# Logic and Knowledge

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# FOREWORD

The theme of this book is the relation between logic and knowledge. This problematic relation has given rise to some of the most important works in the history of philosophy, from Books VI-VII of Plato's *Republic* and Aristotle's *Prior* and *Posterior Analytics*, to Kant's *Critique of Pure Reason* and Mill's *A System of Logic, Ratiocinative and Inductive*. It provides the title of an important collection of papers by Bertrand Russell (*Logic and Knowledge. Essays, 1901-1950*, ed. by Robert Charles Marsh, Allen & Unwin, London 1956). However, it has remained an underdeveloped theme in the last century, because logic has been treated as separate from knowledge.

For example, in the Preface to his *Introduction to Logic and to the Methodology of Deductive Sciences*, Tarski states that although "the methodology of empirical sciences constitutes an important domain of scientific research," one must admit that "logical concepts and methods have not found any specific or fertile applications in this domain. And it is certainly possible that this state of affairs is not simply a reflection of the present stage of methodological research," for one may doubt whether any special 'logic of empirical sciences' as opposed to the 'logic of deductive sciences,' exists at all (Oxford University Press, Oxford 1994, p. xii).

We cannot hope that this book will make up for a century-long absence of discussion. Rather, our ambition is more modest: calling attention to the theme and stimulating renewed reflection upon it. Thus in this collection of essays, we address the theme as a topic of current debate, or as a historical case study, or when appropriate as both. The book is divided into three sections, "Logic and Knowledge," "Logic and Science," and "Logic and Mathematics," without a sharp separation among them. Indeed, some of the essays are borderline, so their inclusion in one rather than another part is somewhat subjective. Each essay is followed by the comments of a younger discussant, in an attempt to transform what might otherwise appear as a monologue into an ongoing dialogue. Each of the three sections begins with a historical paper.

The first section deals with the relation between logic and knowledge in general, without much reference to specific disciplines. Some essays start with logic and address epistemological problems in terms of it; others find that logic generates its own peculiar problems. Logic applies itself to epistemology. Other essays start with epistemology and then are driven to take up logical issues. Epistemology turns to logic for help and clarification, or epistemology applies itself to logic and demands more of logic than mere deduction.

The second section deals with the relation between logic and the sciences. Some essays start with an epistemological problem in the sciences, or a feature of scientific knowledge, and then show how logic can develop it or sort it out or clarify it. Logic applies itself to science. However, essays that favor 'naturalism' see logic as one of those interesting human habits, which allows us to 'survive as the fittest', and which science can study. Science applies itself to logic.

The third section deals with the relation between logic and mathematics. Some essays start with an epistemological problem in mathematics, or a problematic feature of mathematical knowledge, and then show how logic can (or cannot) develop it or sort it out or clarify it. Logic applies itself to mathematics. Other essays treat logic as a mathematical discipline, and try to show what we can learn about logic by algebraizing or topologizing it. Mathematics applies itself to logic.

## Section I. Logic and Knowledge

In “The Cognitive Importance of Sight and Hearing in Seventeenth and Eighteenth-Century Logic,” Mirella Capozzi presents evidence for the importance that the sense of sight acquired in seventeenth and eighteenth century logic, by considering the attempts to transfer the cognitive function of visible signs from algebra to logical calculi by authors such as Leibniz, Lambert, Holland and Ploucquet. This involved a separation of logical calculi from ordinary language and the sense of hearing, and a parallel separation of logic from rhetoric. Capozzi also points out that Leibniz and Lambert investigated not only the cognitive use of visible signs in logical calculi but, in their search for a rational treatment of discovery, looked at the cognitive use of mnemonic and inventive techniques employed in rhetoric to find persuasive arguments in the sphere of the audible signs of ordinary language. She argues that this happened because these authors considered the study of both deduction and discovery as a task of a logic practiced by the embodied minds of human beings, relying on their senses as well as on reason. The battle against ‘psychologism’ in the past century has tried to suppress this dimension of deductive logic, and has tried to separate logic from discovery, which has been attributed to luck or to inexplicable personal qualities of the discoverers. Capozzi concludes that Leibniz and Lambert were able to assign logic a wider role than the study of correct arguments, because they investigated logic within the general context of their conceptions of knowledge. Within that context, it was possible for them to acknowledge the cognitive role played by sensibility in logic and to admit that, if logic relies on the sense of sight for manipulating written signs and symbols in calculi, then it can try to establish a methodology of discovery by borrowing inventive methods from a scholarly tradition having its roots in orality.

In her comments, Chiara Fabbrizi underlines the importance of Wolff for the discussion concerning the cognitive function of signs, and supports Capozzi's reading of Lambert's heuristics by insisting on Lambert's efforts to establish a connection between spoken language and heuristics.

