Teaching and Learning Mathematics Together

Teaching and Learning Mathematics Together: Bringing Collaboration to the Centre of the Mathematics Classroom

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CONTENTS

TABLES AND FIGURES

PREFACE

This book is written for mathematics teachers—for all mathematics teachers in secondary schools, particularly those who find the whole notion of reform daunting or even misguided. I have taught mathematics in a number of different schools where the daily need to prepare students for examinations becomes paramount in one's mind: therefore teaching mathematics in new ways that are consistent with the calls for reform will just have to wait until I have a little more time.

This book is also written for mathematics teachers in training who are yet to experience the pressures that teachers in classroom face to get through large amounts of content in short periods of time. I hope that some of the ideas outlined in this book may inspire you to try some alternative approaches to teaching mathematics. You may even want to use these ideas as a springboard for developing new approaches and different ways of organising classroom activity that can reshape the way students learn mathematics.

The motivation for conducting this research was to somehow find a way of realising the principles of reform in classroom environments that have remained relatively unchanged for many years. There is an inherent simplicity and apparent efficiency in the model that places at the centre of the classroom the teacher who demonstrates how to perform different mathematical tasks which students then attempt on their own. And yet my own experience at high school tended to suggest that learning environments could be structured very differently—where most of the learning took place between students who were able to explain ideas to each other, provide feedback on each other's ideas and motivate each other to seek out new forms of understanding.

Perhaps the first time I became conscious of this potential was during a physics lesson in Year Eleven. I went to school in a small country town in NSW. This particular lesson took place each week in one of the demountable classrooms at the other end of the school to the science staffroom. So it was not unusual for our class (of six students) to be without a teacher for the first half of the lesson as our teacher, who was also the head teacher of science, would often be caught up in the staffroom dealing with some incident that had arisen earlier that day.

On this particular day, we arrived at class and realised that our teacher was going to be late. Typically, we would get out our folders and start on some other work. But this lesson we thought why not try this whole teaching caper out for ourselves? We picked up some left over chalk, stood around at the blackboard and worked our way through the concepts that comprised the topic we were then studying. By the end of that forty minute lesson, I walked away having come to terms with about six weeks' work.

It was the experience of sitting around studying together with other students, working on projects together, discussing ideas during mathematics lessons that provided the motivation for incorporating this particular approach into my own teaching. I remember being taught by my father (which proved somewhat awkward, but that's another story) what was then referred to as 4 unit Mathematics, so we saw a lot of each other at school. His policy on teaching was simple—as the teacher at the front of the classroom you shouldn't talk for more than ten minutes. The rest of the time should be made available for students to work together making sense of the work for themselves, talking with each other, trying different things out and asking the classroom teacher for help with more specific concerns. It was an approach to teaching that I attempted to make use of myself as a mathematics teacher providing opportunities to talk mathematics as well as practise mathematics.

This book retells the story of what happened when one possible way of encouraging more talk in mathematics classrooms was introduced at two different schools. All of the teachers involved would have typically started each lesson with an exposition of what the main concepts were followed by some examples, and then time for students to work through some exercises. Some were trained as mathematics teachers while others were science teachers teaching a few mathematics classes on the side. All of them indicated that they were comfortable with teacher-led classrooms and that they had little experience with student-centred learning environments. So how would they go dealing with a very different paradigm to the one they were used to? Their experiences and perceptions represent one source of information about this approach to teaching mathematics. But the main question, of course, relates to the perceptions of students who participated in this approach. How would they react to being given greater responsibility for their own learning?

Across the six chapters of this book, I have tried to provide the story of the development of this approach to teaching. In doing so, I have presented some theoretical ideas that led to the development of a collaborative learning model in Chapters One and Two. You may wish to skip these chapters and go straight to Chapter Three in which the model is described in detail then going back to read Chapters One and Two when you have more time. Chapter One contains discussion about the nature of mathematics while Chapter Two outlines eight principles of effective mathematics teaching and learning that can be found in research papers, books and journals. While Chapter One might appear more philosophical than practical and Chapter Two more theoretical than practical I hope that you might be motivated to consider these sections once you have digested the model itself and what it looks like in classrooms.

Chapter Three give you the model in a nutshell. Chapter Four provides some background to the two schools who took part in this project and a description of what the model looked like in these different classrooms. I think this background will help you to consider whether such a model might work in your particular classroom. All classrooms have their own unique characteristics and this is certainly the case with the seven classrooms involved in this project. Chapter Five then describes some of the findings—what worked, what didn't work so well and how teachers and students described their experience teaching and learning using this approach. Finally, Chapter Six provides a brief discussion of how this model might be developed and incorporated into classrooms in the future.

