geq 1}$  are independent variables that have an exponential distribution with parameter  $\lambda$ :  $P\{T_{k+1} - T_k \geq t\} = e^{-\lambda t}$ . The average value of  $R(t)$  is  $ER(t) = u + (c - \lambda\mu)t = u + \rho\lambda\mu t$  with coefficient  $\rho = c/(\lambda\mu) - 1$ .

The assumptions formulated in model Eq. (1) can naturally be weakened, and the model itself can be complicated. For example, one can assume that

the risk process has the form  $R(t) = u + (ct + \sigma B_t) - \sum_{k=1}^{N_t} \xi_k$ , where  $(B_t)$  is a Brownian motion, and  $(N_t)$  is a Cox process (i.e., a “counting process” with random intensity).

Sometimes, it is more convenient to use the equation:

$$\zeta(t) = u - R(t) = S(t) - ct, \quad \zeta(0) = 0, \quad c > 0, \quad (2)$$

which is referred to as the claim surplus process.

If, in the classical risk process Eq. (1), we replace the deterministic linear function  $C(t)=ct$  with the stochastic process  $S_1(t) = \sum_{k \leq v_1(t)} \eta_k$ ,  $\eta_k > 0$ ,  $\eta_0 > 0$ ,  $v_1(t)$  is a simple Poisson process independent of  $N_t$  with intensity  $\lambda_1$ , then:

$$R(t) = u + S_1(t) - S(t), \quad \zeta(t) = S(t) - S_1(t). \quad (3)$$

The processes  $R(t)$  and  $\zeta(t)$  are called the reserve and claim surplus risk processes with random premiums, respectively. The scaled cumulant generating function (SCGF) of the process  $\zeta(t)$ , Eq. (2), (see expression (7)) is given by  $k(r) = t^{-1} \ln \mathbf{E} e^{r\zeta(t)} = -cr + \lambda(\mathbf{E} e^{r\xi_1} - 1)$ , Eq. (0.3).

Furthermore,

$$R(t) = u + S_1(t) - S(t), \quad u > 0, \quad S_1(t) = \sum_{k \leq v_1(t)} \eta_k, \quad S(t) = \sum_{k=1}^{N_t} \xi_k, \quad P\{\xi_k \neq 0\} = P\{\eta_k \neq 0\} = 1. \quad (4)$$

If the premiums  $\eta_k$  have the characteristic function  $\varphi_1(\alpha) = c(c - i\alpha)^{-1}$ , then  $R(t) = S_1(t) - S(t)$  is an upper almost continuous risk process with exponentially distributed premiums and initial capital  $u=0$ . Such processes with random premiums include the fluctuations of trajectory observables, as discussed in work [62]. These issues are considered in greater detail in Refs. such as [7, 12, 19, 21].

Let us dwell briefly on the question of the nature of the distributions  $F=F(x)$  of the amounts of payments. Events associated with payments are usually (albeit rather conditionally) attributed to one of three types: a) normal, b) extreme, c) catastrophic. To describe normal events, distributions with rapidly decreasing “tails” are used (for example, an exponential distribution with the condition  $1-F(x) \sim e^{-x}$ ,  $x \rightarrow \infty$ ). To describe “extreme” events, distributions  $F=F(x)$  with “heavy tails” are used, for example,  $1-F(x) \sim x^{-\alpha}$ ,  $x \rightarrow \infty$ ,  $\alpha > 0$  (Pareto-type distribution) or  $1 - F(x) = \exp\{-\left(\frac{x-\mu}{\sigma}\right)^p\}$ ,  $x > \mu$  (Weibull distribution) with  $p \in (0,1)$ .

In the context of risk theory, instead of representing initial capital, the value  $u$  in equations (1)-(4) can also hold a physical interpretation, such as representing energy (for example, the internal energy of a statistical system).

### 2.3. Boundary functionals

Let us indicate what boundary functionals are. In book [11], it is written: "Most problems for such processes are related to the so-called boundary problems, in which boundary functionals of process trajectories are studied, associated with the exit of the process trajectory beyond a certain level. In this case, two types of boundaries are distinguished: absorbing and repulsive. The intersection of the trajectories of the absorbing boundary process determines the boundary functionals from the trajectory of the process to the moment of intersection. In particular, the moment of intersection of a given level also serves as a boundary functional. Repulsive boundaries "distort" the trajectory of the process, i.e., change its initial characteristics, and, consequently, the boundary functionals determined by another absorbing level." In book [13], it is written: "... boundary functionals (i.e., functionals from selective trajectories associated with the exit of these trajectories beyond certain boundaries)".

Below, various random times will be defined. Following the formal definition (book [10]), we will also provide an informal definition of random time. Let us consider stochastic experiments conducted continuously over time, and let event  $A$  be such that its occurrence can be determined by observing the results of the experiments up to some random moment in time. Such a moment in time will be referred to as a random time. It is also sometimes referred to as a random variable that is independent of the future, a Markov moment, or a stopping time.