In "Nominalistic Content," Jody Azzouni considers Quine's indispensability argument. Quine proposed the following criterion of existence: if we write up a theory as a first-order theory, its universe of discourse will exhibit its ontological commitments: what the theory asserts to exist are just those individuals over which the quantifiers range. He used this criterion to argue that mathematical objects must exist, because they are indispensable to our best scientific theories. This argument has troubled nominalists like Azzouni ever since, who wish to keep ontological commitments to a minimum and prefer an ontology that consists only of empirically detectable objects with empirically detectable properties. Azzouni himself rejects Quine's criterion, but in this essay he examines the objections of another kind of nominalist who accepts the criterion but tries to find a way around it by claiming that for every sentence S (with nominalistically inadmissible content) in the theory, we can identify a corresponding C (with only admissible content). Then we can just say that S is a proxy for C, and we don't have to take the ontological commitments of S literally or seriously. Azzouni cites variations on this theme offered by Melia, Balaguer, Rosen and van Fraassen: in all these cases, we need only be committed to the content of C, even though it is often hard to articulate what that content is. Azzouni illustrates the difficulties inherent in this approach by constructing examples that use particles moving in space-time; nominalists have an especially hard time accounting for continua, since when a continuum is considered as a set of points the cardinality is non-denumerable. In this case, claims about such small systems of particles (C, which would have to report empirically collected evidence) are not understandable independently of the proxy (S, which might assert for example the existence of a continuous trajectory). But then S must still be considered part of the theory; it has not really been eliminated along with its objectionable ontological commitments. Many candidates C for reducing S to proxy status have the same limitation, so Azzouni concludes by suggesting that a dedicated nominalist should join him in rejecting Quine's criterion, and see where that project leads. Another possibility, of course, is to wonder whether realism in mathematics might deserve reconsideration.

In her comments on Azzouni's paper, Silvia De Bianchi claims that Azzouni does not really explain how a mathematical fiction can be relevant to our understanding of the physical world, by challenging the ground of his distinction between concreta and abstracta.

In "A Garden of Grounding Trees," Göran Sundholm distinguishes between epistemic and ontological grounding trees in Aristotle, Frege, Leibniz and Bolzano. Epistemic grounding trees are demonstrations which serve to ground a judgment in terms of immediate evidences. They concern the *ordo cognoscendi* and are finite. Conversely, ontological grounding trees are trees with respect to propositions which concern the *ordo essendi* and may be infinite. For example, in the case of Frege, an epistemic grounding tree for a general arithmetic truth of the form $(\forall x \in \mathrm{N})A(x)$ is a finite grounding tree, with inferences, which includes the principle of mathematical induction whose validity reduces ultimately to suitable chains of logical inferences. Conversely, an ontological grounding tree for $(\forall x \in \mathrm{N})A(x)$ is not grounded by the principle of mathematical induction but by all truths that are its instances, $A(0)$, $A(1)$, $A(2)$, ... , so it is a tree with an infinite branching. The tree is well-founded, each branch is finite but the bound on lengths of the branches may be infinite. Such a tree gives an ontological grounding, but not necessarily an epistemic one. Sundholm points out that, somewhat surprisingly, such infinite well-founded trees are found not only in the realist approaches of Aristotle, Frege, Leibniz and Bolzano, but also in the anti-realist approach of Martin-Löf. In his Constructive Type Theory there is a distinction between epistemic demonstrations, which serve to establish judgments, and proof-objects, which can be infinite. This comes about because, for example, in the case of an implication $A \supset B$, we continue upwards from '$\lambda x.b$ is a canonical proof of $A \supset B$' to: 'For each canonical proof $a_i$ of $A$, the tree is continued upwards with proof-objects of $B$: $b[a_0/x]$, $b[a_1/x]$, $b[a_2/x]$, ... that we obtain by applying the proof of $A \supset B$ to $a_i$.' Unlike epistemic demonstrations, these trees cannot be effectively generated and are not decidable.

In his response, Luca Incurvati critically evaluates two claims made by Sundholm on how to interpret natural deduction: that the objects appearing in a derivation cannot be taken to represent sentences, and that one should account for assertion under an assumption by conditionalizing content.