CHAPTER ONE

I HATED MATHEMATICS AT SCHOOL

In the meritocratic society within which we live it is perhaps not surprising that the first question we ask when we meet someone is "what do you do?" It is almost as if the answer to this question will provide us with the necessary information to make a decision about whether or not the person we are talking to is worth our time and effort. It is at this early point in the acquaintance that I would regularly baulk at providing a straight answer knowing only too well what impression I would be creating by telling the truth. As a mathematics teacher I would get used to conversations that came to a grinding halt when you mention what you do only to receive the reply "I hated mathematics at school". Once associated in this manner with the dark side it takes a considerable effort to regain someone's interest in, perhaps, some other aspect of your life.

Why is it that so many have such negative recollections of their time at school learning mathematics? More than any other subject mathematics tends to evoke such strong responses, particularly negative responses. Perhaps this is not surprising when considering the nature of mathematics as a subject area and how it is taught in school. For many students, mathematics is static rather than dynamic, rigid rather than flexible, abstract rather than practical, and concerned more with learning procedures to solve problems rather than creative thinking.

Is there something about the subject matter that necessitates mathematics being viewed as a fixed, abstract body of knowledge for appropriation? Certainly, it has the oldest history of any of the disciplines taught in secondary school. When compared with the other core subjects of science, English, social sciences and technology mathematics is by far the oldest discipline with central concepts remaining unchanged for millennia. Euclid's five axioms and associated system of geometrical proofs are still taught in schools today two and a half thousand years after they were first developed. Problems involving algebra have been found in documents four thousand years old and the word algebra is drawn from a textbook written around the year 830 AD. Even the more recent invention/discovery of the integral calculus by Isaac Newton took place in 1665 over four hundred years ago (although there are some who identify Eudoxus from the 4th century BC as the father of calculus) making this relatively new aspect of mathematics older than the theory of evolution, the development of the Periodic Table and almost older than Shakespeare. In terms of the mathematics taught in high schools, not a great deal has changed in recent times.

As students become aware of these ideas for the first time in the 21st century their initial reaction may not be that of awe that such ideas have survived for so long. For many students of mathematics (and for many teachers as well, I am sure) the fact that many of these ideas appear to have remained unchallenged for millennia suggests that these ideas will remain so into the future. The degree to which one can interact with these ideas and, in doing so, develop new ways of thinking about these mathematical constructs is therefore limited. You do not question the intellectual giants of our civilisation such as Newton, Archimedes, Euler and Descartes. You respect them through the careful reproduction of their methods and techniques. You make every effort to replicate their findings rather than set about reinventing the wheel. While it may be within the scope of the mathematics community to question the methods of the ancients (see Wildberger, 2005 for an example of this) this activity is not typically pursued by students of mathematics in high school. The preferred approach to learning mathematics could perhaps be crudely summarised as "monkey see, monkey do".

Perhaps the most typical affective response to mathematics as a subject area (which could, in fact, be described as a lack of any affective response) is boredom. Mathematics represents a boring practice, tedious in its repetition and strict adherence to set algorithms for solving problems dealing with a set of abstract concepts. "Good" students of mathematics are those who don't need a calculator because they are able to reproduce the rule-based procedures of a calculator on their own. Being able to follow rules is also often not a very desirable character trait. School students value spontaneity, creativity, the ability to question authority and the ability to work outside the established rules.

Even if you could engage meaningfully with these concepts and develop new ways of thinking about mathematics, what would be the point since mathematics seems so abstract, so removed from the real world and so alien? After all, what we are dealing with in mathematics are abstract systems that have no necessary connection with the world of experience. Many mathematical systems develop independently of the real world. The integral calculus, real and complex number systems, algebraic rules and formulations and different approaches to geometry for example, were all developed to solve specifically mathematical problems rather than real world problems. And as students reach the senior years of mathematics and attempt the more advanced courses, the more abstract becomes the mathematics they encounter.

Is the subject matter of mathematics fundamentally static, abstract and rule bound and therefore students' role in the mathematics classroom is one of reproduction rather than invention? Many teachers of mathematics may concur with this assessment of mathematics as a body of knowledge that is there to be presented and appropriated. Students need to learn how to follow rules to achieve results and they need to be made aware of the great achievements of mathematics that are part of our heritage. To this last sentence I want to say "amen"—but I think doing mathematics and learning mathematics can be much more than this—incorporating investigation, communication, experimentation and even creativity. The purpose of this book is to outline just one approach to teaching and learning mathematics that enables students to experience mathematics as something more than training to be human calculators.

If mathematics is rule bound and fundamentally unattractive to adolescents who value breaking rules rather than following them how do we go about motivating students to learn mathematics? Are we as mathematics teachers fighting a losing battle when we try and motivate students to learn mathematics—a subject which they find boring and which has no bearing on their everyday lives? Many teachers aiming to motivate students head straight for the extrinsic reward—the doors that a good grade in mathematics can open. It is true—doing well in mathematics is a necessary prerequisite for tertiary studies and so students need to be reminded of the importance of mathematics for their future career prospects. Unfortunately, though, many teachers struggle to make mathematics a subject of intrinsic interest.

