From Foundations to Philosophy of Mathematics

From Foundations to Philosophy of Mathematics: An Historical Account of their Development in the XX Century and Beyond

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To My Father

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INTRODUCTION

The fascination of philosophy with mathematics goes back to the origins of Western thought, as it is clear from the interest showed by Pythagoras (ca. 569 BC-ca. 475 BC) and the so-called Pythagoreans for mathematics. Plato (427 BC-347 BC), greatly influenced by them, had written on the front of the entrance to the Academy, the slogan, “Let no one ignorant of geometry enter here,” and his belief that mathematics is the best preparation for the statesman was extremely beneficial for the further development of it. Indeed, the most important mathematicians of the fourth century, such as Eudoxus of Cnidus (ca. 390 BC-ca. 338 BC) or Archytas of Tarentum (ca. 428 BC-ca. 350 BC), were friends and disciples of Plato.

Plato does not place significant mathematical discoveries, but the controversies of the Academy and the views of Plato himself concerning the nature of mathematical knowledge (the nature of proofs, the sorts of admissible principles, etc.) and mathematical objects decisively influenced the further course of mathematics and philosophical reflection on them. The preponderance of the study of mathematics within the Academy was such that it led Aristotle (384 BC-ca. 322 BC), the most famous pupil of Plato, to complain bitterly that “mathematics has come to be identical with philosophy for modern thinkers, though they say that it should be studied for the sake of other things.”[1] However, Aristotle also contributed to the development of mathematics and his philosophy of mathematics is an important alternative to Platonism. Thus, for example, Aristotle’s theory of science decisively influenced the *Elements* of Euclid (fl. 300 BC) and the subsequent development of the axiomatic method, whereas his distinctions between actual and potential infinity or between discrete and continuous magnitudes determined the conceptual framework within which Western mathematics developed until the late nineteenth century.

The interest in mathematics remained unchanged in the Middle Ages and the Renaissance, but it was not until the dawn of the modern age, especially under the influence of rationalist authors such as René Descartes (1596-1650), Gottfried Wilhelm Leibniz (1646-1716) or Blaise Pascal (1623-1662), but also of astronomers and physicists like Johannes

[1] Aristotle. *Metaphysics* A, 9: 992a32-b1.

Kepler (1571-1630), Galileo Galilei (1564-1642) and Isaac Newton (1643-1727), that mathematics finally became the paradigm of knowledge. All these authors are known for their important scientific discoveries, a fact which could lead us to believe that they were interested only in developing new techniques of calculation that allow the application of mathematics to the domain of nature. However, as was noted by Abraham Robinson (1918-1974), the creator of *non-standard analysis*, a new approach to the infinitesimal analysis of Newton and Leibniz, "the picture is incomplete. It ignores the fact that, from the seventeenth to the nineteenth century, the history of the Philosophy of Mathematics is largely identical with the history of the foundations of the Calculus."[2] Is there anything more philosophical, for example, than the idea of building a *characteristica generalis*, namely, a general symbolic language by which anyone could write with symbols and formulas all processes of reasoning used in mathematics, which led to Leibniz' discovery of infinitesimal calculus? And does the debate on the foundations of calculus in the seventeenth century not have a strong philosophical character (Are there infinitely small quantities? Is it safe to use them in the calculations?), which largely determined the progress in the foundations of mathematical analysis that occurred in the next two centuries?

In any case, thanks to the efforts of Augustin Louis Cauchy (1789-1857), Bernard Bolzano (1781-1848), Bernhard Riemann (1826-1866) and Karl Weierstrass (1815-1897) among others, towards the end of the nineteenth century analysis had already reached the conceptual clarity and rigor in its proofs whose absence philosophers and mathematicians had lamented in the previous century. Much of this process of *rigorization of analysis* was inextricably linked to the progressive abandonment of the use of temporary or space intuition that had dominated the infinitesimal calculus since its creation by Newton and Leibniz. Moreover, in the late nineteenth century the so-called *non-Euclidean geometries* such as the geometries of Riemann, Nikolai Ivanovich Lobachevski (1792-1856) and János Bolyai (1802-1860) emerged, with which the Euclidean space largely lost the place it had been granted so far as the ultimate source of mathematical intuition and, ultimately, that intuition no longer played the fundamental role it had been given in mathematics until then. As a result, the traditional definition of real numbers as continuous magnitudes of Euclidean geometry grasped intuitively was replaced by arithmetic definitions, in which the real numbers were defined from infinite sequences or sets of rational numbers and, ultimately, from natural numbers. This

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[2] Robinson 1966, 280.

process of *arithmetization of analysis* left the door open to question: What are natural numbers reducible to? And the answer was immediate, as for Georg Cantor (1845-1918) and Richard Dedekind (1831-1916) natural numbers (and with them, do not forget, the other numbers) were reduced to sets, while for Gottlob Frege (1848-1925) were reduced to concepts.