In “Logics and Metalogics,” Timothy Williamson discusses Dummett’s claim that the logic of the object-language should be as insensitive as possible to the logic of the metalanguage, in order to avoid begging questions about the former in the latter. According to Williamson, this claim conflicts with the goal of expressing the meaning of expressions of the object-language in the metalanguage. As a test case for Dummett’s claim, Williamson considers the standard metatheory of modal logic, since its metalanguage is an extensional language with enough non-logical primitives to express set theory and the syntax of the modal object-language but with no modal expressions at all. Questions in the model theory of modal logic are answered by purely mathematical proofs. Philosophical disputes about possibility or necessity are irrelevant to this process. The result is, however, that the model theory of modal logic does not resolve those disputes about the correct logic of genuine modal notions. Williamson concludes that semantics is no royal road to the resolution of disputes in modal logic. The role of semantics with respect to logic is similar to its role with respect to, for example, physics. There is no reason for physicists to expect much help from a good semantic theory for the language of physics. We should not be too quick to assume that the case of logic is radically different. Indeed, according to Williamson, one simple line of thought suggests that the role of semantics will indeed be secondary. Logical principles in general are not metalinguistic in content, and so should not be explained in semantic terms. Although we may need semantics to clear away confusions which block us from accepting valid logical principles, it is not semantics that explains the principles themselves. The same applies to the rejection of invalid logical principles, once they are put in the form of universal generalizations, perhaps in higher-order logic. We should not expect semantics to exceed its proper task.

In his response, Cesare Cozzo argues that Dummett’s view is different from the one criticized by Williamson. Dummett does not claim that a semantic theory alone can settle a dispute over the validity of a fundamental logical law, for a semantic theory is usually supposed to be a theory of logical consequence, while, according to Dummett, such disputes should be settled by a theory of meaning, which is a theory of understanding.

In “Is Knowledge the Most General Factive Stative Attitude?” Cesare Cozzo discusses Williamson’s view that knowing is merely a state of mind, more precisely, knowing is the most general factive stative attitude, that which one has to a proposition if one has any factive stative attitude to it at all. In a language the characteristic expression of a factive stative attitude is a factive mental state operator (FMSO). Williamson’s proposal is that “if Φ is any FMSO, then ‘S Φs that A’ entails ‘S knows that A.’” However, Williamson does not show that *every* FMSO conforms to his principle that factive-stative attitudes entail knowledge. Cozzo gives a counterexample. The counterexample is a new verb ‘yig’ whose meaning can be given by saying that ‘S yigs that A’ is true if, and only if, ((A & S grasps the proposition that A) & S does not know that A). As defined in English, ‘yig’ is semantically analysable, but there could be a language Yig-English, otherwise identical to English, that took ‘yig’ as semantically unanalysable and ‘know’ as semantically analysable in terms of ‘yig’. Cozzo argues that ‘yig’ is a factive mental state operator (FMSO), and maintains that if knowing is a mental state, then the state of yigging (i.e. the state of grasping a true proposition in ignorance of its truth) is a mental state and a factive stative attitude. If this is right, knowledge is not the most general factive stative attitude.

In his response, Timothy Williamson argues that, by contrast with knowing, yigging lacks the unity and naturalness to play the required causal-explanatory role or to constitute a genuine mental state. He compares Cozzo’s attempts to establish a more than purely formal symmetry between “know” and “yig” to Goodman’s attempts to establish a more than purely formal symmetry between the pair ‘green’ and ‘blue’ and the pair ‘grue’ and ‘bleen’, and argues that they fail for similar reasons.

In “Classifying and Justifying Inference Rules,” Carlo Cellucci discusses the view that inference rules must be divided into deductive, that is, necessarily truth preserving, and ampliative, that is, not necessarily truth preserving, and that there is a basic asymmetry between them: while deductive rules can be justified, ampliative rules cannot be justified. According to Cellucci, this view is unsatisfactory because, on the one hand, there are inference rules, such as abduction, which are neither truth preserving nor ampliative, and, on the other hand, all justifications of deductive rules, non-deductive rules and abduction that have been given are inadequate. In particular, the truth-functional and the inferential justification of deductive rules are circular, the intuitional justification of non-deductive rules is untenable because intuition is fallible, and a justification of such rules in terms of fallible intuition make their acceptance ultimately depend upon plausibility rather than intuition. Therefore Cellucci proposes an alternative classification of inference rules, one which takes into account their role in knowledge. He argues that inference rules must be divided into ampliative and non-ampliative. Ampliative rules consist of non-deductive rules and are not necessarily truth preserving. Non-ampliative rules must be divided into deductive rules and abduction, where the former are truth preserving while the latter is not necessarily truth preserving. Moreover, he argues that deductive rules and abduction can be vindicated with respect to the end of making explicit the content or part of the content that is implicit in the premises, while non-deductive rules can be vindicated with respect to the end of discovering hypotheses. As to the alleged basic asymmetry between deductive and non-deductive rules, Cellucci argues that there is no such asymmetry because the justification of deductive and non-deductive rules raises similar problems and must be approached in much the same way.

