Fundamentals of Quantum and Classical Optics

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Squeezed State and Super Resolutions

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Conflict of Interest

The author declares no conflict of interest. Artificial AI-assisted technology was not used.

INTRODUCTION

The aim of the present book is to present central topics in quantum and classical optics, and to develop the theoretical framework for each topic from first principles familiar to graduate students and researchers. The analysis is simplified as much as possible without losing the important physical features of each system. For cases in which the analysis becomes too complex, we refer the reader to the relevant works of the present author and others. While the extensive body of literature on these topics is not fully covered, a focused selection of references is provided to support and complement the mathematical derivations presented here.

The analysis in this book falls into two main categories. The first focuses on the fundamentals of quantum optics. Although the material introduced in these sections is inherently complex, we adopt the most straightforward approach for their interpretations. In some of these topics, where conceptual issues arise, the author presents a specific perspective to address these issues. The second category includes the examination of selected topics from different scientific areas, which are typically absent from textbooks on quantum and classical optics, and may be of particular interest to researchers and graduate students.

In the second part of the book, five complex quantum optics systems are treated. In Chapter 10, we explore how squeezed light interacting with atoms can produce squeezed atomic states and examine the implications of such interactions. In Chapter 11, it is shown that local dipole-dipole interactions lead to changes in the Maxwell-Bloch equations of dense two-level systems, resulting in intrinsic bistability effects. In Chapter 12, the separability and entanglement of 2, 3, and 4 qubits are treated, a core topic in the field of quantum computation. In Chapter 13, we treat the use of pseudo-Hermitian Hamiltonians for physical systems exhibiting parity-time symmetry and possessing real eigenvalues. In Chapter 14, we treat quantum engines, where unitary transformations of their mixed states can lead to geometric phases.

In the third part of the book, we show how super resolutions can be obtained through label-free nanoscale object scattering (i.e., without fluorescent atoms, which can enhance resolutions, but can distort the imaging process). In Chapter 15, we treat super-resolution effects observed in hot spots, where extremely high electromagnetic intensity is confined to a small region, enabling Raman spectroscopy at the single molecule level. In Chapter 16, we examine super-resolution obtained by microsphere systems, where their explicit geometry helps the analysis.

We hope that the additional topics in this book will raise scientific interest beyond the interest in the fundamentals of quantum optics.

LIST OF ABBREVIATIONS

AB	Aharonov and Bohm
AN	Anti-normally
BCH	Baker-Campbell-Hausdorf
BS	Beam splitter
CR	Commutation relations
CS	Coherent state
DDA	Discrete dipole approximation
EF	Enhancement factor
EM	Electromagnetic
EPR	Einstein-Podolsky-Rosen
EW	Entanglement witnesses
FS	Fully separable
HS	Hilbert–Schmidt
IOB	Intrinsic optical bistability
IS	Intelligent states
JCM	Jaynes–Cummings model
LHV	Local hidden variable
LO	Local oscillator
NA	Numerical aperture
NO	Normal order
OPG	Optical parametric generator

PB	Pegg–Barnett
PC	Photon counting
PDC	Parametric down conversion
PJ	Photonic jet
PM	Photomultipliers
POVM	Positive-operator valued measure
PT	Partial transpose
PVM	Projection-valued measure
QM	Quantum mechanical
QPD	Quasi probability density
SERS	Surface-enhanced Raman scattering
SG	Susskind and Glogower
SNOM	Scanning near-field optical microscopy
SQL	Standard quantum limit
SS	Squeezed states
SV	Squeezed vacuum
SVA	Slowly varying approximation
WN	White noise

PART I

FUNDAMENTALS OF QUANTUM OPTICS

CHAPTER 1

FUNDAMENTAL PROPERTIES OF COHERENT AND SQUEEZED ELECTROMAGNETIC (EM) STATES

The present chapter reviews the fundamental properties of coherent and squeezed electromagnetic (EM) states, focusing on their quantum characteristics. The creation and annihilation operators are central to the quantization of EM field operators (Glauber 1963a; Glauber 1963b; Loudon 1983; Scully and Zubairy 1997; Walls and Milburn 2008) in relation to the decrease of the quantum noise of squeezed states (SS) relative to that of coherent states (Teich and Saleh 1989). The emphasis in the analysis is placed on SS, which have been obtained and discussed in many experiments (see, e.g., Heidmann et al. 1987; Reynaud et al. 1987; Shelby et al. 1986; Slusher et al. 1986; Wu et al. 1986). An extensive treatment of this topic, supported by numerous calculations, is provided by Loudon and Knight (1987), including the references therein.