The more proximal but equally extrinsic motivation for working hard in mathematics is the prospect of future examinations. External examinations are realities that loom large in the minds of participants in schooling systems. It is not just students who are focused on examinations. Classroom teachers are also focused on preparing students for examinations, as are school executives, parental organisations and government departments. Examination results are the most commonly used metric for measuring success or failure of individuals, teachers, schools and school systems.

Whether we like it or not, the reality of external examinations will impact upon activities in the mathematics classroom. For most classroom teachers, this reality is a fixed point around which their teaching practice revolves—while we might wish to try new approaches to teaching mathematics, it is the ability of our students to perform well on these examinations that will determine what education and work-based opportunities they will have access to in the future. As a teacher of mathematics you may feel some concern about the importance placed on these examinations. Irrespective of what we think about these examinations, however, they are a reality in schools that can open or shut doors for students in the future.

What do teachers do in classrooms to ensure that students are sufficiently prepared for their exams? Typically, they waste no time with peripheral matters; instead, they focus on providing students with efficacious strategies for solving problems that they are likely to be examined on. Classroom activity, therefore, is directed very clearly and carefully by the teacher who knows what is likely to be assessed and concentrates on giving students the necessary knowledge to succeed at these exams. I don't mean to suggest that most teachers "teach to the test", but rather that they give priority in their classrooms to giving students strategies that they can use to obtain the right answer.

To analyse this from a commercial perspective, the teacher is, after all, providing a service within a free market where customers are free (to a certain extent) to choose who will provide them with this particular service. The customers, being students, parents, schools and governments expect results for their money (whether that money is paid directly by the customers or not). What type of results is most easily quantified for comparison with other service providers? Answer—examination results.

This approach to teaching mathematics raises many concerns for education researchers who have been arguing for many years now that reform is needed if students are to be empowered to think mathematically in a wide range of different contexts. Consider the following description of mathematical classroom activity made by Welch (1978) in American schools

> In all math classes I visited, the sequence of activities was the same. First, answers were given for the previous day's assignment. The more difficult problems were worked through by the teacher or a student at the chalkboard. A brief explanation, sometimes none at all, was given of the new material, and problems were assigned for the next day. The remainder of the class was devoted to working on the homework while the teacher moved about the classroom answering questions. The most noticeable thing about math classes was the repetition of this routine (Welch, 1978, p. 6)

Reform documents emphasise learning to make sense of mathematics, talking mathematics, engaging in mathematical activities rather than simply practising pre-packaged techniques. Clearly, the reality of mathematical practice described by Welch is a long way from the form of classroom practice recommended by theorists aiming to reform mathematics classroom practice. The 1991 document released by the National Council for Teachers of Mathematics (NCTM) in the United States, for example, outlines five shifts in mathematical practice that need to take place in mathematics classrooms. The first of these involves reshaping the way we view classroom practice—not as a group of individuals working in isolation but as mathematical communities. Second, the NCTM document encourages using logic and mathematical evidence as the process of verification rather than the teacher's authority. Third, teachers need to support a shift in students' mathematical practice from merely memorising procedures toward appropriating mathematical reasoning. Fourth, a shift is required from mechanistic answer finding towards conjecturing and problem solving as the principle aspects of classroom mathematical practice. Finally, the NCTM document recommends a shift towards connecting mathematics, its ideas and application instead of treating mathematics as a body of isolated concepts and procedures.

These proposed changes place a greater emphasis on students working together rather than individually and making sense of mathematical concepts for themselves rather than being given mathematical procedures to be memorised. Within the reform classroom students spend time discussing with each other how to do different questions, asking questions about different approaches and making connections between what they are learning and other mathematical ideas.

This process of guiding student discussions, giving students opportunities to form hypotheses and ask questions of each other is far less rigid and controllable (at least from the teacher's perspective) than what happens in most mathematics classrooms. Perhaps it is not so surprising that this approach to mathematics remains the exception in mathematics classrooms rather than the rule. It does require teachers to take risks, to give students opportunities to direct mathematical activity and make decisions about where their own learning might take them. As teachers ponder how they are going to cover the crowded curriculum that is in place in most secondary schools, taking such risks is probably going to appear somewhat unnecessary—there is already an approach that ensures good exam results at the end of the year and that is the teacher-led approach whereby students appropriate methods and strategies for solving problems likely to appear on the exam.

And yet, what is being proposed is a radical reshaping of how we understand what mathematical practice should look like in the classroom—

that is, what *teaching and learning* mathematics looks like and what *doing* mathematics looks like. Before looking at some of the recent theoretical approaches to mathematics teaching, therefore, the rest of this chapter will be devoted to examining different ideas about how we understand the discipline called mathematics.

What is this thing called mathematics?

So what are we talking about when we use the term "mathematics"? We might be talking about mathematical processes or mathematical products or perhaps both at the same time. Debates focusing on the truth value of mathematical statements (particularly between the Platonic and formulist schools of thought) focus on mathematical products—asking questions about the relationship between systems of mathematics and reality. However, for the mathematics teacher, it is the process of mathematics rather than the products which is of more immediate concern. What does it mean to *do* mathematics—what is this activity called mathematics all about? Before looking more closely at the activity of mathematics, I will briefly outline what is referred to as the fallibilist approach to mathematics which brings together ideas about processes and products. This approach suggests that mathematics is not necessarily the fixed body of knowledge that most people presume.