In her response, Norma B. Goethe points out that, in his *Treatise*, Hume attacks the assumption of general principles as ultimate reasons which may be discovered without consulting any form of experience. But in that work there is still no asymmetry between deductive and inductive reasoning, as both forms of reasoning raise similar objections. Only with his *Enquiries*, Hume allows the asymmetry view to enter the scene hand in hand with the distinction between moral and demonstrative certainty.

## Section II. Logic and the Sciences

In "The Universal Generalization Problem and the Epistemic Status of Ancient Medicine: Aristotle and Galen", Riccardo Chiaradonna faces the problem of generalization, examining some issues in ancient Greek medicine, in particular the fact that we cannot attain "precise rational knowledge of an individual in his/her singularity (*individuum ineffabile*)." The tension between generalization and observation of particular entities in medicine lies at the core of the epistemology of Galen. As a matter of fact, he states that the facts investigated by medicine are not necessary, but allow for exceptions and are intrinsically uncertain, and exceptions cannot be explained away as residual, since medicine aims to heal every single patient in his/her individuality. Galen aims to show that medicine is a genuine demonstrative science, but he shares the overall Aristotelian account of scientific knowledge, which considers medicine as something outside the domain of science: the lack of complete generalization affects the status of medicine. In order to resolve this tension, Galen looked at methods such as *inventio medii*, the geometrical method of analysis, and non-deductive kinds of inferences (abduction and analogy). Chiaradonna remarks that unlike Aristotle, Galen argues that medicine deals with facts which are neither intrinsically uncertain nor impossible to determine. Like Aristotle, Galen maintains that medical skill is ultimately directed towards individuals and that therapies cannot be repeated without exceptions for all individuals of the same kind. However, unlike Aristotle, Galen argues that via technical conjecture the doctor is able to attain a virtually exact knowledge of each individual and of the particular circumstances affecting his or her medical treatment.

In her comments on Chiaradonna’s paper, Diana Quarantotto faces the challenge posed by Galen to Aristotle’s epistemology and defends Aristotle’s conception of the epistemic status of medicine. She does this mainly by focusing on the different views held by Aristotle and Galen on the standards a discipline must satisfy to qualify as a science, on the problem of the relation between individuals and universals within medicine, on the notion of ‘exact knowledge of individuals', and on the definition of medicine's aim.

In his essay “The Empiricist View of Logic” Donald Gillies shows that scientific practice continuously generates knowledge that requires new rational tools to solve problems in specific domains: these new logics, and their principles, are always exposed to revision, as Quine notably argued concerning the law of the excluded middle in quantum mechanics. Accordingly, there is no privileged, ‘central’ and certain method to acquire knowledge about the world. In particular, Gillies argues that scientific practice is the test for logic and there is no other way to justify it but by means of its fruitful applications to the problems: “there is not a single universal logic”, but “different logics may be appropriate in different contexts and problem-situations” and “the use of a particular logic in a particular context is justified by the experience of its successful application in that context.” Specific cases can best illustrate this kind of bottom-up generation of knowledge, conceptual revolutions and logics. A decisive example, according to Gillies, is PROLOG, which “turns out to be the logic appropriate for dealing with some everyday problems such as using databases for travel planning.” So by adapting classical logic for computer specific problems, “the developers of PROLOG inadvertently changed it” and, unlike the quantum logicians, “conscious revolutionaries who failed to carry out their revolution,” the computer programmers, without revolutionary intentions, nonetheless started a revolution in logic. Gillies concludes that classical logic is the logic that underlies a body of mathematics which has an enormous number of practical applications in physics and other areas. Clearly classical logic is of great importance, but it is not universal because there are some applications for which standard mathematics is not the appropriate tool.

In his discussion on Gillies' paper, Paolo Pecere proposes a distinction between the latter's empirical view of logic and the view supported by Quine and Putnam, adding some historical and critical remarks on Quantum Logic

In “Artificial Intelligence and Evolutionary Theory: Herbert Simon’s Unifying Framework”, Roberto Cordeschi discusses the connection between rationality and problem-solving from a cognitive point of view. In particular, the essay examines the notion of bounded rationality developed by Simon and shows how such a notion, with its impact on strategies in heuristic problem-solving is compatible with a refined version of evolutionary theory. He also shows how it might unify the study of both natural and artificial systems. He notes that criticism of naive versions of adaptationism is reminiscent of Simon’s criticism of optimizing classical rationality. One of the main problems with this analogy is the tension between a *random* (or *blind*) and a *guided* heuristic process. Discussing the differences between the notions of satisfaction and optimization, the idea of *selective* trial and error, Cordeschi examines the key idea of *exaptation* as a way to overcome this tension: in fact, he supports the adoption of a mixed strategy as suggested by Simon himself. To support this point Cordeschi takes as example the evolution of robotics from the 1990s to the present time: something like an intermediate strategy has characterized research in robotics during this period. So his conclusion is in line with the foresight of Simon, who argued that the departure from the classical view of rationality “should not be mistaken for a claim that people are generally irrational”, i.e blind, but that “they usually have reasons for what they do.”