Considering a cavity with a volume V, cross-section A and length L along the z-direction. From the boundary conditions, the allowed wave vectors are discrete: $k_m = \pi m / L$. A single cavity mode of the EM field behaves as a simple harmonic oscillator of unit mass. The electric field operator $\hat{E}(z,t)$ is given by:

$$\begin{aligned}\hat{E}(z,t) &= E^{(+)}(z,t) + E^{(-)}(z,t)\\ &= i\sum_m \left(\frac{\hbar\omega_m}{2\varepsilon_0 V}\right)\{\hat{a}(k_m.t)\sin(k_m z) - \hat{a}^\dagger(k_m.t)\sin(k_m z)\},\end{aligned} \tag{1.1}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, ω_m is the frequency, and

$$E^{(-)}(z,t) = E^{(+)}(z,t)^{\dagger} . \qquad (1.2)$$

The coherent state (CS) $|\alpha\rangle$ can be defined as:

$$|\alpha\rangle = exp\left(-|\alpha|^2/2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n\rangle , \qquad (1.3)$$

where the number state $|n\rangle$ is given as:

$$|n\rangle = \frac{(\hat{a}^{\dagger})^n}{(n!)^{1/2}}|0\rangle , \qquad (1.4)$$

and $|0\rangle$ is the vacuum state. For a CS, the photon number follows a Poisson distribution, where the variance is given by:

$$\left\langle (\nabla n)^2 \right\rangle = \langle\alpha|\hat{n}^2|\alpha\rangle - \langle\alpha|\hat{n}|\alpha\rangle^2 = \langle\alpha|\hat{n}|\alpha\rangle , \qquad (1.5)$$

where $\hat{n} = \hat{a}^{\dagger}\hat{a}$ is the number operator. We define the quadrature operators (Walls 1983).

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^{\dagger}) = (\omega/2\hbar)^{1/2}\,\hat{q}$$
$$\hat{Y} = \frac{1}{2}(\hat{a} - \hat{a}^{\dagger}) = (2\hbar\omega)^{1/2}\,\hat{p}, \qquad (1.6)$$

where $\hat{q}$ and $\hat{p}$ are the space and momentum operators, respectively. For CS, we obtain the following relations:

$$\left\langle (\Delta X)^2 \right\rangle = \left\langle (\Delta Y)^2 \right\rangle = 1/4 . \qquad (1.7)$$

The CS can be generated from the vacuum states by the action of the Glauber unitary displacement operator $\hat{D}(\alpha)$ as:

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a});\ |\alpha\rangle = \hat{D}(\alpha)|0\rangle$$
$$\hat{D}^{-1}(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha;\ \hat{D}^{-1}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha) = \hat{a}^{\dagger} + \alpha^{*} \qquad (1.8)$$

The *two-mode squeezed vacuum* (*SV*) *state* is given by:

$$|r,\theta\rangle = \hat{S}(r,\theta)|0\rangle_1|0\rangle_2$$
$$\hat{S}(r,\theta) = \exp\left(\frac{1}{2}\varsigma^* \hat{a}_1\hat{a}_2 - \frac{1}{2}\varsigma \hat{a}_1{}^\dagger \hat{a}_2{}^\dagger\right);\ \varsigma = re^{i\theta}, \tag{1.9}$$

where $\varsigma = re^{i\theta}$, $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for the two modes with frequencies $\Omega_1 = \Omega + \delta$ and $\Omega_2 = \Omega - \delta$. One method for producing this state is by using a parametric amplifier (which will be analyzed in the next chapter), where 2Ω and ϕ represent the frequency and phase of the pump field, respectively. As δ approaches zero, the two-mode SV state can be reduced to a *degenerate one-mode SV state*. The operator $\hat{S}(r,\theta)$ is a unitary operator that transforms the creation and annihilation operators according to the following relations:

$$S^{-1}(\varsigma)\hat{a}_1 S(\varsigma) = \hat{a}_1\cosh(r) - a_2{}^\dagger \sinh(r)\exp(i\phi) = \hat{a}_{1,S}$$
$$S^{-1}(\varsigma)\hat{a}_1{}^\dagger S(\varsigma) = \hat{a}_1{}^\dagger\cosh(r) - \hat{a}_2 \sinh(r)\exp(-i\phi) = \hat{a}_{1,S}{}^\dagger. \tag{1.10}$$