Current approaches to teaching mathematics arise from this particular view relating to the nature of mathematical knowledge (Lampert, 1990). The common view of mathematics within our culture, and subsequently in schools, is that mathematics is associated with certainty, with being able to get the right answer. These assumptions are shaped by school experience in which mathematics is commonly taught as a fixed, static body of knowledge (Romberg and Kaput, 1999). Doing mathematics means following certain rules, applying certain algorithms, and knowing mathematics means remembering and applying appropriate rules to answer certain questions (Lampert, 1990). However, it is the open character of mathematical findings that allows mathematics to grow and develop (Lampert, 1990). Instead of viewing mathematical understanding as a fixed all-or-nothing concept mathematical understanding can be more accurately described as cumulative, fragmented and incomplete (Lerman, 1998). Mathematical understanding is always open to refutation, potentially advanced through hypothesis testing and strengthened through the development of proofs.

Recent conceptions of mathematics have promoted this *fallibilist* approach to mathematical knowledge (Tymozcko, 1986; Lerman, 1998;

Davis and Hersh, 1980; Lakatos, 1976; Ernest, 1991). Instead of viewing mathematics as the gradual accumulation of truth statements which are irrefutable, mathematical knowledge remains uncertain, dependent on axiomatic systems and open to question. Accepted proofs remain open to refutation (even if many such proofs have remained unquestioned for centuries). Historical examples of mathematical knowledge being reconstructed include the development of complex numbers and the development of alternative axioms to Euclid's five axioms and associated geometries.

Paul Kitcher (1984) provides a definition of mathematical knowledge that reflects this fallibilist perspective. Kitcher argues that mathematical knowledge consists of five components—a language, a set of accepted statements (including mathematical statements, diagrams, definitions, theories and proofs), a set of accepted reasonings accepted by the mathematical community, a set of questions selected as important, and a set of meta-mathematical views (including standards for proofs and definition and claims about the scope and structure of mathematics). Standards for proofs are not open for explicit description but rather proof standards are exemplified in texts taken as paradigmatic for proof making. Exemplary problems, solutions, definitions and proofs provide the foundation for accepted norms and criteria that such aspects of mathematical practice are expected to satisfy.

Kitcher's five components of mathematical knowledge are emergent properties of historical mathematical activity. Mathematical understanding is inextricably linked to the activity of mathematical communities (Toulmin, 1999). This activity can be described as "pattern-seeking" (Schoenfeld, 1992; NCTM, 1989) and the term "mathematics" refers to this form of activity—a science of patterns, rather than a set of statements that are either true or false (Schoenfeld, 1992). Mathematical activity produces "language games" and mathematical "forms of life" (Wittgenstein, 1953) within which mathematical concepts have meaning and make sense to participants in that activity. Classroom communities are contexts in which such activity can be reproduced developing new "language games" and new "forms of life" for students to participate in, giving birth to new forms of mathematical understanding.

A feature of mathematical activity overlooked until recently by theorists working in the field of mathematics education is its collective nature. Doing and thinking mathematics constitute a social practice (Stein, Silver and Smith, 1998) rather than an individual pursuit. Mathematical understanding resides within communities of practice, artefacts and interactions between individuals and their environments (Schoenfeld, 1992). Mental events and activities can be external to the body—suggesting that the concept of the individual mind needs to be replaced with the concept of collective knowledge (Toulmin, 1999; Ernest, 1998).

The dialectical relationship between the individual and their cultural heritage also contributes to the distributed nature of mind and cognition. Vygotsky (1978) suggests that wider cultural forces shape intellectual development and are in turn shaped by the products of human activity. Participating in a cultural practice results in the transformation of individual consciousness, just as the individual contributes to the ongoing evolution of the same cultural practice. Mathematical understanding, therefore, exists in Popper's World 3 knowledge realms (Popper, 1972; Bereiter, 1994). Popper's World 3 knowledge refers to knowledge that exists as a public, collective object outside an individual's mind as part of the public domain. Such knowledge structures have histories and are open to criticism and falsification. Effective mathematics classroom environments support students' engagement in this process of ongoing mathematical practice.

So much for the philosophers' view of mathematics—they don't have to teach Year Nine trigonometry! What does such a view have to do with teaching mathematics? The answer to this question as it turns out is quite a lot. If mathematics is open to refutation, then the practice of mathematics in the classroom should reflect the questioning of assumptions and the testing of ideas to see where they work and where they might not work (Romberg and Kaput, 1999). Students should be encouraged to find out why techniques work, to invent new techniques and justify assertions in the same manner that research mathematicians operate. Furthermore, rather than engaging in this activity on their own, students should be given opportunities to be a part of collective mathematical activities that incorporate mathematical practices that have evolved over time. This involves, amongst other things, being familiar with mathematical tools, conventions, methods of analysis and agreed-upon approaches to mathematical knowledge-building.