For certain functionals of the random process  $\xi(t)$ , definitions are provided in books [12, 67]. These include:

$$\begin{aligned}
 \tau^+(x) &= \inf\{t: \xi(t) > x\}, \quad x > 0 \text{ is the moment of the first exit for the level } x > 0; \\
 \tau^-(x) &= \inf\{t: \xi(t) < x\}, \quad x < 0 \text{ is the moment of the first exit for the level } x < 0; \\
 \gamma^+(x) &= \xi(\tau^+(x)) - x \text{ is first overjump over } x > 0; \\
 \gamma_+(x) &= x - \xi(\tau^+(x) - 0) \text{ is the first under jump over } x > 0; \\
 \gamma^+_x(x) &= \gamma^+(x) + \gamma_+(x) \text{ is the first jump that covers } x > 0; \\
 \gamma^-(x) &= \xi(\tau^-(x)) - x \text{ is first overjump over } x < 0.
 \end{aligned}
 \tag{5}$$

In risk theory,  $\tau^+(u) = \tau(u)$  is defined as the ruin time. In physics, this is the moment when the value  $R(t)$  from equation (1) reaches zero, representing the moment of degeneration or the end of the lifetime, also referred to as the first-passage time (*FPT*). This is related to the fact that in physical problems, the attainment of a zero level by a certain process is often referred to as the end of the lifetime of the quantity described by that process, or as the *FPT* of the zero level. In risk theory, this is not the case; the system can recover, as in queueing theory, where idle periods of downtime and busy periods alternate with one another. Physical analogues of such phenomena also exist. For the claim surplus process Eq. (2), this is the *FPT* when the process  $\zeta(t)$  reaches the level  $u$ :

$$\tau(u) = \inf\{t: R(t) < 0\}, \quad \tau^+(u) = \inf\{t: \zeta(t) > u\}. \quad (6)$$

The characteristic function of a random variable  $\xi$  with a distribution function  $F(x) = P\{\xi < x\}$  is referred to as a complex-valued function:

$$\varphi(t) = \mathbf{E}e^{it\xi(t)} = \int_{-\infty}^{\infty} e^{itx} dF(x).$$

A process with independent increments  $\xi(t)$  is called homogeneous if it is defined on  $[0, \infty)$ ,  $\xi(0) = 0$ , and the distribution  $\xi(t+h) - \xi(t)$  coincides with the distribution  $\xi(h)$  for all  $t > 0$  and  $h > 0$ . The characteristic function of a homogeneous process  $\xi(t)$ ,  $t \geq 0$  is determined in the theory of random processes (for  $\xi(0) = 0$ ) (works [10, 12, 17, 18]) by the relation:

$$\mathbf{E}e^{i\alpha\xi(t)} := \int_{-\infty}^{\infty} e^{i\alpha x} dF(x) = e^{t\Psi(\alpha)}, \quad t \geq 0, \quad (7)$$

where  $F(x) = P\{\xi < x\}$  is the distribution function of a random process  $\xi(t)$ ,  $t \geq 0$ , and the function  $\psi(\alpha)$  represents the scaled cumulant generating function (*SCGF*) of the process  $\xi(t)$ ,  $t \geq 0$ . The *SCGF* does not depend on  $t$  and characterizes all finite-dimensional distributions of the process.

If for a process  $\xi(t)$ ,  $t \geq 0$ , the function  $\psi(\alpha)$  at  $i\alpha = r$  is equal to  $s$ , then we obtain the equation:

$$\Psi(\alpha) \Big|_{i\alpha=r} := k(r) = s, \quad \pm \operatorname{Re} r \geq 0, \quad (8)$$

This equation, in risk theory (works [12, 19, 21, 67]), is referred to as the fundamental Lundberg equation.

We will now rewrite the definitions from Eq. (5) using the notations from work [67]:

$$\begin{aligned} Y^+(u) &= -R(\tau(u)), \quad \gamma^+(u) = \zeta(\tau^+(u)) - u, \\ X^+(u) &= R(\tau(u) - 0), \quad \gamma_+(u) = u - \zeta(\tau^+(u) - 0), \\ X^+(u) + Y^+(u) &= R(\tau(u) - 0) - R(\tau(u)), \\ \gamma^+_u &= \gamma^+(u) + \gamma_+(u). \end{aligned} \quad (9)$$

Here,  $Y^+(u) = \gamma^+(u)$  denotes the severity of ruin,  $X^+(u) = \gamma_+(u)$  is the value of the process  $R(t)$  just before the onset of bankruptcy, referred to as the surplus prior to ruin,  $\gamma^+_u$  is the amount of the claim causing ruin;  $x=u$ ,  $x$  from Eq. (5),  $u$  from Eq. (9).