Similarly, relations for $\hat{a}_2$ and $\hat{a}_2{}^\dagger$, where the new operators are denoted as $\hat{a}_{1,S}$ and $\hat{a}_{1,S}{}^\dagger$, respectively, obey the same *CR*. The *degenerate one-mode SV state* is given by (Walls 1983):

$$\hat{S}(\varsigma)|0\rangle = \exp\left(\frac{1}{2}\varsigma^* \hat{a}^2 - \frac{1}{2}\varsigma \hat{a}^{\dagger 2}\right)|0\rangle$$
$$S^{-1}(\varsigma)\hat{a}S(\varsigma) = \hat{a}\cosh(r) - \hat{a}^\dagger \sinh(r)\exp(i\theta) = \hat{a}_S$$
$$S^{-1}(\varsigma)\hat{a}^\dagger S(\varsigma) = \hat{a}^\dagger\cosh(r) - \hat{a}\ \sinh(r)\exp(-i\theta) = \hat{a}_S{}^\dagger. \tag{1.11}$$

The expectation values for the *quadrature over the degenerate SV state* vanish:

$$\langle 0|\hat{S}^{-1}(r,\theta)\hat{X}\hat{S}(r,\theta)|0\rangle = 0\ ;\ \hat{X} = \left(\frac{\hat{a}+\hat{a}^\dagger}{2}\right)$$
$$\langle 0|\hat{S}^{-1}(r,\theta)\hat{Y}\hat{S}(r,\theta)|0\rangle = 0\ ;\ \hat{Y} = \left(\frac{\hat{a}-\hat{a}^\dagger}{2i}\right), \tag{1.12}$$

as can be verified using Eq. (1.11). The expectation value of the *squared quadrature operator* $\hat{X}^2$ over the degenerate SV state is given by using Eq. (1.11):

$$\begin{aligned}&\langle 0|\hat{S}^{-1}(r,\theta)\hat{X}^2\hat{S}(r,\theta)|0\rangle \\ &=\frac{1}{4}\langle 0|\{(\hat{a}+\hat{a}^\dagger)\cosh r-(e^{i\theta}a+e^{-i\theta}a^\dagger)\sinh r\}^2|0\rangle \\ &=\frac{1}{4}(\cosh^2 r+\sinh^2 r-2\sinh r\cosh r\cos\theta).\end{aligned} \tag{1.13}$$

Assuming that at $t=0$, we have $\theta=0$, obtaining:

$$\langle 0|\hat{S}^{-1}(r,\phi)\hat{X}^2\hat{S}(r,\phi=0)|0\rangle=\frac{1}{4}e^{-2r}. \tag{1.14}$$

We develop, in a similar way, the equation for the *squared quadrature operator* $\hat{Y}^2$ over the degenerate SV state, obtaining:

$$\begin{aligned}&\langle 0|\hat{S}^{-1}(r,\phi)\hat{Y}^2\hat{S}(r,\phi=0)|0\rangle=\frac{1}{4}e^{2r} \\ &\left\langle(\Delta X)^2(\Delta Y)^2\right\rangle=\frac{1}{16}.\end{aligned} \tag{1.15}$$

The above calculations reveal that for $\theta=0$ the measurement of quadrature $\hat{X}$ is more accurate at the expense of increasing the noise in the other quadrature $\hat{Y}$. By returning to Eq. (1.13), we find that the quadrature operators in the $X-Y$ plane depends on the angle θ (Fisher et al. 1984). Without the approximation $\theta=0$ more general calculations lead to the equations:

$$\begin{aligned}&\left\langle(\Delta X)^2\right\rangle=\frac{1}{4}\left\{\exp(-2r)\cos^2\frac{1}{2}\theta+\exp(2r)\sin^2\frac{1}{2}\theta\right\}; \\ &\left\langle(\Delta Y)^2\right\rangle=\frac{1}{4}\left\{\exp(-2r)\sin^2\frac{1}{2}\theta+\exp(2r)\cos^2\frac{1}{2}\theta\right\}.\end{aligned} \tag{1.16}$$