What, then, distinguishes mathematics from other domains of knowledge taught at schools? First and foremost, mathematics involves a higher level of abstraction than other subjects studied at school. Kinard and Kozulin (2008) suggest that mathematical activity is fundamentally about abstraction. They describe mathematical activity in terms of "...(making) meaning from aspects of patterns and relationships through abstraction" (p. 24). According to Kinard and Kozulin mathematics involves a "detachment" from specific examples of such patterns and relationships working with abstractions, compatible with a larger conceptual framework, reflecting the framework shared by mathematicians. Mathematical learning activity, therefore, represents a distinct form of learning activity within the classroom concerning the appropriating of mathematical abstractions.

Mathematical practice within the school context—what students actually do in the classroom rather than what they learn about—is also quite different from what they typically do in other disciplines studied at school such as science, history and studies of literature. More than any other discipline studied at school, mathematical understanding develops principally through a process of deduction—that is, starting with certain premises, mathematical practice involves identifying logical consequences and using these consequences as the premises for further deductive arguments. This is in contrast to the process of induction that begins with specific exemplars to build a case for more general statements. Deduction and induction are necessary for the development of understanding in all domains, including mathematics. Schoenfeld's approach to developing an effective knowledge base for solving mathematical problems, for example, emphasises the value of both experience at different types of problems (developing expertise through induction) and the development of effective methods (appropriating deductive principles) (Schoenfeld, 1987). The relative significance of these two reasoning processes may vary across different mathematical domains, however, the cumulative process of building a body of knowledge from first principles is characteristic of most mathematical practice. Students learn about trigonometry, for example, by building on other concepts such as similar triangles, right-angled triangles and algebra rather than identifying multiple examples of different triangles with the same angles and drawing out general statements about each type of triangle.

The value of deduction as a logical procedure, however, is dependent on the viability of the initial assumptions. Included in the category of assumptions are definitions, conventions and mathematical tools such as systems of notation that carry implicit assumptions. In the case of notation, for example, certain systems simplify the practice of mathematics and therefore become accepted as viable systems. For students and professional mathematicians it is the assumptions upon which deduction proceeds that remain open to question rather than the procedures of mathematical logic (Lampert, 1990). Developments in mathematics often occur through the questioning of these assumptions. Historical examples of particular note include the development of complex analysis, non-Euclidean geometries and the mathematics of the infinitesimal. The development of symbolic representations is also commonly associated with advancements in the development of mathematical practice. The development of algebra, the Cartesian plane, the Argand diagram and the Hindu-Arabic number system itself have all resulted in progress in mathematical understanding.

Similarly, within the classroom, quantum leaps in mathematical understanding are associated with the questioning of assumptions (Richards, 1991). The questioning of assumptions represents a significant challenge for most students of mathematics, and students are unlikely to see the need for such revisionist activity without the cultural insights afforded by the teacher into the viability of different assumptions. Students need to appreciate the cultural reasons for such mathematical activities that make use of negative numbers, complex numbers, the replacing of numbers with pronumerals and other accepted mathematical notions. The acceptance of such concepts historically has occured through the negotiation of mathematical understanding by the community of mathematicians. This same process of negotiation is required in the classroom between members of the classroom community.

Mathematical understanding, therefore, develops through deduction and the reformulation of assumptions. Most mathematical practice in classrooms, however, focuses on induction—developing mathematical understanding through multiple examples and exercises. Practising mathematical techniques encourages students to ask “how” questions. Mathematical discussions, however, also have the potential to raise “why” questions about mathematical structures and procedures. Deducing mathematical principles and procedures in traditional classrooms is typically the activity of the teacher alone who presents certain procedures, including proofs underlying such procedures, for students to learn for later reproduction in examinations. By reshaping classroom practice to place greater emphasis on classroom discussion, perhaps it is possible for students to generate their own deductions, to form their own conclusions about the validity of their assumptions and assess those put forward by other students. Perhaps it is possible to realise the principles put forward by the reform documents in classrooms where discussion rather than practise is the dominant feature of classroom activity.

CHAPTER TWO

WHAT CONSTITUTES EFFECTIVE TEACHING OF MATHEMATICS?

Teaching mathematics is an art form. On one level it can be thought of as solving ill-defined problems. Or perhaps the mathematically inclined would describe the classroom as a chaotic system within which there are multiple patterns and factors interacting with each other. As a mathematician this can be a frightening prospect given that mathematicians typically feel more comfortable working in systems that are well-defined and in which each different factor can be readily isolated and understood in isolation. But in reality, mathematics teachers are artists rather than technicians. They become teachers through experience, learning about the individuals in each class and the dynamics of each group of students, constantly making assessments about students' progress, determining strategies for presenting information in new and appropriate ways, planning programs, developing and administering assessments as well as motivating students to keep going even when they struggle with difficult concepts.