In her comment on Cordeschi’s paper, Francesca Ervas argues that the classical view of rationality assumes nothing about human psychological features, while an evolutionary-oriented theory of bounded rationality should presuppose a cognitive inferential mechanism able to “read” others’ minds.

In “Evolutionary Psychology and Morality: The Renaissance of Emotivism?,” Mario De Caro discusses the tenet of the ‘disenchantment of the world’ and the difficulties arising in the scientific account for social sciences. In particular he faces the problems of scientific-naturalist approaches to the moral-ethical domain. This approach, characterized by the ontological principle that the world consists of nothing but the entities to which successful scientific explanations commit us, the epistemological principle that scientific inquiry is, in principle, our only genuine source of knowledge or understanding, and a corollary that ontologically and epistemologically philosophy must be continuous with science. Moreover naturalism poses the ‘placement problem’, namely the problem of locating normative, intentional, modal, phenomenological and abstract entities in the scientific view of the world. De Caro remarks that many philosophical programs are nowadays based on the idea that a suitable version of evolution or cognitive science or neuroscience can illuminate, if not solve, the ‘placement problem’. The paper concentrates on an attempt to naturalize the realm of morality and offers “an evolutionary account of the genesis of morality, by bringing new, and allegedly very relevant, evidence of its origins”. De Caro examines the case of moral disgust as an example for testing the naturalization of morality and concludes that the evolutionary hypothesis “may have something interesting to say as to the enabling biological conditions of morality—something that may contribute to an explanation of how our ancestors evolved in order for us to be biologically able to assume the moral perspective”. But this does not mean that it is relevant to explaining morality as such. The theory of evolution is a strong scientific theory, but we cannot expect too much from it, i.e. that it could explain everything.

In commenting on De Caro’s paper, Annalisa Paese expands on his point that moral reactions are discursive, in the sense of open to rational criticism, in a way in which physiological reactions of disgust are not. She draws on Martha Nussbaum’s work on this topic to argue that the capacities involved in experiencing the emotions that play a role in human moral life need not to be conceived as evolutionarily more primitive than rational capacities but are arguably best understood as further, distinctive, manifestations of those very rational capacities. Once the assumption that equates emotions to thoughtless physiological reactions like oral disgust is rejected, emotions turn out to be quite inadequate materials for a reductionist program.

In his essay “Between Data and Hypotheses,” Emiliano Ippoliti faces the problem of the generation of knowledge, as instantiated in the relationship between data and hypotheses. The paper uses an informational approach to the relationship between data and hypotheses; in particular it considers this relationship as an information-driven interaction, namely a modification coming from data to hypothesis and vice versa. Ippoliti argues that the problem of the ampliation of knowledge must be treated in terms of content and information, by means of a bottom-up, problem-oriented approach. Ippoliti shows the limit of the notions of information proposed by Simon, Popper, Shannon and Kolmogorov and argues for a heuristic conception of information, based on ampliative associations. In particular he argues that hypotheses and data are not separate, as discovery and justification also are not. The paper shows how ampliation of knowledge is a problem-solving and problem-formulating activity based on a specific way to process information, i.e. data-integration, which allows us to modify and create constraints and conditions of solvability. The author remarks that the objects at the frontier of research are unstable and unsaturated, and require redundancy (not elimination) of information in order to be treated. Ippoliti discusses the case of the Feynman path integral to support his view. He argues that this fruitful hypothesis, i.e. sum-over-history, elaborated by Feynman to account for problems in quantum mechanics, is a clear example of a bottom-up, problem-solving process which shows (i) how data are integrated by means of ampliative stages and (ii) the provisional nature of the entities generated to solve problems, which sheds new light on the notions of equivalence, consequence, and knowledge.

In his discussion on Ippoliti's paper, Fabio Sterpetti criticizes Ippoliti's use of the concept of information in addition to those of data and hypothesis, in order to account for the process of ampliation of knowledge, and proposes a semiotic-oriented approach to the issue.