By considering the zeroth-order time dependence $a\rightarrow ae^{-i\omega t}$, $a^\dagger\rightarrow a^\dagger e^{i\omega t}$, the CS rotates in the $X-Y$ plane with frequency ω. As a result, it becomes a minimum-uncertainty state only at times $nT/2$, where n is an integer, and T is the time period of the EM field. At all other times $\Delta X \Delta Y > 1/4$. By performing interference between the SV state and the two-mode CS, the result can be described as a minimum uncertainty state in a rotating reference frame, where the frame rotates at the angular frequency of the coherent state.

The excited (or displaced) SV state is given by:

$$|\alpha,\varsigma\rangle = \hat{D}(\alpha)\hat{S}(\varsigma)|0\rangle \tag{1.17}$$

The application of Eq. (1.17) is usually based on the conditions that the uncertainties are obtained by the SV state, i.e., by Eq. (1.16), while the light intensity is given mainly by $|\alpha^2|$, resulting in:

$$\begin{gathered} \langle\alpha,\varsigma|\hat{a}^\dagger\hat{a}|\alpha,\varsigma\rangle = |\alpha|^2 + \sinh^2 r \\ \sinh^2 r \ll |\alpha|^2 . \end{gathered} \tag{1.18}$$

In this framework, the distribution of the SV state in the complex α plane is represented as an ellipse centered at the origin. The major and minor axes, denoted by ΔX and ΔY are given by Eq. (1.16). The state described in Eq. (1.17) is obtained by displacing the ellipse center by the vector α with the same distribution of uncertainties. Generalization of Eqs. (1.17) and (1.18) to two modes with detailed calculations are given, for example, in (Loudon and Knight 1987). Additional material on squeezed states can be found in other reviews, e.g., Dodonov (2002).

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CHAPTER 2

PHOTON STATISTICS, NONCLASSICAL STATES, AND HOMODYNE DETECTION

Let us consider a photoelectric detector illuminated by a beam of light. The photodetector is based on the photoionization process: an incident photon ionizes an atom of the detector, and the emitted electron is accelerated and amplified to create a pulse. A shutter in front of the phototube controls the length of time for which light falls on the detector. The number of counted photons is recorded, and the experiment is repeated many times for the same opening *T*. The measured statistical distribution $P_m(t)$ contains information about the statistical properties of light.

The standard theory of photodetection is based on the multi-coincidence rate (Kelly and Kleiner 1964):

$$W^{(n)}(t_1 \cdots t_n) \\ = \eta^n Tr\left[\rho \hat{E}^{(-)}(t_1) \cdots \hat{E}^{(-)}(t_n) \hat{E}^{(+)}(t_n) \cdots \hat{E}^{(+)}(t_1)\right], \tag{2.1}$$

where ρ is the density matrix of the field, and η is the quantum efficiency of the detector. The $\hat{E}$ operators are defined, for example, in Eq. (1.1). However, in this case, we assume propagating light rather than light within a cavity.

Instead of expressing the field in terms of spatial modes (which involves performing a Fourier analysis of the spatial variable into wave vectors k_m) we can write it in terms of temporal modes, where we perform a Fourier analysis of the time variable t into discrete frequencies ω_m, where $\omega_m = 2m\pi / T$. The electric field for a plane wave can then be written as (Collett, Loudon, and Gardiner 1987; Huttner and Ben-Aryeh 1990; Collett and Gardiner 1984):

$$\hat{E}(t)=\hat{E}^{(+)}(t)+\hat{E}^{(-)}(t)=\sum_m\left[\frac{\hbar\omega_m}{2\varepsilon_0 cT}\right]^{1/2}[\hat{a}(\omega_m)e^{-i\omega_m t}+H.C.], \tag{2.2}$$

where $\hat{a}(\omega_m)\to\hat{a}(z,\omega_m)$ and their conjugates form a set of localized creation and annihilation operators (instead of the usual $\hat{a}(k_m,t)$). The cross-section of the beam should be included in the denominator of Eq. (2.2); however, we set it to unity. Following this description, the operator

$$\hat{n}_T=\hat{a}^\dagger(\omega_m)\hat{a}(\omega_m), \tag{2.3}$$

is the number operator of the photons of frequency ω_m passing through a certain plane z during a time period T and with a unit cross-sectional area. Here, we focus on the quantization of temporal modes (however, a more general treatment of temporal modes, including the effects of dispersion, is treated in Chapter 3).