However, as was demonstrated by Bertrand Russell (1872-1970) in the early twentieth century, set theory and logic, to which it was intended to reduce mathematics by Dedekind and Frege, were inconsistent or contradictory. Therefore, when it seemed that it was possible to close once and for all the old problem of the foundation of mathematics, the problem re-emerged again, but now with an even greater urgency. In any case, as has happened throughout the history of mathematics, the answer to this foundational crisis was swift, and so emerged Ernst Zermelo's (1871-1953) *axiomatic set theory* and Russell's *theory of logical types*, through which it was possible to resolve the paradoxes hitherto known. These theories were respectively the solutions to the problem of the paradoxes of logic and set theory of the *logicist* and *formalist* schools, which conform with the *intuitionist* school, the *big three* schools or currents in the philosophy of mathematics that have dominated the twentieth century.

Nonetheless, although the debate about the nature of mathematics among these three schools and their respective foundational programs was encouraged by the appearance of the paradoxes of logic and set theory, it would be wrong to think that these schools emerged as a direct response to these antinomies. For their emergence can be explained basically as a result, on the one hand, of the search for a rigorous formulation of calculus and of the progressive arithmetization of analysis carried out in the eighteenth and nineteenth centuries and, on the other, as a response to what could be called a growth crisis in mathematics caused by the emergence of non-Euclidean geometries and Cantorian set theory in the late nineteenth century. This is the case, as we shall see later, of Frege's logicism and Hilbert's formalism, but also of Brouwer's intuitionism, which can be explained largely as a reaction to the loss of the privileged place that intuition had occupied until then, caused by the attempts to bring to completion the arithmetization of analysis and the development of non-Euclidean geometries.

From what has been said, it follows that some of the problems addressed by the philosophy of mathematics, as we understand it in this book, are problems related with their foundations, such as: What is a number? What is a set? What is a mathematical axiom? What is a mathematical definition? What is a mathematical proof? What are the appropriate axioms to characterize the concept of number? What are the appropriate axioms to characterize the concept of set? Etc. However, the range of problems which attempts to answer the philosophy of mathematics extends beyond problems relating to their foundations and includes a series of problems that have, so to speak, a more philosophical taste, such as: What is the nature of mathematical objects? What is the nature of mathematical propositions? Which is, ultimately, the source from which the truth of the propositions of mathematics arises? How and why mathematics is applicable to the empirical world? Is mathematics merely a linguistic game? Is there truth, after all, in mathematics? Etc. Although apparently this latter group of questions may seem remote from the first, the fact is that very often both types of questions are intertwined in the philosophy of mathematics of the twentieth century and therefore will also have a place in this work.

For example, an essential aspect of the mathematics developed between the seventeenth and nineteenth centuries and, in general, the mathematics of all ages, is its applicability to experience. However, another no less essential aspect is its universality and necessity, as it was seen by Leibniz and David Hume (1711-1776). According to these authors, the universality and necessity of the propositions of mathematics come from their *a priori* nature, namely from the fact that they are true regardless of experience. But, how can mathematics be independent of experience and at the same time, be applied to it? The solution of Immanuel Kant (1724-1804) was that mathematical propositions are based on our pure intuitions of space and time and thus have an *a priori* component that makes them universal and necessary. However, since space and time are our way of categorizing the phenomena of experience, these propositions are likely to be applied to sensitive data coming from experience.

Kant is without doubt the most influential philosopher for the philosophy of mathematics of the twentieth century, for all the mainstream schools have tried to confirm or refute the Kantian thesis that the propositions of mathematics are based on the intuitions of space and time. For example, according to Frege and Russell the logical propositions of arithmetic are *a priori*, but not because they are based on the pure intuition of time, but rather the opposite, that is, because they are analytical truths (i.e., because they are provable from the laws of logic and its concepts can be defined in terms of logical notions) and are thus independent of intuition. However, for Egbertus Luitzen Jan Brouwer (1881-1966) and the intuitionists, the propositions of arithmetic are mental constructs based on pure intuition of time and therefore it will only be possible to provide a secure basis to reconstruct mathematics from the "apriorism of time".