## Section III. Logic and Mathematics

This section begins with Michael Detlefsen's careful look into the historical motives behind Dedekind's commitment to a certain kind of logicism, based on the writings of algebraists like Wallis, Bolzano and Frege. In "Dedekind Against Intuition: Rigor, Scope, and the Motives of his Logicism," Detlefsen argues that Dedekind challenges the notion of mathematical intuition, taken to be an apprehension of particulars that is immediate and passive. Dedekind, he argues, believed that many judgments thought to be "immediate" were really acquired on the basis of elaborate reasonings that sustain them. He was especially interested in the justificative development of arithmetical beliefs, which he thought should be traced back to their basis in the laws of pure thought: counting, for example, should be traced back to our ability to classify and associate. Moreover, if we could locate and articulate the ultimate conditions of thought, they would be true starting points, laws not susceptible of proof because their truth or validity would be required for any argument that might support them. Thus Detlefsen elicits from his writings a new standard of rigor: 'Dedekind's Principle,' which claims that the only proper justification of a principle is a proof, in which all the basic premises are unprovable. One important motivation for Dedekind's Principle, Detlefsen continues, was his concern for the proper scope of truth. Two hundred years before, Wallis argued that geometry was too special (limited in scope) compared to algebra and arithmetic. Since in a rigorous proof we cannot derive a more general truth from less general premises, proofs that begin with geometry are bound to lack rigor. Bolzano made similar arguments: geometrical proofs invert the proper order of reasons and thus exhibit an important type of circularity. Dedekind took these arguments to heart; as Wallis promoted the arithmetization of geometry, so Dedekind promoted the logicist program for justifying the truths of mathematics. Detlefsen concludes, “Seen this way, his logicist program was conservative and traditional rather than radical,” the continuation of an older ideal. Detlefsen’s argument leaves open a number of questions: Is geometry less general than arithmetic? Can mathematical subject matters be elicited from logic, which has no subject matter? Does logic have no subject matter? Can we locate a mathematical intuition associated with arithmetic (or even perhaps logic) as well as geometry? The essays that follow defend intuition in a variety of ways.

In her comment, Marianna Antonutti brings some of the methodological and epistemological issues raised in Detlefsen’s paper into sharper focus and considers how a study of Dedekind's work may be relevant to the contemporary debate in the philosophy of mathematics.

In “Mathematical Intuition: Poincaré, Polya, Dewey,” Reuben Hersh shows that mathematical practice involves knowledge that can’t be demonstrated. He calls it ‘plausible reasoning,’ and argues that it includes mathematical knowledge that we believe in strongly enough to act on it. Not only does mathematical research and instruction depend upon it, we often entrust our lives to the ultimately indemonstrable truths of mathematics that inform the engineering of bridges and airplanes. The practice of the infinitesimal calculus includes much reasoning that is essentially inductive. How, for example, can we know that an infinite sequence converges? Convergence depends only on the (infinite) end of a sequence; a sequence converges to a limit if and only if it gets within a distance $\varepsilon$ of the limit and stays there, for any positive $\varepsilon$ no matter how small. The convergence of the sequence has nothing to do with the beginning; as Hersh observes, “you can change the first hundred million terms of the sequence, and that won’t affect whether it converges, or what the limit is.” But the beginning of the sequence is the only thing we can calculate! In pure and applied mathematics alike, we assume that when a sequence reliably appears to converge to a limit, it does. We trust to the regularities of custom, nature and mathematics; even though these regularities are different in kind, our trust in them is not a matter of deduction but stems from experience. We might want to call the wisdom of mathematical experience “intuition.” There is a role for deductive proof, but as Jacques Hadamard (quoted by George Pólya) once said, “The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.”

In his discussion of Hersh's paper, Claudio Bernardi does not deny the importance of intuition and plausible reasoning in mathematics, but he lays stress on rigor and proofs especially in mathematical education, because often in mathematics, "it is easier to prove than to see." Doing rigorous mathematics at school is an important way to train students' intuition in dealing with mathematical abstract concepts.

In "On the Finite: Kant and the Paradoxes of Knowledge," Carl Posy raises the same issue, but in Kantian and Intuitionist terms. Posy argues that Kant was right to separate the empirical from the transcendental, the investigation of the world from discursive reflection on the world. As we carry out empirical investigations (say, the astronomical investigation of the extent of the cosmos) we are receptive and our understanding is incomplete: we must put ourselves in a position to find things and wait for the outcome. Thus we send out the Hubble Space Telescope and wait for signals, not knowing what we will find; the project of cosmological exploration remains open-ended and indeterminate. The universe as a whole can never be an object of knowledge. By contrast, the transcendental, discursive project of cosmology—and reason—regards every indeterminacy as capable of determination: the universe as a whole must be an object of knowledge. We are caught in a contradiction. The project of transcendental philosophy is important because it is regulative, organizing and driving research forward, but Kant warns us against treating the transcendental as if it were empirical. And Posy shows that this warning applies not only to cases like Kant's First Antinomy or the Fourth Paralogism where we try to think the infinitary, but also to certain finitary cases (exemplified by the Hangman's Paradox and Moore's paradox), where the empirical and transcendental perspectives collide. This warning (to be careful about treating the transcendental as if it were empirical) and exhortation (to see how fruitful the combination of perspectives can be) applies equally to mathematics, according to Reuben Hersh and Emily Grosholz, for part of mathematical research is receptive. We cannot know what the objects of mathematics are like until we go out and investigate them, and this is true even in cases where they are precipitated by our own discourse. The interplay between logic and knowledge thus has to address the tension between two ways of proceeding, the investigation of objects of knowledge that we explore receptively and incompletely, and the demands of reason, which seeks truth and conclusive proof.