It is one of those areas of expertise that draws on both sides of the supposed right brain/left brain divide—mathematics teachers need to be comfortable with a wide range of mathematical abstractions, techniques, concepts, ideas and generalisations. Mathematics teachers, however, also need to feel comfortable working with individuals, with people who are fundamentally unpredictable, beyond complete understanding, each person representing a unique exemplar of multiple overlapping abstractions.

It is perhaps not surprising, then, that research in mathematics education doesn't always present us with safe, easy to follow sets of instructions. On one level this might be disappointing to the teacher looking for the quick fix that will ensure everyone in their class passes their next examination. But if such a formula did exist it would be far more economical for education systems to program computers to present lessons rather than pay teachers every week. It would be even more economical to simply make a teaching program available to students to apply at home saving money on school buildings as well. As it turns out, the principles promoted in the literature are contrary to either computer-focused scenario. The classroom and the teacher support learning environments that are beyond the algorithmic capacities of computers.

Instead of giving teachers a foolproof method, then, research in the field of mathematics contributes to classroom practice principles which teachers are able to use to inform their approach to teaching. So what are these principles currently being promoted by mathematics education journals? Given that there are many journals devoted to mathematics education and an even greater number of journals devoted to education in general, the classroom teacher is faced with an overwhelming task of extracting principles that she can appropriate in her classroom. The purpose of this chapter is to provide a brief summary of some of the main ideas about classroom practice that can be found in the mathematics literature and the broader field of educational psychology. Because this book is designed primarily for use by classroom teachers, however, issues relating to social equity, gender and curriculum that have an impact on classroom practice at the level of policy as well as practice have not been covered. Furthermore, the intention of this book is not to provide ideas and strategies for designing specific learning tasks for different areas of mathematics. Instead, this book looks at ways of reshaping classroom practice more broadly, within which specific learning tasks can be introduced as appropriate for different aspects of the mathematics curriculum.

Approaches to teaching mathematics have their origins in broader theories of education that are themselves derived from psychological theories of learning. There are many different approaches to teaching that have their adherents in the field of education—the field of direct instruction, for example, that draws heavily on behaviourist concepts of learning (Adams and Engelmann, 1996), constructivist approaches to education that draw on the work of Piaget (Steffe and Gale, 1995) and information processing models that use ideas from cognitive psychology (Anderson, Greeno, Reder and Simon, 2000). However, the reform movement in mathematics education has drawn most heavily from constructivist approaches—in particular the social constructivist approach to education that has its theoretical roots in the work of Russian psychologists such as Lev Vygotsky.

Vygotsky was a Russian psychologist who died in 1934 at the age of 37, however it was not until the 1960s that his works were translated into English. Vygotsky's approach to learning proposed that individual development is social in origin. He argued that social interaction with others provides the foundation for individuals coming to understand and appropriate ideas for themselves. Classroom discussion, dialogue and collaboration, therefore, are critical components of Vygotskian-based approaches to teaching and learning. Instead of students working through set exercises on their own, Vygotskian approaches encourage interaction between students—which occurs most easily and naturally in a classroom. The role of the teacher as someone who can model different practices and assist individuals to perform tasks just beyond their current level of understanding is also emphasised in Vygotskian approaches.

As well as focusing on the social nature of development, Vygotsky also argued that learning is shaped by the use of different tools, both physical and mental, that reshape individual mental processes. Instead of a stimulus-response mechanism at the heart of learning, Vygotsky suggested that tools which are cultural constructions radically reshape the interaction between individuals and their environment. In the field of mathematics education, Vygotskian approaches have recently received considerable attention (Lerman, 2006; Radford, Bardini, Sabena, Diallo and Simbagoye, 2005; Yoshida, 2004). Research has examined how different technological tools can impact on the learning that takes place in classrooms (Hollebrands, Laborde and Strässer, 2008; Benson, Lawler and Whitworth, 2008) as well as how using linguistic tools and different forms of language play a role in shaping learning activities (Bartolini-Bussi and Mariotti, 2008; Mercer, 1995; Wegerif, Mercer and Dawes, 1999).

Vygotsky's ideas have also shaped sociocultural approaches to education that describe learning as participation in cultural practices (Verschaffel, Greer and Torbeyn, 2006; Franke, Kazemi and Battey, 2007; Lave and Wenger, 1991; Greeno 1997) and emphasise the importance of engaging in collective activity that is shaped by the shared goals of its participants (Leont'ev, 1978; Engeström, 1987). Such approaches to understanding learning make use of the concept of activity. "Activity", or *deyatel'nost* in Russian, does not refer to individual actions but rather refers to a system that incorporates individual agency, mediation of action through tool use, and the recognition of historically developed rules and practices of divisions of labour.