Frege's *logicism* (Chapter 1), David Hilbert's (1862-1943) *formalism* and *finitism* (Chapter 5) and Brouwer's *intuitionism* (Chapter 4) will occupy an important part of the historical detour in the philosophy of mathematics of the twentieth century to be drawn in this book. However, at the beginning of the twentieth century emerged a fourth school, *predicativism*, which due to historical circumstances explained later, was abandoned until the early 60s. Anyway, from our standpoint, it easily deserves a place beside the three traditional schools and we will also deal with it in this book (Chapter 6). From a philosophical point of view, these four schools are a response to the distrust in the existence of an intellectual intuition by which it is possible to grasp the existence of mathematical entities supposedly independent of our sensory faculties (here, again, the influence of Kant). So, all these schools tried to formulate a philosophy of mathematics which doesn't presuppose the existence of abstract entities of Platonic nature.

Apart from the philosophical theories developed within the schools mentioned above, we will also look into some technical developments that have largely shaped our understanding of the nature of mathematics, such as Russell's *type theory* (Chapter 2) or Zermelo's *set theory* (Chapter 3), to which we have referred before, or Kurt Gödel's (1906-1978) *incompleteness theorems* (Chapter 7), whose impact on the philosophy of mathematics has been enormous, particularly in Hilbert's finitism and his program for the foundation of mathematics, the so-called *Hilbert's program*. Indeed, towards the end of the 30s in the last century it was clear that not only Hilbert's finitism, but also Brouwer's intuitionism and Frege's and Russell logicism, at least as these programs were originally designed, were not feasible (predicativism was an exception, but at that time was a program in hibernation). As a result, in the 40s a renewed interest in Platonism began to crystallize and his most genuine representative at that precise time was Gödel.

Finally, we will look at some current programs in the foundation of mathematics which, in a certain way, try to show the feasibility of the main programs in the foundations that appeared in the first half of the twentieth century. These programs are basically *neologicism*, *constructivism*, *predicativist reductionism* and what has been called *partial realizations of Hilbert's program* (Chapter 8). The latter program has been developed under the research program known as *reverse mathematics*, led by Harvey Friedman (1948 -), which is surely the most interesting research program on the foundations of mathematics that has emerged in recent years and, therefore, we will focus largely on it in the same chapter. In the *Coda* we will sketch how the different approaches to mathematical foundations respond to philosophical problems about the nature of mathematical objects and how they have shaped our understanding of mathematical knowledge.

CHAPTER ONE

FREGE'S LOGIC AND LOGICISM

1.1 Fregean Logicism

Gottlob Frege was born on November 8th, 1848, in Wismar, Mecklenburg-Schwerin, Germany, where he studied until 1869. That year he passed the *Abitur*, which allowed him to go to the University of Jena, where he spent the first four semesters studying chemistry, mathematics and philosophy. He spent the last five semesters at the University of Göttingen, where he studied physics, mathematics and philosophy of religion. In 1873 he received his doctorate at this university with a dissertation entitled *Über eine Darstellung der imaginäre geometrische Gebilde in der Ebene* [*On the geometrical representation of imaginary forms in the plane*] and the following year he obtained the *Venia docendi* by the University of Jena with the dissertation *Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegrifes Grunden* [*Methods of calculation based on an extension of the concept of quantity*]. In the latter work, Frege says:

> There is a noticeable difference between geometry and arithmetic in the way they lay the foundation of their basic propositions. The elements of all geometric constructions are intuitions and geometry refers to intuition as the source of its axioms. However, since the object of arithmetic does not have an intuitive nature, not its basic propositions can be derived from intuition.[1]

Although in his early writings Frege says that arithmetic is not based on intuition, he doesn't say that the basic propositions of arithmetic are based on logic. Indeed, it seems clear that he came to this idea between 1874 and 1879 and that in order to arrive at this it played a decisive role the elucidation of the concept of number, that is, the discovery that every statement about numbers is actually a statement about a concept. For Frege derived later from this thesis that numbers are logical objects and therefore

[1] Frege 1967, 50.

they have no intuitive nature. Furthermore, Frege's research on the concept of number made him clearly see the need to formulate a new language, the so called *Begriffschrift* [*Conceptscript*], for he soon realized that everyday language is not suitable for these investigations. This is not only for its vagueness and ambiguity, but also because he wanted to avoid expressions like "obviously," "because," etc., which are normally used in mathematical proofs and could be interpreted as an appeal to intuition.