Additionally:

- $\zeta^\pm(t) = \sup_{0 \leq t' \leq t} (\inf) \zeta(t')$  represents the extremes of the process  $\zeta(t')$  on the interval  $[0, t]$ .
- $\zeta^\pm = \sup_{0 \leq t < \infty'} (\inf) \zeta(t)$  represents the absolute extremes of the process  $\zeta(t)$ .
- $\tau'(u) = \inf\{t > \tau(u), R(t) > 0\}$  is the moment of return  $R(t)$  after bankruptcy to the half-plane  $\Pi^+ = \{y > 0\}$ .

Let us also denote:

$$T'(u) = \begin{cases} \tau'(u) - \tau(u), & \tau(u) < \infty \\ \infty, & \tau(u) = \infty \end{cases}. \quad (10)$$

This is called the "red time," which determines the duration of stay of the process  $R(t)$  in the half-plane where  $\Pi^- = \{x < 0\}$ , the amount of time the risk reserve process  $R(t)$  stays below zero after the ruin time  $\tau(u)$ , work [67], the interval between moments  $\tau'(u)$  and  $\tau(u)$ , and the process  $\zeta(t)$  is below the level  $u$ ;

$$\begin{aligned} Z^+(u) &= \sup_{\tau^+(u) \leq t < \infty} \zeta(t) = \sup_{\tau(u) \leq t < \infty} \{-R(t)\}, \\ Z_1^+(u) &= \sup_{\tau^+(u) \leq t < \tau'(u)} \zeta(t) = \sup_{\tau^+(u) \leq t < \tau'(u)} \{-R(t)\}, \end{aligned} \quad (11)$$

$Z^+(u)$  defines the total maximal deficit,  $Z_1^+(u)$  the maximal deficit on  $T'(u)$ ;  $Z^+(u)$  defines the total maximal deficit, which is the largest shortfall encountered by the process  $T'(u)$  during its lifetime. In other words,  $Z^+(u)$  is the most negative value (or largest deficit) reached by the process when it goes below zero. Similarly,  $Z_1^+(u)$  defines the maximal deficit on a finite time interval  $[0, \tau(u)]$ . This is the largest deficit encountered by the process

$T'(u)$  during the time interval  $[0, \tau(u)]$ . There are other boundary functionals, for example Eqs. (0.6)-(0.7).

All of these quantities, in addition to their values in risk theory, also have physical interpretations. For example, in problems involving achieving a certain particle threshold necessary for a reaction or biochemical process, the value of the jump over a given level represents the excess beyond this threshold. In such cases, the conditions for the reaction or biochemical process may change. The moment of bankruptcy corresponds to the first passage time (*FPT*). In chemical reactions, the time spent above the threshold determines whether there is sufficient time for the reaction to occur, provided that the number of molecules (level  $x=u$ ) is adequate. An insurance company can continue operating after bankruptcy by taking out a loan, and distributions of jump functionals are used to determine the loan amount. There are also physical analogs for these quantities.

It is also possible to indicate the physical meaning of the probabilistic interpretation of the Laplace transform, noted between expressions (0.5) and (0.6). The Laplace transform  $Ee^{-\gamma\xi}$  at  $\gamma \geq 0$  represents the probability that the moment of reaching a given level  $x$  by the process  $\xi$  will occur before the moment of stopping the observation  $\tau$ , which has an exponential distribution.

### **3. Characteristic functions (CF) and MGF of boundary functionals**

In works [7, 12, 67], expressions for the distributions of the functionals specified in Section 2 were obtained.

Following work [12], we consider the case of the risk process being in the survival and risk zones. We examine the claim surplus risk process Eq. (2) with an initial capital  $u > 0$ . We assume  $m = E\zeta(1) = \frac{\lambda - cb}{b} < 0$ ; parameters  $\lambda, c$  is defined in Eqs. (1)-(2), parameter  $b$  in Eq (12). The moment  $\tau^+(u) = \tau(u)$  Eq. (6) determines not only the time of bankruptcy (i.e., the first exit  $\zeta(t)$  into the “red” zone,  $\{y > u\}$ , the risk zone), but also the duration of the initial stay in the survival zone  $\{y \leq u\}$ , which can be conditionally referred to as the “green” zone. The value  $\tau'(u) = \inf\{t > \tau^+(u), \zeta(t) < u\}$  determines the moment of the first return to this zone. The duration of the first stay  $\zeta(t)$  in the “red” zone is determined by the value Eq. (10). The probability of bankruptcy, which in risk theory is denoted by  $\Psi(u)$ , is given by the “tail” of the distribution  $\zeta^+$ ,  $\Psi(u) = P\{\zeta^+ > u\} = \bar{P}_+(u)$ , book [12].

### 3.1. Approximation used. Lundberg equation

We focus on the example of the claim surplus risk process Eq. (2) with an exponential distribution of claims  $\xi_k$ ,  $S(t) = \sum_{k \leq \nu(t)} \xi_k$ ,  $P\{\xi_k > 0\} = 1$ , as in the Kramers-Lundberg model (Refs. [7, 12]), where:

$$\bar{F}(x) = P\{\xi_k > x\} = e^{-bx}, \quad x > 0, \quad \mu = E\xi_k = b^{-1}, \quad b > 0. \quad (12)$$

It is possible to consider the general case, but it leads to cumbersome expressions that make it difficult to derive an explicit final result. Therefore, we restrict our analysis to this example. In Section 5.6, some other distributions of the magnitude of process jumps are considered. In Section 5.3, a comparison is made between the results obtained using approximation (12) and without it.