In her comment, Silvia Di Paolo argues that if we want to apply Kant's warnings to certain finitary situations, we have to inquire whether we can retain some of those Kantian insights which Posy appeals to in giving a solution to the paradoxes exemplifying those situations.

In “Assimilation: Not Only Indiscernibles are Identified,” Robert Thomas characterizes mathematical objects by contrasting them with empirical objects. Our ability to characterize and classify empirical objects, he argues, depends on processes of assimilation. Experience furnishes initial examples of a kind of thing (like cats) and we go on to make decisions about what other kinds of objects to place in the same grouping; because such assimilation is often based on a shifting, fuzzy, and socially constructed notion of sufficient likeness, assimilation classes are not determinate enough to be sets. Since the world is relatively orderly, our individual and cultural assimilation classes overlap enough that we can communicate and live together. Thomas thus formulates a ‘principle of assimilation,’ which states pragmatically that we can ignore the differences among things that have been assimilated on the basis of salient features relevant to present interests and context, and notes that in ordinary life we cannot do without it. He notes further that in mathematical research, this principle is rarely needed because most mathematical objects can be “fully” characterized by a short list of relational features. A mathematical object may have other features, but we suppose that they could be deduced from this short list. Anti-realists then advocate structuralism: mathematical objects are no more than discourse about them; realists claim that mathematical objects exist, but have fewer characteristics that physical objects: they are ‘thinner.’ The procedures of classification in mathematics seem to be distinctive. Ernst Cassirer, reflecting on Descartes’ algebraization of the conic sections, saw that some mathematical general concepts do not abolish the determinations of the special cases but retain them in all strictness: that is, the particulars can be deduced from the general concept. And when mathematicians abstract in the Aristotelian sense, as in group theory, they sum up common features (e.g. the definition of a finite group) whose further deducible consequences can then be correctly and precisely applied back to the specific instances. However, if this were all that goes on in modern mathematics, it would have become trivial. At the end of his essay, Thomas gives five lists of modern cases where processes of assimilation do take place in mathematics, for as Posy and Hersh suggest, much mathematical work is receptive and exploratory.

In his discussion of Thomas’s paper, Diego De Simone looks for the cognitive and evolutionary roots of the “principle of assimilation”.

In “Proofs and Perfect Syllogisms,” Dag Prawitz makes the important observation that the definition of deductive proof requires more than the criterion of validity. Otherwise one-step proofs would suffice even for the most difficult theorems, since the last conclusion of a proof follows from the initial premises of the proof. Practicing mathematicians know that proofs must be fully articulated, because the conclusion must be *seen* to follow from the premises. A proof is related to a subject, and a community of working mathematicians. So too systems of formalized proof do not provide an 'objective' answer to the philosophical question of what a proof is, for formal proof is based on the study of, and codifies, 'real' proofs. Prawitz takes his inspiration from Aristotle's distinction between perfect and imperfect syllogisms. Perfect syllogisms compel us to affirm an argument, that is, to accept the conclusion on the basis of the premises: it needs nothing other than the premises to make the conclusion evident. Thus we should look for atomic inferences from which more complex inferences can be constructed; but how do we know when an inference is perfect? The criterion of truth-preservation offered by Tarski and Bolzano does not satisfy Prawitz, nor the criterion of constructibility offered by the Intuitionists, nor Gentzen's inferentialism. Instead, he proposes a kind of extended Intuitionism, revised to include classical reasoning and a notion of "grounds" broader that Brouwerian construction. By performing an acceptable (perfect) inference, a subject who has grounds for its premises thereby comes into possession of grounds for the conclusion. This is the program on which he is currently embarked, to work out objectively the conditions under which validity and legitimacy coincide.

In his response, Julien Murzi argues that Prawitz's approach faces the same problems besetting the inferentialists, and proposes moderate inferentialism as an alternative to Prawitz's extended intuitionism.