The first-order coincidence rate is related to the average photon flux and the quantum efficiency as:

$$W^{(1)}(t)=\left\langle\hat{E}^{(-)}(t)\hat{E}^{(+)}(t)\right\rangle\eta=I(t)\eta. \tag{2.4}$$

where *I(t)* represents the photon flux per unit time and unit area. This quantity is related to the average number of counts during a time interval $[0,T]$ by:

$$\langle\hat{n}\rangle_T=\int_0^T W^{(1)}(t)dt \tag{2.5}$$

The second-order coincidence rate is related to the second factorial moment (Dagenais and Mandel 1978):

$$\begin{gathered}\left\langle n^{(2)}\right\rangle_T=\langle n(n-1)\rangle_T=\int_0^T\int_0^T dt_1dt_2W^{(2)}(t_1,t_2)\\ W^{(2)}=\eta^2\left\langle\hat{E}^{(-)}(t_1)\hat{E}^{(-)}(t_2)\hat{E}^{(+)}(t_2)\hat{E}^{(+)}(t_1)\right\rangle.\end{gathered} \tag{2.6}$$

For stationary light, we apply the following approximations:

$$W^{(2)} = \eta^2 \left\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t) \right\rangle$$
$$\left\langle W^{(1)} \right\rangle = \frac{\langle n \rangle_T}{T}, \tag{2.7}$$

which leads to the second-order normalization condition:

$$g^{(2)}(\tau) = \frac{W^{(2)}(t,t+\tau)}{\langle W^{(1)}(t)\rangle \langle W^{(1)}(t)\rangle}$$
$$= \frac{\eta^2 \left\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t) \right\rangle}{\langle n \rangle_T^{\,2} / T^2}. \tag{2.8}$$

Substituting Eq. (2.8) into Eq. (2.6) and changing the variables t_1 and t_2 to t_1 and $\tau = t_2 - t_1$ results in:

$$\langle n(n-1) \rangle = \langle n \rangle_T^{\,2} / T^2 \int_0^{T-\tau} dt_1 \int_0^T g^{(2)}(\tau) d\tau \quad . \tag{2.9}$$

The upper limit in the integration over t_1 is given by $t_1 = T - \tau$. By rearranging terms, Eq. (2.9) can be rewritten into the following very useful form (Gardiner and Collett 1985; Gardiner and Zoller 2004):

$$\langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle + \langle n \rangle^2 \left[\frac{2}{T} \int_0^T \left(1 - \frac{\tau}{T} \right) (g^{(2)}(\tau) - 1) d\tau \right]. \tag{2.10}$$

We note that:

$$\langle n \rangle^2 \left[\frac{2}{T} \int_0^T \left(1 - \frac{\tau}{T} \right) (-1) d\tau \right] = -\langle n \rangle^2 . \tag{2.11}$$

This term is inserted on both sides of Eq. (2.10), leading to the following important result: $\langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle$ that represents the variance for a CS (see Eq. (1.5)). The last term on the right-hand side of Eq. (2.10) shows the deviation of the occupation number distribution from Poisson statistics. This deviation is measured by the Mandel *Q*-factor (Mandel 1986; Mandel 1979):

$$Q = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle} - 1 \quad . \tag{2.12}$$

For short-times, we apply the approximation $g^{(2)}(\tau) \to g^{(2)}(0)$ in Eq. (2.10), and after performing an integration, we get the following:

$$Q = \langle n \rangle (g^{(2)}(0) - 1) \quad , \tag{2.13}$$

where n is the photon number operator, and $g^{(2)}(0)$ is the normalized second-order correlation function at short time intervals. Negative values of Q corresponds to a state whose variance of photon number is less than the mean (i.e., sub-Poisson statistics). Here $-1 \le Q < 0$ presents nonclassical light. The minimum value, $Q = -1$ is obtained for a number state.