The moment-generating function (MGF, Eqs. (0.1), (0.2)) of the process  $\zeta(t)$  (or  $\zeta(\theta_s)$ ), see the definition of a randomly stopped process in Eq. (0.5)), is determined by the SCGF Eq. (7) at  $\Psi(\alpha) \Big|_{i\alpha=r} = k(r)$ , as in Eq. (8),  $k(r) = t^{-1} \ln \mathbf{E} e^{r\zeta(t)} = -cr + \lambda(\mathbf{E} e^{r\xi_1} - 1)$ , Eq. (0.3),  $\mathbf{E} e^{r\xi_1} = b/(b-r)$ ,  $b > r$ , Eq. (12) where:

$$\begin{aligned} \mathbf{E} e^{r\zeta(t)} &= e^{tk(r)}, \quad \mathbf{E} e^{r\zeta(\theta_s)} = \frac{s}{s-k(r)}, \quad (13) \\ k(r) &= \frac{\lambda r - cr(b-r)}{b-r}, \quad m = \mathbf{E}\zeta(1) = \frac{\partial k(r)}{\partial r} \Big|_{r=0} = \frac{\lambda - cb}{b} < 0. \end{aligned}$$

The Lundberg equation (8) can be derived by setting the denominator in equation (13) equal to zero. In the case of the function  $k(r)$  given by Eq. (0.3) and Eq. (13), the Lundberg equation Eq. (8) simplifies to a quadratic equation (for  $r < b$ ):

$$s - k(r) = 0 \sim cr^2 + (s + mb)r - sb = 0. \quad (14)$$

This equation has two roots:  $r_1 = -\rho_-(s)$ ,  $r_2(s) = \rho_+(s) > 0$ :

$$\rho_+(s) = \frac{1}{2c} [-(s + mb) + \sqrt{(s + mb)^2 + 4scb}],$$

$$\rho_-(s) = \frac{1}{2c} [(s + mb) + \sqrt{(s + mb)^2 + 4scb}].$$

Knowledge of roots  $\rho_-(s)$ ,  $\rho_+(s)$  is important for the study of boundary functionals.

### 3.2. Distribution of extreme values of the process. Extremes of the claim surplus process

The roots  $r_1 = -\rho_-(s)$  and  $r_2(s) = \rho_+(s) > 0$  of equation (14) determine the *MGF* of the extrema  $\zeta^\pm(\theta_s)$  of the process Eq. (2) and their distributions:

$$\begin{aligned} \mathbf{E}e^{-z\zeta^+(\theta_s)} &= \frac{\rho_+(s)(b+z)}{\rho_+(s)+z}, & \rho_+(s) &= bp_+(s), \\ \mathbf{E}e^{-z\zeta^-(\theta_s)} &= \frac{\rho_-(s)[1-\rho_-(s)/b]}{\rho_-(s)+z}, & cp_+(s)\rho_-(s) &= s. \end{aligned} \quad (15)$$

Here,  $b$  are the parameters (indices) of the exponential distributions (in accordance with Eq. (12)) of claims (jumps  $\xi_k$  in the process  $S(t)$  from equation (2)), and  $c$  represents the intensity of insurance premium receipts, as described in Eqs. (1)-(2). The probability of reaching the minimum extremum is given by work [12]:

$$P_-(s, x) = P\{\zeta^-(\theta_s) < x\} = e^{\rho_-(s)x}, \quad x \leq 0.$$

The probability of the maximum extremum is given by:

$$\begin{aligned} \bar{P}_+(s, x) &= 1 - P_+(s, x), & P_+(s, x) &= P\{\zeta^+(\theta_s) < x\} = 1 - \\ & \mathbf{E}\{e^{-s\tau^+(x)}, \tau^+(x) < \infty\}, \\ \bar{P}_+(s, x) &= q_+(s)e^{-\rho_+(s)x}, & x > 0, & \quad p_+(s) = P\{\zeta^+(\theta_s) = 0\} > 0, \\ q_+(s) &= 1 - p_+(s). \end{aligned}$$

If  $m = dk(r)/dr|r = 0 < 0$ , then  $\rho_-(s)_{s \rightarrow 0} \rightarrow 0$ ,  $\rho'_-(0) = \frac{1}{|m|}$ ,  $\rho_+(s)_{s \rightarrow 0} \rightarrow \frac{b|m|}{c}$ , the probability of bankruptcy, which in risk theory is denoted by  $\Psi(u)$ , is given by work [12]:

$$\bar{P}_+(s, u)_{s \rightarrow 0} \rightarrow \Psi(u) = q_+ e^{-\rho_+ u}, \quad u > 0, \quad q_+(s=0) = q_+, \quad \rho_+(s=0) = \rho_+. \quad (16)$$

Expression (16) describes the Kramers-Lundberg approximation (in the approximation [12]) for the probability of bankruptcy (surplus-level process).