Thus Frege's research on the concept of number is closely related to his earliest piece of important work, and certainly one of the most remarkable works in the history of logic: *Begriffschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* [*Conceptscript: a formula language of pure thought modelled after that of arithmetic*] (1879). Frege's *Conceptscript* is primarily a system of notation intended to represent what he calls the *conceptual content* [*begrifflichen Inhalt*] of an expression, which is that part of the content that is relevant to the process of deduction. This is the reason Frege says that the *Conceptscript* is a *formula language of pure thought*, that is to say, a symbolic language used to express pure thought and the laws that derive from the self-determination of thought and, therefore, they are fully general. That's why, according to Frege, the *Conceptscript* provides the only means to support solid scientific truths, "a means that regardless of the particular characteristics of objects depends only on the laws in which all knowledge rests."[2]

In the preface of *Begriffschrift*, Frege divided scientific truths that require proof into two different kinds, according to whether his proof can be carried out exclusively by logical means or, conversely, is based on experience. Now, continues Frege:

> In considering the question as to which of these two kinds the judgements of arithmetic belong, I first had to see how far one could get in arithmetic by means of inferences [*Schlüsse*] alone, based solely on the laws of thought which transcend all particularity. In this sense, the course I took was to attempt to reduce first the concept of order in a series [*Anordnung in eine Reihe*] to that of *logical* consequence [logische *Folge*], in order to progress from here to the concept of number. So that nothing intuitive [*Anschauliches*] could penetrate this process unnoticed, everything had to rely on a deductive chain free of gaps. However, in striving to fulfil this requirement as closely as possible, I found an obstacle in the inadequacy of language: the more complicated the relations became, the more cumbersome the expressions that arose. As a result of this, the precision

[2] Frege (1879) 1964, IX.

> attained was not adequate for what my purpose demanded. Out of this difficulty came the idea of the present *Conceptscript* [*Begriffschrift*].[3]

So, in order to prove the *analyticity* of arithmetic, namely, the fact that its propositions can be proved by appealing only to the means of logic, Frege set out in the first two parts of *Begriffschrift* a logical system, the *Conceptscript* proper, within which a logical language, axioms and inference rules are formulated. From this logical system he proceeds in the third part of *Begriffschrift* to reduce the concept of *order in a series* to that of *logical consequence,* which he understands as the first step toward defining the concept of number in strictly logical terms.

1.2 Frege's Logic

A *formal system* is essentially a collection of formulas defined solely by its syntax (*formal language*) and a set of axioms and inference rules that allow us to deduce specific formulas from collections of previously given formulas (*deductive system*). A *logical system* is a formal system in which the formal language is a *logical language* and the deductive system consists only of axioms and inference rules expressible or referred to that language, that is, *axioms and inference rules of logic*.

Logical languages are characterized by the fact of having a set of *common symbols*: *variables, parentheses,* the *connectives* $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ ("and," "or," "not," "if … then" and "if and only if" respectively), the *quantifiers* $\forall, \exists$ ("for all" and "there are") and the *identity* sign $=$ (optional). *First-order languages* have only one variable type, the so-called *first-order variables,* whose range is a collection of elements or individuals, often called the *universe* or *domain of discourse*. *Second-order languages* have, in addition to first-order variables, *second-order variables,* i.e., variables whose range are properties, sets or relations between elements of the universe. And so on. Generally speaking, *higher-order languages* are those with *higher-order variables,* i.e., variables of second- or higher-order.

The logical language from which Frege proceeds to the reduction of arithmetic to logic in *Begriffschrift* essentially coincides with the language of modern second-order logic and, therefore, includes not only the first-order variables $x, y, z, ...$ but also the second-order variables $F, G, ..., Q, R, ...$ As usual, we will use the letters $F, G, ...$ as variables whose range are *properties* or *concepts* (in Frege's terminology) and write,

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[3] Ibid., X.

for example, Fx to express the fact that the value of *F* for the argument *x* is true or, as Frege says, that *x falls under* the concept *F* or *has* the property *F*. On the other hand, we will use the letters $Q, R, \ldots$ as variables whose range are *relations* and write, for example, Rxy to express the fact that *x* and *y are in* the relation *R*.