In classical light sources, such as thermal light, photons tend to arrive in bunches. However, the phenomenon of antibunching, where photoelectric counts exhibit anti-correlations, was first observed in resonance fluorescence experiments (Kimble, Dagenais, and Mandel 1977; Mandel and Wolf 1995). The number of recorded pulsed pairs as a function of the time delay τ between them showed growth from $\tau = 0$ demonstrating antibunching. While photon bunching has semiclassical interpretations, antibunching can only be understood in terms of a quantized EM field. The antibunching of photons can also be related to the property of the second-order correlation function with $\hat{E}$ operators as a function of τ in Eq. (2.8).

2.1. Photocurrent spectrum

We follow the approach given by Kelly and Kleiner (1964) and define the random function $O(t)$ representing the output of the photomultipliers (PM):

$$O(t) = \sum_{i=0}^{N} eG\delta(t - t_i)Y_i \tag{2.14}$$

where the measurement time T is divided into N intervals: $[t_0,t_1];[t_1,t_2];\cdots[t_{N-1},t_N]$, and N is large enough to ensure that two photoelectrons are not emitted simultaneously. The variable Y_i is a random variable with values of $[t_i,t_{i+1}]: Y_i=1$ if detection occurs and $Y_i=0$ if no detection occurs. Here, $eG\delta(t-t_i)$ represents the intensity of the sharp pulse occurring when a detection event takes place, and G is the gain of the PM.

To obtain the spectrum, we introduce the random variable correlation function for the operator $\hat{O}$ in Eq. (2.14):

$$C(\tau)=\frac{1}{T}\int_0^T O(t)O(t+\tau)d\tau, \tag{2.15}$$

using the relation $\delta(t-t_i)\delta(t-t_j+\tau)\rightarrow\delta(t_i+\tau-t_j)$ we obtain:

$$C(\tau)=\frac{e^2G^2}{T}\sum_{i=0}^{N}\sum_{j=0}^{N}\delta(t_i+\tau-t_j)Y_iY_j \tag{2.16}$$

This leads to two types of terms: for $Y_iY_j=Y_i^2\,(Y_i=Y_j)$ we obtain the average term for $C_1(\tau)$:

$$\langle C_1(\tau)\rangle=\frac{e^2G^2}{T}\left[\int_0^T W^{(1)}(t)\,dt\right]\delta(\tau) \tag{2.17}$$

Applying Eqs. (2.4-2.5), we get:

$$\int_0^T W^{(1)}dt=\langle n\rangle_T=\langle I\rangle T\eta, \tag{2.18}$$

where $\langle I\rangle\eta$ represents the average photon flux. For stationary light, we have:

$$\langle W^{(1)}\rangle=\frac{1}{T}\int_0^T W^{(1)}dt=\langle I\rangle\eta\ ;\ \frac{\langle n\rangle_T}{T}=\langle I\rangle\eta \tag{2.19}$$

The average current $\langle i\rangle$ is given by:

$$\langle i\rangle = \eta eG\langle I\rangle \tag{2.20}$$

Thus, Eq. (2.17) can be transformed by substituting Eqs. (2.19-2.20) as:

$$\langle C_1(\tau)\rangle = eG\langle i\rangle\delta(\tau)\,;\langle i\rangle = \eta eG\langle I\rangle\,. \tag{2.21}$$

For $Y_1Y_j\ (i\neq j)$ we evaluate the average of the term $C_2(\tau)$:

$$C_2(\tau) = \frac{e^2G^2}{T}\int_0^T W^{(2)}(t,t+\tau)dt = e^2G^2W^{(2)}(\tau), \tag{2.22}$$

where for stationary light, using Eqs. (2.8) and (2.19-2.20), we obtain:

$$C_2(\tau) = \langle i\rangle^2 g^{(2)}(\tau)\,. \tag{2.23}$$

By subtracting $\langle i\rangle^2$ from $\langle C(\tau)\rangle = \langle C_1(\tau)\rangle + \langle C_2(\tau)\rangle$ and applying the Fourier transform on $\langle C(\tau)\rangle$ the intensity fluctuation spectrum $N(\omega)$ is obtained as:

$$N^2(\omega) = \frac{eG}{2\pi}\langle i\rangle + \frac{1}{2\pi}\langle i\rangle^2\int_{-\infty}^{\infty}[g^{(2)}(\tau)-1]e^{-i\omega\tau}d\tau\,. \tag{2.24}$$