### 3.3. Characteristic function (CF) and MGF of other boundary functionals

Taking into account that for model Eq. (12)  $q_+ = \frac{\lambda}{cb}$ ,  $\rho_+ = bp_+ = \frac{bc-\lambda}{c}$ , the *MGF* of  $\tau^+(u)$ ,  $\tau'(u)$ ,  $\gamma^+(u)$ ,  $T'(u)$ , Eqs, (5)-(6), (9)-(10), are determined by books [12, 67]:

$$q_+(s, u) = \mathbf{E}[e^{-s\tau^+(u)}, \tau^+(u) < \infty] = q_+(s)e^{-\rho_+(s)u}, \quad u \geq 0, \quad (17)$$

$$u = 0, \quad m < 0, \quad q_+(s) = \mathbf{E}[e^{-s\tau^+(0)}, \tau^+(0) < \infty] =$$

$$\frac{\lambda}{c} \int_0^\infty e^{-z\rho_-(s)} \bar{F}(z) dz = \frac{\lambda}{c} \frac{1}{\rho_-(s) + b},$$

$$\lim_{s \rightarrow 0} q_+(s, u) = \Psi(u) = q_+ e^{-\rho_+ u}, \quad u \geq 0,$$

$$\varphi_u(s) = \mathbf{E}[e^{-s\tau^+(u)}, \tau^+(u) < \infty] = q_+(s, u) \frac{b}{b + \rho_-(s)},$$

$$\tilde{q}_+(z, u) = \mathbf{E}[e^{-z\gamma_+(u)}, \tau^+(u) < \infty] = \Psi(u) \frac{b}{b+z},$$

$$\tilde{q}_u(s) = \mathbf{E}[e^{-sT^+(u)}, \tau^+(u) < \infty] = \Psi(u) \frac{b}{b + \rho_-(s)}.$$

From the last two formulas, it follows that  $\gamma_+(u)$  and  $T^+(u)$  do not depend on  $u$ . This follows from Eqs. (16)-(17) when  $\mathbf{E}[\tau^+(u) < \infty] = q_+ e^{-\rho_+ u} = \Psi(u)$ . Therefore, only for this example of the exponential distribution Eq. (12), we express the MGF as:

$$\mathbf{E}[e^{-z\gamma_+(u)} / \tau^+(u) < \infty] = \mathbf{E}e^{-z\tilde{\gamma}^+(u)} = \frac{b}{b+z}, \quad (18)$$

$$\mathbf{E}e^{-sT^+(u)} = \mathbf{E}e^{-\rho_-(s)\tilde{\gamma}^+(u)} = \frac{b}{b + \rho_-(s)}, \quad u > 0;$$

$$\tilde{q}_+(z, 0) = \varphi(0) \frac{b}{b+z}, \quad \tilde{q}(s) = \Psi(0) \frac{b}{b + \rho_-(s)}, \quad \text{for } u = 0,$$

$$\varphi(s) = q_+(s) \frac{b}{b + \rho_-(s)}, \quad \varphi(0) = \tilde{q}(0) = \Psi(0) = q_+.$$

Let us denote the following:

- $\theta_{N_u} = \theta^{(u)}_{N_u}$  is the total duration of survival periods (by analogy with queuing theory), in the survival zone  $\{y \leq u\}$ ;
- $\sigma_{N_u} = \sigma^{(u)}_{N_u}$  is the total duration of "red" periods, in the risk zone  $\{y > u\}$  (as defined in Eq. (10));
- $S_{N_u} = S^{(u)}_{N_u}$  is the total duration of regeneration periods, when  $\zeta(t) < u$ .

The moment of the first regeneration of the claim surplus process  $\zeta(t)$  is given by:

$$\tau'_1(u) = \inf\{t: t > \tau^+(u); \zeta(t) < u\} = \tau^+_1(u) + T'_1(u), \quad (20)$$

$$\theta^{(u)}_n = \tau^+_1(u) + \sum_{k=2}^n \tau^+_k(0), \quad (\theta_n = \sum_{k=1}^n \tau'_k(0), \quad u = 0),$$

$$\sigma^{(u)}_n = T'_1(u) + \sum_{k=2}^n T'_k(0), \quad (\sigma_n = \sum_{k=2}^n T'_k(0), \quad u = 0),$$

where  $T'_k$  and  $\tau^+_k$  describe  $k$ -th regeneration. The value  $S_{N_u}$  is determined through a random walk  $S_n = \sum_{k=0}^n \xi_k$ ,  $S_0 = 0$ , described by the sequence:

$$\{\tau'_k(0)\}_{k \geq 1}; \quad S_n = \sum_{k \leq n} \tau'_k(0), \quad u = 0,$$

$$\{\tau'_k(u)\}_{k \geq 1}: S_n^{(u)} = \tau'_1(u) + \sum_{k=2}^n \tau'_k(0), \quad u > 0.$$

The renewal process determines the number of renewals on the interval  $[0, t]$  for  $S_n$ , which we denote by:

$$N(t) = \max\{n: S_n < t\}, \quad H(t) = EN(t),$$

$N(\theta_s)$  is a randomly stopped process  $N(t)$ , Eq. (0.5).