According to Frege, concepts and relations can be of different *levels* [*Stufe*], according to the *type* [*Art*] of the objects, concepts or relations that fall under them, but we will only focus on first- and second-level concepts and relations since the reduction of mathematics to logic only requires this kind of concepts and relations. For example, "*x* is *father*" or "*x* is *father of y*" are respectively a first-level concept and relation, because under them fall only objects, whereas "*F* is a *first-level concept*" and "*F corresponds one-to-one* with *G*" are respectively a second-level concept and relation, since under them only fall first-level concepts.

Another basic notion of Frege's logic is the notion of the *extension* of a concept or relation, first introduced in *Die Grundlagen der Arithmetik* [*The foundations of arithmetic*] (1884). Basically, the extension of a concept is the set or class of all objects or concepts that fall under that concept. For example, the extension of the concept "*x* is a *prime number*" is the set of prime numbers and the extension of "*F* is a *first-level concept*" is the set of all first-level concepts (*father*, *moons of Jupiter*, etc.). Similarly, the extension of a relation is the set of ordered pairs of objects or concepts that are in that relation. For example, the extension of the first-level relation "*x* is *father of y*" is the set of ordered pairs $\langle x, y \rangle$ of individuals such that *x* is father of *y*, namely, the set $\langle \text{Barack Obama, Malia A. Obama} \rangle, \langle \text{Benjamin Peirce, Charles S. Peirce} \rangle$, etc.

The axioms and inference rules presented by Frege in *Begriffschrift* cover both propositional logic and first-order logic with identity, but because the reduction of arithmetic to logic is performed in the language of second-order logic, this deductive system is clearly inadequate and needs to be supplemented by specific rules and axioms of second-order logic. This flaw will be corrected in *Grundgesetze der Arithmetik* [*Basic Laws of Arithmetic*] (1893, 1903), where Frege introduces for the first time a logical system which includes essentially all the axioms and inference rules necessary to carry out the reduction of arithmetic to logic. These axioms and rules are the same as those of first-order logic, with the exception of the axioms and rules governing the use of the second-order quantifiers $\forall F$ and $\exists F$, which are very similar to those governing the use of the first-order quantifiers $\forall x$ and $\exists x$. For example, next to the axiom $\forall xFx \rightarrow Fa$ of first-order logic, we also find in *Grundgesetze* the axiom $\forall FFa \rightarrow Ga$, specific of second-order logic. These axioms are equivalent (in the presence of the rule of *modus ponens*, which allows us to infer from the formulas φ and $\varphi \rightarrow \psi$ the formula ψ) to the rules of *universal instantiation* of first- and second-order logic, that is to say, the rules that allow us to infer from the sentences $\forall xFx$ and $\forall FFa$, the sentences Fa and Ga respectively.

Apart from the *modus ponens* rule, Frege introduces in *Begriffschrift* and *Grundgesetze* another rule to which we shall also refer later, namely, the rule of *universal generalization*, which allows us to infer $\forall xFx$ from Fa when *a* is arbitrary (as in the previous case, we suppose that we also have a similar rule for second-order quantifiers). In general, in what follows we will assume that the axioms and inference rules of second-order logic that the reader can find in any text-book of modern logic apply to Frege's mature logical system and, therefore, we won't concern ourselves with the subject any longer. There are, however, two axioms that are worth highlighting the importance of in the work of Frege. These are the following:

Axiom of comprehension for concepts (CA): $\exists F \forall x (Fx \leftrightarrow \varphi(x))$

Axiom V of Grundgesetze (AV): $\hat{x}Fx = \hat{x}Gx \leftrightarrow \forall x (Fx \leftrightarrow Gx)$

The first axiom asserts the existence of a concept for each formula φ with one free variable *x*. For example, corresponding to the formulas $x < 5$ and $x \neq x$ we have, according to CA, the concepts "less than 5" and "number that is not equal to itself" (a concept in which obviously no object falls). Frege does not explicitly introduce this axiom in any of his works, but both in *Grundgesetze* and *Begriffschrift* he constantly uses a rule of substitution equivalent to the previous axiom and, therefore, either the rule or the axiom should be considered as part of Frege's deductive system.