> Activity is a molar, not an additive unit of the life of the physical, material subject. In a narrower sense, that is, at the psychological level, it is a unit of life, mediated by psychic reflection, the real function of which is that it orients the subject in the objective world. In other words, activity is not a reaction and not a totality of reactions but a system that has structure, its own internal transitions and transformations, its own development. (Leont'ev, 1978, p.50)

This system, rather than its individual components, is the unit of analysis within activity theory since activities involve human actors who are motivated towards an *object*, their actions being *mediated* by *tools* and their *community* (Engeström, 1987). The *object* (or goal) of an activity system is the defining feature of the system and the notion of goal-directed behaviour is central to the concept of activity (Wertsch, 1981).

Several principles arise from this activity-oriented approach to cognitive processes. The first principle is that human activity reshapes the physical and the mental world. By engaging in practical activity or *labour*, and using instrumental as well as psychological tools, both the physical and the mental world are transformed. Secondly, activity theory emphasises the cultural and historical processes that underlie activity systems. Throughout this book, whenever I use the word “activity” I am thinking of activity in this theoretical sense—as a collective human enterprise that incorporates ways of doing and ways of thinking that have historical antecedents.

Sociocultural perspectives can be identified in the different reform documents which focus on collective, practical activity rather than individual cognition as the fundamental form of mathematical activity (Franke, Kazemi and Battey, 2007). The *National Statement on Mathematics for Australian Schools*, for example, sees mathematics as an emergent property of purposeful, student activity (AEC, 1991). The NCTM document *Professional Standards for Teaching Mathematics* (NCTM, 1991) encourages teachers to view classrooms as mathematical communities instead of a collection of individuals. The NCTM document *Principles and Standards for Mathematics Teaching* (NCTM, 2000) also supports providing students with opportunities to engage in collaborative activities that encourage students to justify their thinking, develop conjectures, conduct experimentation with various approaches, and construct and respond to mathematical arguments.

Documents encouraging reform as well as the wider mathematics education literature outline a number of principles relating to classroom practice. Across different documents and theoretical works eight principles for the effective teaching of mathematics are evident. These eight principles are fostering collaboration, developing a dialogical classroom in which many different voices are heard, providing opportunities for students to become more central participants in communities of practice, embedding learning experiences in cultural practices, encouraging students to become self-regulated learners, making mathematics meaningful, supporting intentional learning, and developing learning experiences that are appropriate given students' prior experience and development.

Principle One: Fostering collaboration in the classroom

Let's start with the most challenging idea regarding effective mathematics teaching—that effective mathematics teachers support students' collaboration with one another. The word "collaborative" has replaced the word "co-operative" in more recent theoretical approaches to learning together to emphasise a different approach from that promoted by co-operative learning theorists. Co-operative learning involves each member of the group having a set role or task to perform which then contributes to the overall learning of the group. Collaborative learning, however, describes learning in which a group of people have a shared goal that they strive towards together. Each member of the group has the same role within the group and that is to assist the group to achieve its collective goal. It involves creating a shared social world (Palincsar and Herrenkohl, 1998) thus providing students with many challenges as they deal with different perspectives and opinions and work towards achieving consensus.

Part of working in groups is often structured or unstructured peer tutoring. When students teach other students research has demonstrated benefits for all students who take part—those who are tutoring and those who are being tutored. For those who take on responsibility for teaching others, there are benefits when these students are encouraged to clarify and organise their own thinking as well as present ideas in a number of different ways (Bossert, 1988; Webb, 1989). For students receiving assistance, when this assistance is targeted specifically towards areas of misunderstanding and the student has an opportunity to use the explanation to solve a problem, these students have also benefited from peer tutoring (Webb, 1989).

Every document on reforming mathematics teaching includes some acknowledgement that collaborative learning is a critical component of good practice. Collaborative discussions provide students with multiple perspectives from which they can develop their own understanding as well as develop their capacity to understand others. Drawing on multiple perspectives encourages a flexible approach to learning. In collaborative contexts, students also have opportunities to explain their ideas to other students. As students verbalise or write down their ideas they make them public, open to scrutiny and challenges from other students.

Goos (2004) reports on the role of one teacher in transforming a mathematics classroom from fostering learning mathematics through memorisation and rote learning to one in which collaboration amongst classroom participants created a "climate of intellectual challenge" (p.259). Within this emergent mathematical community of practice, students learnt how to participate in mathematical discussions focused on solving problems, set forth conjectures, and respond appropriately to the mathematical ideas of others. Goos' "inquiry mathematics" approach places collaboration and discussion amongst students at the centre of classroom activity designed to facilitate students' participation in a certain cultural practice. Drawing on Vygotsky's concept of the zone of proximal development Goos describes how students are "pulled forward" (Goos, 2004, p.262) into different communities of inquiry through their interaction with other peers and the classroom teacher who gradually introduces cultural forms of knowing.

Mathematical understanding in collaborative groups emerges through articulating what one knows, communicating one's knowledge and reflecting on one's conceptual understanding. Articulation of conceptual understandings in collaborative groups represents a public form of reflection (Carpenter and Lehrer, 1999). Through encouraging student discussion and collaboration classroom teachers can develop a supportive environment for students in which serious mathematical thinking is the norm. Collaborative activities provide a forum in which students are expected to justify their thinking, develop conjectures, conduct experimentation with various approaches, and construct and respond to mathematical arguments (NCTM 2000).