In book [12], the distributions for  $N_u(\theta_s)$  and  $N_u(N)$  are found, and subsequently, from the distribution of  $N_u$ , the MGFs can be expressed:

$$\begin{aligned} E[e^{-s\sigma_{N_u}}, \sigma_{N_u} < \infty] &= 1 - \Psi(u) + \frac{p_+ \Psi(u)}{1 - q_+ q_+(s)} \frac{b \Psi(u)}{b + \rho_-(s)}, \\ E[e^{-s\theta_{N_u}}, \theta_{N_u} < \infty] &= 1 - \Psi(u) + \frac{p_+ \Psi(u)}{1 - q_+ q_+(s)} q_+(s) e^{-\rho_+(s)u}, \\ E[e^{-sS_{N_u}}, S_{N_u} < \infty] &= 1 - \Psi(u) + \frac{p_+ \Psi(u)}{1 - q_+ q_+(s)} \frac{b q_+(s)}{b + \rho_-(s)} e^{-\rho_+(s)u}, \\ Ee^{-s\tilde{\sigma}_{N_u}} &= 1 - \frac{\rho_-(s) \Psi(u)}{\rho_-(s) + \rho_+} \Big|_{s \rightarrow 0} \rightarrow 1, \quad \frac{q_+ \rho_-(s)}{\rho_-(s) + \rho_+} = \frac{\rho_+(s) - \rho_+}{\rho_+(s)}, \end{aligned} \quad (21)$$

where unlike Eq. (19)  $\tilde{\sigma}_n = \sum_{k \leq n} [T'_k(u) / \tau^+(u) < \infty]$ .

From Eq. (21) at  $s \rightarrow 0$  it follows that

$$\begin{aligned} P\{\sigma_{N_u} < \infty\} &= \dots = 1 - \Psi(u) + \frac{\Psi^2(u)}{1 + q_+}, \\ P\{\sigma_N < \infty\} &= \frac{1}{1 + q_+} = \frac{cb}{\lambda + cb} \quad (\lambda \leq cb). \end{aligned}$$

The density of the marginal bankruptcy functions (for  $u=0$ ) is determined by the following relations:

$$\begin{aligned} p_1(s, 0, x) &= \frac{\lambda b}{c} \frac{e^{-bx}}{b + \rho_-(s)} = b q_+(s) e^{-bx}, \\ p_2(s, 0, x) &= \frac{\lambda}{c} e^{-(b + \rho_-(s))x} \neq p_1(s, 0, x), \\ p_3(s, 0, x) &= \frac{\lambda b}{c \rho_-(s)} (1 - e^{-\rho_-(s)x}) e^{-bx}, \quad q_+(s) = \frac{\lambda}{c} \frac{1}{b + \rho_-(s)}, \end{aligned} \quad (22)$$

where in the general case, for any distributions  $F(x)$ :

$$\begin{aligned} p_1(s, 0, x) &= \frac{\lambda}{c} \int_0^\infty e^{-\rho_-(s)y} dF(x + y), \\ p_2(s, 0, x) &= \frac{\lambda}{c} e^{-\rho_-(s)x} \bar{F}(x) \neq p_1(s, 0, x), \\ p_3(s, 0, x) &= \frac{\lambda}{c \rho_-(s)} (1 - e^{-\rho_-(s)x}) F'(x), \quad x > 0, \end{aligned}$$

are the densities of distributions of quantities  $\gamma^+(0)$ ,  $\gamma_+(0)$  from expressions (9), in Eq. (22)  $\bar{F}(x) = P\{\xi_k > x\}$ ,  $F(x) = P\{\xi_k < x\} = 1 - e^{-bx}$ . From the pre-boundary marginal bankruptcy functions  $\bar{P}_k(s, u, x) = P\{\zeta^+(\theta_s) > u, \gamma_k(u) > x\}$ ,  $(\gamma^+(u) = \gamma_1(u), \gamma_+(u) = \gamma_2(u), \gamma_u^+(u) =$

$\gamma_3(u)$ ), (integrals over  $x$  from Eq. (22) and more general expressions written below Eq. (22)) we obtain for  $x=0$  the relation  $q_+(s) = \frac{\lambda}{c} \frac{1}{b+\rho-(s)}$ .