For small time intervals, we can exchange $[g^{(2)}(\tau)-1]\rightarrow[g^{(0)}(0)-1]$. For coherent light, we obtain the standard limit $N^2(\omega)=(eG/2\pi)\langle i\rangle$. Nonclassical light is obtained under the condition $[g^{(0)}(0)-1]<1$, where maximal noise reduction occurs in a number state, which satisfies $[g^{(0)}(0)-1]=-1/n$.

2.2. Homodyne detection of nonclassical light

The general theory of homodyne and heterodyne detection was developed by Huttner and Ben-Aryeh (1989). In the following discussion, we limit our focus to applying this theory to the photocurrent spectrum for analyzing the detection of nonclassical light. In a typical experimental scheme, a coherent local oscillator (*LO*) field and a SS are mixed at a beam splitter (*BS*). The resulting field generates a current $\langle i_1 \rangle$ and $\langle i_2 \rangle$ at the two photomultipliers (*PM1* and *PM2*). These currents are then sent to a frequency analyzer to obtain the noise spectrum. We assume a simple transformation of the fields after splitting at the BS (Huttner and Ben-Aryeh 1988):

$$E_1^{(+)}(t) = S\varepsilon^{(+)}(t) + RE^{(+)}(t)$$
$$E_2^{(+)}(t) = -R\varepsilon^{(+)}(t) + SE^{(+)}(t) \tag{2.25}$$

where $S\,(R)$ is the real transmission (reflection) coefficient, $\varepsilon(t)$ is the SS field before splitting, and $E(t)$ is the coherent *LO* field. We assume steady-state conditions for the fields, such that time dependence enters only via the time differences in the various correlation functions.

To determine the noise in the current, we consider three correlation functions: $g_1^{(2)}(\tau), g_2^{(2)}(\tau), g_{12}(\tau)$ for *PM1* and *PM2*, and the correlation between the two photomultipliers, respectively:

$$g_1^{(2)}(\tau) = \frac{\left\langle \hat{E}_1^{(-)}(t)\hat{E}_1^{(-)}(t+\tau)\hat{E}_1^{(+)}(t+\tau)\hat{E}_1^{(+)}(t)\right\rangle}{\left\langle \hat{E}_1^{(-)}(t)\hat{E}_1^{(+)}(t)\right\rangle\left\langle \hat{E}_1^{(-)}(t+\tau)\hat{E}_1^{(+)}(t+\tau)\right\rangle}, \tag{2.26}$$

and similarly, for $g_2^{(2)}(\tau)$, we have:

$$g_{1,2}^{(2)}(\tau) = \frac{\left\langle \hat{E}_1^{(-)}(t)\hat{E}_2^{(-)}(t+\tau)\hat{E}_2^{(+)}(t+\tau)\hat{E}_1^{(+)}(t)\right\rangle}{\left\langle \hat{E}_1^{(-)}(t)\hat{E}_1^{(+)}(t)\right\rangle\left\langle \hat{E}_2^{(-)}(t+\tau)\hat{E}_2^{(+)}(t+\tau)\right\rangle}. \tag{2.27}$$

Then, by sending $i_1 - i_2$ to a spectrum analyzer, we obtain the noise spectrum:

$$N^2(\omega) = \frac{eG}{2\pi}(\langle i_1 \rangle + \langle i_2 \rangle) + \frac{\langle i_1 \rangle^2}{2\pi}\int_{-\infty}^{\infty}[g_1^{(2)}(\tau) - 1]e^{-i\omega\tau}d\tau$$
$$+\frac{\langle i_2 \rangle^2}{2\pi}\int_{-\infty}^{\infty}[g_2^{(2)}(\tau) - 1]e^{-i\omega\tau}d\tau - 2\frac{\langle i_1 \rangle\langle i_2 \rangle}{2\pi}\int_{-\infty}^{\infty}[g_{1,2}^{(2)}(\tau) - 1]e^{-i\omega\tau}d\tau, \tag{2.28}$$