The second axiom states that the extensions of the concepts *F* and *G* are equal if, and only if, exactly the same elements fall under these concepts. For example, under the concepts $x < 5$ and $x \leq 4$ fall exactly the same numbers and, therefore, they have the same extension. As explained below, this axiom is responsible for introducing the extensions from the concepts and for regulating their use (see Section 1.6). Frege introduced for the first time AV in *Grundgesetze* and the presence of it is, together with the implicit use of CA, the cause of the inconsistency of the logical system presented in this work (see Section 1.7).

1.3 The Reduction of the Concept of Order in a Series

From a modern viewpoint, we can understand what Frege calls a *series* [*Reihe*] Q as an ordered pair $\langle X, Q \rangle$, where X represents any non-empty set and Q a binary relation on X, i.e., $Q \subseteq X \times X$. To reduce the mathematical concept of order in a series to logic, Frege introduces first a relation that Russell will call later the *ancestral relation*. To define the ancestral of a given arbitrary set Q, Frege defines previously a property symbolized by $\mathrm{Her}(F)$ (the property F is hereditary in the Q series) through the condition (Proposition 69):

$$\forall u \forall v (Fu \wedge Quv \rightarrow Fv),$$

or, in other words: the property F is hereditary in the Q series if, and only if, for each pair of objects in the Q series, if u has property F, then v also has the property F. Intuitively, the idea is that F is hereditary in the Q series if F is inherited by v whenever it is held by any element with which v is in the relation Q. Frege then defines the *ancestral* of Q, $Q^{*}xy$ ("y follows x in the Q series" or "x precedes y in the Q series") through the following condition (Proposition 76):

$$\forall F \left[\mathrm{Her}(F) \wedge \mathrm{In}(x, F) \rightarrow Fy \right],$$

where $\mathrm{In}(x, F)$ is an abbreviation for the condition $\forall \mathrm{z} (Qxz \rightarrow Fz)$ ("after x, the property F is inherited in the Q series"). So, "y follows x in the Q series" if, and only if, y has inherited all the hereditary properties F that has any object z with which x is in the relation Q. For example, if Q is the *successor* relation (namely, the relation *immediately succeeds to* or *immediately preceding to*) in the natural numbers, then $\mathit{successor}^{*}$ is the relation *less than* in the natural number series (the successor series beginning with zero). Of course, Frege has not yet defined the successor relation or the concept of natural number, but it is worth keeping in mind the example above in order to intuitively understand the results discussed below. For example, Frege proves in *Begriffschrift* that $\neg (Q^{*}xy \rightarrow Qxy)$ (Proposition 91) and also that the ancestral relation is transitive:

$Q^{*}xy \wedge Q^{*}yz \to Q^{*}xz$ (Proposition 98). Now if we consider what happens when Q is the *successor* relation, then the first sentence states that from the fact that x is *less than* y, it does not follow that x *immediately precedes* y, while the second states that if x is *less than* y and y is *less than* z, then x is *less than* z.

The above propositions follow immediately from the definition of the ancestral relation. This is not the case however with Proposition 133, which is probably the most important result proved by Frege in *Begriffschrift*. To state this proposition, Frege introduces a couple of new definitions. The first is the definition of $Q^{*}_{=}xz$, the *weak ancestral* of Q, which Frege read as either "z belongs to the Q series beginning with x" or "x belongs to the Q series ending with z." Frege defines this relation by the condition (Proposition 99):

$$Q^{*}xz \vee z = x,$$

that is, z belongs to the Q series beginning with x if, and only if, z follows x in the Q series or is equal to x. Clearly, if Q is the successor relation on the natural numbers, since Q^{*} is then the relation *less than*, $Q^{*}_{=}$ will represent the relation *less than or equal to* in the natural numbers. The second definition is the definition of the property "Q is functional" or simply "Q is a function," denoted by $\mathrm{Func}(Q)$. This property is defined by the condition (Proposition 115):

$$\forall x \forall y \forall z \left(Qxy \wedge Qxz \to y = z \right),$$

which expresses the fact that a relation Q is functional if, and only if, x is related by Q with y and z, then y is identical with z. Finally, proposition 133 expresses the property of the ancestral sometimes called *connectivity* and is the proposition with which Frege closes *Begriffschrift*. This proposition can now be formalized as follows:

$$\left[\mathrm{Func}(Q) \wedge Q^{*}xz \wedge Q^{*}xy \right] \to Q^{*}yz \vee Q^{*}_{=}zy,$$

that is, if Q is functional, and if z and y follow x in the Q series, then y precedes z in the Q series or belongs to the Q series that begins with z. For example, if Q is the successor relation on the natural numbers, then the above proposition states that if this relation is a function (as it actually happens), then from $x < z$ and $x < y$ it follows that $y < z$ o $z \leq y$ (for which reason it is often said that z and y are *connected* by the relation *less than*).