The *MGF* of the quantity  $Q_x(\infty)$ , where  $Q_x(t) = \int_0^t I\{\xi(u) > x\} du$ ,  $I\{A\} = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$  representing the time spent above the level  $x$  in a semi-continuous as well as an almost semi-continuous from above process  $\zeta(t)$ , is given by the relation:

$$D_x(\mu) = \mathbf{E}e^{-\mu Q_x(\infty)} = 1 - \frac{\rho+(\mu)-\rho_+}{\rho_+(\mu)} e^{-\rho_+x}, \quad x > 0. \quad (23)$$

This agrees with the last relation from Eq. (21) for  $x=u$ , i.e.,  $Q_u(\infty) = \tilde{\sigma}_{N_u}$  for Eq. (12). Furthermore, the integral transformation  $Q_x(\infty)$  of the time spent in the risk zone  $\{y > u\}$  is also defined.

#### ***4. Example. Unicyclic network with affinity $A$ , an asymmetric random walk***

##### **4.1. Statement of the problem. Approximations used.**

We apply the relations developed in Sections 2 and 3 to the unicyclic network with affinity  $A$  (works [69-71]) in periodically controlled systems. These results are applicable, for example, to the description of the behavior of a colloidal particle controlled by a periodic field, molecular pumps, enzymatic reactions with stochastic substrate concentrations, and networks of enzymatic reactions or small electronic circuits.

Current fluctuations that are universally observed in systems can be represented in terms of continuous-time Markov networks with a scaled cumulant generating function  $\lambda(z)$ , Eqs. (24) and (25).

A unicyclic graph is a type of graph that contains exactly one cycle, which is a closed path of edges that begins and ends at the same vertex. In other words, a unicyclic graph is a connected graph with exactly one cycle. It can have any number of vertices and edges, as long as it satisfies this condition.

"Unicyclic networks" are graphs obtained from a binary leaf labeled as a ("phylogenetic") tree by adding an edge to introduce a cycle. This class of graphs provides a useful representation of hybrid evolution in molecular evolutionary biology.

In Ref. [69], a Markovian network consisting of  $N$  discrete states  $\{i\}$  is considered, allowing for transitions with rates  $k_{ij} \geq 0$  from state  $i$  to state  $j$ . All transitions are assumed to be reversible, meaning that  $k_{ij} > 0$  implies  $k_{ji} > 0$ . In work [73], a complete set of fundamental cycles  $\{\beta\}$  within the network is identified. Each cycle is associated with an affinity  $A_\beta$ , a fluctuating current  $X_\beta(t)$  that counts the number of cycle completions after time  $t$  (the so-called integrated current), and an average current  $J_\beta \equiv \langle X_\beta(t) \rangle / t$ , where the brackets indicate an average over stochastic trajectories. This average is independent of  $t$  for initial conditions drawn from the steady-state distribution.

A generalized distance  $d^{\beta}_{ij} = -d^{\beta}_{ji}$  is introduced, which quantifies how much  $X_\beta$  increases during the transition from  $i$  to  $j$ . These increments are constrained to sum to one for each closed loop that completes the cycle once in the forward direction. For example,  $X_\beta$  could represent the scaled displacement of a molecular motor. In this context, the affinity  $A_\beta$  would be given by the external force multiplied by the length of the full step of the motor, while  $d^{\beta}_{ij}$  denotes the relative length of the substep associated with the conformational change from  $i$  to  $j$ .

Adopting a vector notation  $\vec{X}$  for the set of all cycle currents  $X_\beta$  the scaled cumulant generating function is defined as:

$$\lambda(\vec{z}) \equiv \lim_{t \rightarrow \infty} (1/t) \ln \mathbf{E} \exp[\vec{z} \cdot \vec{X}(t)], \quad (24)$$

where  $\vec{z}$  is a real vector.

Comparing Eq. (24) with equations (7), (8), and Eq. (13), we see that  $\lambda(\vec{z})$  coincides with the function  $k(\vec{r})$  when  $\vec{z} = \vec{r}$ , which appears in equations (8), (13)-(14).

For unicyclic networks, where there is only a single affinity  $A \equiv A_\alpha$  and a single fluctuating current  $X \equiv X_\alpha$ , we consider a network consisting of a single cycle with  $N$  vertices and affinity  $A$ , as illustrated in Fig. 1c (Fig. 1a in Ref. [69]).

The transition rates for an arbitrary unicyclic model with  $N$  states and periodic boundary conditions are denoted by  $k_{i,i+1} = k_i^+$  and  $k_{i,i-1} = k_i^-$ , where  $i=1, 2, \dots, N$ . A fixed affinity  $A$  imposes the constraint  $(\prod_{i=1}^N k_i^+ / \prod_{i=1}^N k_i^-) = e^A$  on the transition rates. Different choices of transition rates that satisfy this condition can lead to different SCGF. In particular, if the transition rates are uniform, i.e.,  $k_i^+ = k^+$  and  $k_i^- = k^-$ ,

the *SCGF* divided by the average current  $\lambda(z)/J$  becomes  $\lambda_{ARW}(z, A, N)$ , as given in Eq. (25).