In "Logic, Mathematics, Heterogeneity," Emily Grosholz argues that the advance of knowledge must combine the empirical, 'receptive' investigation of objects with an abstract, analytical search for conditions of intelligibility. These two enterprises, she argues, are usually recorded in disparate idioms, so the challenge is how to integrate them while avoiding inconsistency and loss of meaning. Since the task of logic is to exhibit valid reasoning, and valid reasoning requires a homogeneous and unambiguous language in order to articulate and impose its rules, logic however useful cannot by itself exhaust rationality. Rationality also includes the integration of disparate concepts, lines of research, and modes of representation, and this combination is a kind of reasoning, whose philosophical assessment requires not just the resources of logic, but also the history of mathematics. Strategies of combination change over time, as problem-contexts and conceptions of method change. Moreover, the tension between rational description and taxonomy, and systematization and explanation, is an epistemological problem not just for science, but also for mathematics and logic itself, once logic becomes part of mathematics, a claim she explores in terms of Gödel's two Incompleteness Theorems.

In her comment on Grosholz' paper, Valeria Giardino discusses the issue of the heterogeneity of reasoning, and criticizes Grosholz' choice of the word 'ambiguity' to describe what goes on in mathematical discourse.

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# SECTION I:

# LOGIC AND KNOWLEDGE

# CHAPTER ONE

# THE COGNITIVE IMPORTANCE OF SIGHT AND HEARING IN SEVENTEENTH- AND EIGHTEENTH-CENTURY LOGIC

MIRELLA CAPOZZI

SUMMARY The relation between logic and sight was favoured by many authors of logical calculi. However some of these authors believed that logic should also investigate methods of discovery by taking inspiration from the arts of discourse, so strictly connected to hearing and to the vague meanings of spoken words.

KEYWORDS Logical calculi, written signs, discovery, rhetoric, Leibniz, Lambert, Holland, Ploucquet

## I. Logic and sight

I.1. The relation between logic and the sense of sight was established well before the seventeenth and eighteenth centuries. It was certainly active around the year 1500 when a number of logicians working in Paris decided to focus their attention on written terms with the explicit aim of disconnecting them from their vocal and audible root. Thanks to a well documented historical reconstruction (Meier-Oeser 1997), we know that, according to these Paris logicians, a written term could be any sensible sign, provided that it could be perceived by a sense different from hearing. Even a material object would do, as already maintained by Paul of Venice, according to whom "we can form syllogisms with sticks and draw conclusions with stones [*possemus cum baculis syllogizare et cum lapidibus concludere*]" (Paulus Venetus 1979, p. 78). Particularly clear in this respect was Pedro Margahlo who defined the *terminus scriptus* as "a term perceptible by a sense other than hearing [*terminus alio sensu quam auditus perceptibilis*]" (Margallus 1965, p. 92), and concluded that "every sensible body perceivable by the four external senses can be a written term [*omne sensibile corpus quattuor externis sensibus posse esse terminum scriptum*]" (Margallus 1965, p. 162f).

As could be expected, many logicians established a preferential connection between the written term and the sense of sight. Thus John Major (or Mair or Maior), a St. Andrews logician, defined the *terminus scriptus* as: "a term that can be perceived by a corporeal eye [*terminus qui oculo corporali percipi potest*]" (Maior 1508, fol. 4, in Meier-Oeser 1997, p. 167. Cf. Broadie 1985, Biard 1998). Juan De Oria too explained that:

> a written term is not called so because of being an inscription made up from characters or letters, but rather because of representing something to the cognitive faculty by means of sight [*non enim dicitur terminus scriptus, quia sit scriptura ex caracteribus aut litteris constans, sed quia potentie cognitive aliquid proprie representat, mediante visu*] (De Oria 1987, p. 106, in Meier-Oeser, p. 167. Cf. Muñoz Delgado 1964).

What all the cited authors have in common is that they maintain that logic deals with written visible terms which are not transcriptions of phonemes.

The tendency of logic to disregard spoken language is associated with an underlying conviction—especially relevant in the tradition of Ramus—to consider language "an accretion to thought, hereupon imagined as ranging noiseless concepts or 'ideas' in a silent field of mental space" (Ong 1958, p. 291. Cf. Hotson 2007, p. xi). This conception of thinking as an activity that deals with noiseless concepts (rather than noisy words), combined with the diffusion of printing, strengthened the connection of logic with the sense of sight during the sixteenth century. For, given that a sensible means of expression of thoughts was unavoidable, it had to consist of visible, written and printed signs, independent from the sounds of ordinary speech. Ong mentions the extensive use made in the logic textbooks of that time of allegorical images (e.g. Murner's) and of complex graphics, such as those representing the technique of the *inventio medii* or the dichotomies typical of the Ramist tradition.

The most notable effects of the prevalence of a visual dimension in logic do not belong, however, to Ramus's time, but to the seventeenth and eighteenth centuries.

I.2. In considering the prevalence of a visual dimension in seventeenth century logic, Leibniz would require a separate study. First, one should not forget that he was also much involved in the investigation of natural *spoken* languages (Cf. Heinekamp 1976, pp. 518−570). Second, though it is true that he focussed on systems of writing not subordinated to the