Few mathematics teachers engage students in a public analysis of the assumptions behind different explanations that lead to "correct" answers (Lampert, 1990). Typically, students are required to provide answers in classrooms without reasons, developing into effective symbol manipulators without necessarily grasping the semantic meaning of the symbols manipulated. Students of mathematics often resemble the interpreter of Chinese symbols in Searle's Chinese room thought experiment (Searle, 1980). This interpreter has no understanding of Chinese, but is able to respond to written requests in Chinese using a book of procedures that describes what to write in response to different sets of characters. In a similar way, many students are able to provide the correct symbolic answer to a question without necessarily understanding the meaning of the question or the symbols used to answer it.

Principle Two: Developing dialogical classrooms

An important learning principle for effective mathematics classrooms is that learning occurs best within the context of dialogue rather than monologue. Shared activity needs to be genuinely discursive activity in which pupils are encouraged to ask questions, seek answers, consider different perspectives, exchange viewpoints, and add their findings to existing understanding (van Oers, 1996). It is conversations between students that foster mathematical argumentation (Lampert, 2001) and create a public knowledge base (Hiebert and Grouws, 2007). While traditional mathematics classrooms typically present a single approach for solving a certain type of problem, reform documents in Australia and overseas encourage the development of alternative solutions (DETYA, 2000; AEC, 1991; NCTM, 2000). The NCTM document *Professional Standards for Teaching Mathematics* (NCTM, 1991) also encourages the development of a dialogic classroom. The NCTM recommends that teachers of mathematics need to view the classroom as a community rather than a set of individuals, in which logic and mathematical evidence become the basis of verification rather than the teacher's authority.

Consider the findings of the report conducted by the Victorian Department of Education in conjunction with the Catholic Education Office and the Association of Independent Schools (DEST, 2004). In their report they concluded that effective teaching involves classroom teachers using fourteen different communication strategies designed to evoke discussion between the teacher and students, to begin a dialogue that provides opportunities for students to develop new understandings. This finding is echoed in the research literature that has documented the positive outcomes from encouraging discursive environments in the mathematics classroom (Lampert, 2001; Franke, Kazemi and Battey, 2007).

But the concept of dialogical classrooms presents a much greater challenge to us as mathematics teachers than merely talking about mathematical ideas with students—after all, this is what happens in every classroom to some degree or other. "Dialogical" refers to the existence of multiple voices in the classroom—a "voice" can be that which is stated by an individual or it can be what is read in a textbook. It might also be the voice of someone from outside the classroom whose ideas are presented by someone inside the classroom. Effective mathematics classrooms support the vocalisation of multiple voices thereby creating a rich learning environment in which there are many dialogues taking place at the same time between different voices.

This is a frightening concept for teachers who may now be visualising a learning environment that is out of control and more like a cacophony of voices than one in which these voices interact with each other to create opportunities for new forms of understanding. One theorist describes dialogical spaces using the metaphor of the "carnival" in which there can be a sense of disorientation rather than a growing awareness (Bakhtin, 1984). Managing these complex conversations between students and multiple voices of the perspectives of the mathematical community is certainly not easy. This, to my mind, is part of the artistic nature of mathematics teaching. As I suggested at the beginning of this chapter, teaching mathematics is not about following a particular script or program—if it were, computers could replace human teachers without too much trouble and students could learn mathematics from home using well-designed teaching programs. Instead, good teaching involves balancing a large number of concerns almost continuously during a mathematics lesson (Lampert, 2001).

The model of teacher-student dialogue that emerges from the classroom model to be outlined more fully later in this book is that of a small group conversation in which the teacher participates as a member. The teacher clearly has a special role within the conversation as a representative of the culture of mathematicians (Kinard and Kozulin, 2008; Lampert, 1990; van Oers, 1996; Forman, 1996) providing students with cultural tools for solving different problems. However, the students within the group have the potential to set the agenda for such conversations. Students bring to these conversations their own opinions formed through discussion amongst themselves for comparison with existing cultural modes of thinking (van Oers, 1996). Within such a context, the teacher is more able to ensure the involvement of all students in the process of negotiating mathematical meaning by monitoring the public articulations of each participant.

Within sociocultural theory, the importance of voices in the public sphere for the development of personal understanding is evident in Vygotsky's notion of internalisation. Social relations, according to Vygotsky, are genetically prior to individual understanding. Vygotsky's famous genetic law of psychological development emphasises the existence of mental functions within the social plane preceding their incorporation into the psychological plane.

> Any function on the child's cultural development appears twice, or on two planes. First it appears on the social plane and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category ... It goes without saying that internalisation transforms the process itself and changes its structure and functions … Social relations or relations among people genetically underlie all *higher* functions and their relationships (italics added). (Vygotsky ,1978, p.57)

The process of internalisation represents a complex process of transforming social phenomena into intrapersonal mental functioning (Cole,