For unicyclic networks, we no longer have to distinguish between the different types of *SCGF*s and can simply write  $\lambda(z) \equiv \lambda_\alpha(z)$ . It has been shown in work [69] that for the unicyclic case, the *SCGF*  $\lambda_\alpha(z)$  exhibits effects of faster-than-quadratic growth at large  $z$  and a pronounced plateau around the minimum of  $\lambda_\alpha(z)$ . This behavior of the *SCGF* is best illustrated using an asymmetric random walk (*ARW*). The *SCGF* is given by [72]:

$$\lambda(z) = k^+ \left[ e^{z/N} + e^{-(z+A)/N} - 1 - e^{-A/N} \right] = J \lambda_{ARW}(z, A, N), \quad (25)$$

$$\lambda_{ARW}(z, A, N) \equiv \{ \cosh[(z+A/2)/N] - \cosh[A/(2N)] \} / (1/N) \sinh[A/(2N)].$$

The hopping rates in the forward and backward directions are denoted by  $k^+$  and  $k^-$ , respectively. The *SCGF* Eq. (25) is bounded by the parabola:

$$\lambda_{ARW}(z, A, N) \geq zJ(1 + z/A).$$

From expressions (15) to (23), it is evident that the characteristic function (*CF*) or *MGF* of functionals such as the time spent above a given level and the extremes of a random function are determined by the positive  $\rho_+$  (and negative) roots of equations (8), (14).

In work [69], the series expansion of the exponential in  $\lambda(z)$  is used. In expression (25), we expand the exponential term  $\exp\{-z/N\} \approx 1 - z/N$  in a series. Let's consider the case  $z/N < 1$ . This condition is satisfied because expression (13) is obtained under the condition  $r < N$  necessary for the convergence of the integral, and below an exponential distribution of jumps with  $z=r$ ,  $b=N$  is used. We represent the term  $\exp\{z/N\} - 1$  as:

$$\int_0^\infty (\exp\{zX_\alpha\} - 1) p(X_\alpha) dX_\alpha, \quad p(X_\alpha) = \delta(X_\alpha - 1/N),$$

where  $X_\alpha$  is a fluctuating current that counts cycle completions after time  $t$  (the so-called integrated current), Ref. [69]. We replace the delta distribution density  $p(X_\alpha) = \delta(X_\alpha - 1/N)$  for  $X_\alpha$  with the density of the exponential distribution  $p(X_\alpha) = c_7 \exp\{-c_7 X_\alpha\}$ ,  $c_7 = N$  and obtain expression (13) with  $b=N$ , i.e., a risk process with an exponential distribution of jumps in incoming demands. This type of process is well studied in both risk theory and stochastic storage theory (a process with an exponential distribution of input jumps). In this case, at  $z = r$  expression (25) transitions to relation Eq. (13), for which functionals Eqs. (15)-(23) are derived.

Let us justify the replacement of the distribution. The clockwise jumps in the flow are equal to  $1/N$ , and the distribution  $p(X_\alpha) = \delta(X_\alpha - 1/N)$  is valid. However, flows fluctuate when the affinity  $A$  changes, and the movement can become counterclockwise, with the jump being equal to  $-1/N$ . This situation is described in work [73]. The exponential distribution has a maximum at zero, which corresponds to a large  $N$  and the proximity of the jump from  $1/N$  to  $-1/N$ .

In Ref. [69], it is noted that the *SCGF*  $\lambda(z)$  is bounded below by an exponential function of the form:  $\lambda(z) \geq R(e^{(\sigma/2+z)l-\sigma/2}/R - 1)$ , where  $\sigma$  is the average entropy production,  $R \equiv \sum_i p^s_i r_i$  is the average number of transitions per unit time in the entire network,  $p^s_i$  is the stationary probability distribution of state  $i$ , and the exit rate from state  $i$  is defined as  $r_i \equiv \sum_l k_{il}$ .

The correspondence between the statistical properties of trajectory observables, referred to in work [69] as the integrated current of finite spatial Markov jump processes, is established by mapping *SCGF* Eq. (25) associated with these generalized currents into a Lévy-Khinchin type representation, Eq (0.4). This representation, with the special case in expression (13), corresponds to the scaled cumulant generating function of a general Lévy process.

The exponential relationship between the flow jumps  $X_\alpha$  and the distribution of jumps is physically motivated by increasing the level of stochasticity, replacing the delta distribution for the flows with an exponential distribution. One might expect the reverse type of relation to hold; that is, one could inquire about generalized currents whose determining coefficients are random functions of the stationary velocities, which are associated with an exponential distribution of jumps. This would correspond to a subclass of trajectories that might be of physical interest or even have realistic applications.

This result aligns with a broad class of processes in risk theory, queue theory, and storage theory (stochastic storage systems, as previously mentioned). The results obtained using expression Eq. (25) and approximation Eq. (26) are compared in Section 5.3. It is demonstrated that the form of the dependence of the mean value on the parameter  $s$  remains unchanged; however, with approximation Eq. (26), the dependence is shifted.

Thus, starting from the *SCGF* of the form Eq. (13):