As we have just seen, the main results Frege gives in *Begriffschrift* in defence of the logicist thesis are a series of propositions that express basic properties of the *ancestral* relation, for example, its transitivity and connectivity, properties which also satisfy the relation *less than*, which is the ancestral of the successor relation in the natural number series. As we shall see, Frege defines in *Grundlagen* the successor relation and the number 0 in terms of pure logic, which in turn allows him to define the natural numbers as the numbers n such that $0 \leq n$ and to prove the infinity of the natural number series in strictly logical terms.

What has just been said clearly indicates that the defence of the logistic thesis in *Begriffschrift* consists of exposing a general theory of series in which, starting from an arbitrary Q series, a relation is logically defined, the ancestral relation, which induces in this series an order structure completely analogous to that induced in the natural number series by the relation *less than*. In this sense, it must be remembered that, in Frege's own words, the course taken by him to prove the logicist thesis *was to attempt to reduce first the concept of order in a series to that of* logical *consequence in order to progress from here to the concept of number.*

Now it is clear that Frege's *Begriffschrift* only deals with the first part of the plan, also proving some basic properties that the concept of logical succession (the ancestral relation) shares with the usual order relation on the natural numbers (the relation *less than*). The second part, the definition of number, is taken up in *Grundlagen* and picked up at the formal level in *Grundgesetze*, where Frege derives from it the basic laws of arithmetic.

1.4 The Definition of the Concept of Number

Frege explained in various places that the *fundamental thought* [*Grundgedanke*] in which he would have based its analysis of the concept of number is the following thesis:

> A numerical statement contains an assertion about a concept.[4]

The *numerical statements* [*Zahlurteile*] are statements such as: "Here are four companies" or "Here are five hundred men" and also "Jupiter has four moons" or "The number of moons of Jupiter is four." Numerical statements are then statements through which one responds to questions

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[4] See, for example, Frege 1969, 273.

like *How many...*? and, therefore, one can count or list the individuals or objects of some kind. It is precisely because of this fact that Frege gives them such importance for the deduction of the concept of number. Frege's basic idea is, in effect, that the characterization or definition of cardinal number should reflect the fact that these numbers are primarily used for counting. Now, what is stated in a numerical statement?

According to Frege, when we say, for example, that there are five hundred men or four companies, we are saying something of the concepts "man" or "company" respectively and is precisely the fact that both concepts are different that gives rise to two different statements (although obviously equivalent). Thus, a numerical statement contains an assertion about a concept, namely the assertion that a certain number corresponds to this concept. In short, to have a certain number is a property of concepts and, therefore, a second-level concept. For example, the numerical statement "Jupiter has four moons" states that the first-level concept "moon of Jupiter" has the property that four objects fall under it or, equivalently, that it falls under the second-level concept "concept under which fall four objects." For while under the first-level concept "moon of Jupiter" fall four objects (moons), under the second-level concept "concept under which fall four objects" fall all first-level concepts under which fall exactly four objects: "Jupiter's moon," "prime number less than 8," "letter of the word *York*," etc. Thus the numerical statement "Jupiter has four moons" will be analysed in terms of the statement "The number of moons of Jupiter is four" or, more accurately, as "the number that corresponds to the concept "moon of Jupiter" is four,"

The example just mentioned also shows that all numerical statements actually express a numerical equality [*Gleichung*], since in them the word "is" has the meaning of "equals" or "is the same as," Therefore, according to Frege, if we define the *meaning* [*Sinn*] of a numerical equality, which will have, as we have explained above, the following form:

> The number that corresponds to the concept *F* is the same as for the concept *G*,

we will have a general criterion to know if the numbers denoted by the previous numerical expressions are equal, which for Frege is the same as to define them. According to Frege, a definition of this class is what is now called *Hume's principle*, which states that:

> The number that corresponds to the concept *F* is the same as for the concept *G* if, and only if, we can establish a one-to-one correspondence between the elements that fall under *F* and those that fall under *G*.[5]

In symbols:

$$Nx : Fx = Nx : Gx \equiv F \approx G.$$

Frege himself defines the term "one-to-one correspondence" in the above definition as follows:

> *R* is a one-to-one correspondence between *F* and *G* if, and only if, for every object falling under *F* there is a single object in the relation *R* that falls under *G*, and for every object falling under *G* there is a single object in the relation *R* which falls under *F*.[6]

In logical notation, *R* is a one-to-one correspondence between the *Fs* and *Gs* if, and only if:

$$\forall x\left[Fx \to \exists!y\left(Gy \wedge Rxy\right)\right] \wedge \forall x\left[Gx \to \exists!y\left(Fy \wedge Ryx\right)\right],$$

where $\exists!y\,\varphi y \equiv \exists y\left[\varphi y \wedge \forall z\left(\varphi z \to \varphi y\right)\right]$ and, therefore, $\exists!y\,\varphi y$ reads: "there is a unique y such that φy." From this definition we can also formulate *Hume's principle* in strictly logical terms as:

$$Nx : Fx = Nx : Gx \equiv \exists R\begin{bmatrix}\forall x\left(Fx \to \exists!y\left(Gy \wedge Rxy\right)\right) \wedge \\ \forall x\left(Gx \to \exists!y\left(Fy \wedge Ryx\right)\right)\end{bmatrix}.$$

Clearly, the existence of some one-to-one correspondence between *F* and *G* appropriately defines the meaning of $Nx : Fx = Nx : Gx$, for if the number of *Fs* and *Gs* coincides we can always find a one-to-one correspondence between *F* and *G*, while if the number of *Fs* and *Gs* is different there is no such correspondence between these concepts. We see then that *Hume's principle* provides a criterion for determining the validity of the statements of the form $Nx : Fx = Nx : Gx$ and, therefore, a criterion for the identification of the reference of the numerical expressions that appear in the statements of this kind. In this sense, Hume's principle is a *contextual definition* of "the number that corresponds to the concept *F*,"

[5] See Frege 1959, § 63, 73-74.
[6] Ibid., §§ 71, 72, 83-4.

because it determines the reference of this type of expression in the context of statements of the form $Nx:Fx = Nx:Gx$.

From the above definition, Frege defines the expression "*n* is a number" as equivalent to the expression: "there is a concept such that *n* is the number corresponding to it." In symbols: $\text{Num}n \equiv \exists F\left(Nx:Fx = n\right)$. In other words, a *cardinal number* is any object that is the number that corresponds to some concept. Now, it follows immediately from here that *Hume's principle* not only provides a criterion for deciding the validity of the statements of the form $Nx:Fx = Nx:Gx$, but also of the form $Nx:Fx = n$, where *n* is any finite cardinal, since under the definition of "*n* is a number" the latter type of statements are assimilated to the former.

However, *Hume's principle* does not determine the full meaning of the expressions of the form $Nx:Fx$ (and, therefore, does not determine completely the concept of number), as evidenced by the fact that this principle cannot decide the validity of a statement as "the number of *F* is "..."", where "..." is a proper name such as "Julius Caesar." In general, the contextual definition of number cannot decide the validity of the statements of the form $Nx:Fx = a$ when *a* is no longer of the form $Nx:Gx$ for some concept *G*. This is called the *Julius Caesar problem* and is basically a problem of applicability of the language of arithmetic. In the case of the language of pure arithmetic, which contains no non-logical constants such as *a*, *b*, *c*, ... this problem cannot arise. But as soon as we want to apply this language, we will have to extend it by introducing a stock of non-logical constants, for which the *Julius Caesar problem* will be raised immediately. In any case, the existence of a one-to-one correspondence with the concept *F* can be used to define the second-level concept "equinumerous [*Gleichzahlig*] with *F*" and then the number corresponding to *F* can be defined in a completely satisfactory way as the set or class of all concepts equinumerous with *F*. This is precisely the meaning of the definition of number that Frege finally proposes in *Grundlagen*:

> The number that corresponds to the concept *F* is the extension of the concept "equinumerous with the concept *F*."[7]

The above definition is an *explicit definition* of the concept of number, because it defines the number corresponding to the concept *F* as the extension (set or class) of the second-level concept "equinumerous with the concept *F*" and, therefore, as an object of some kind. For example, the

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[7] Ibid., § 68, 79.

number that corresponds to the concept "moon of Jupiter" is the set or class of all concepts that are in a one-to-one correspondence with the concept "moon of Jupiter" ("prime number less than 8," "letter of the word *York*," etc.).

Once he has explicitly defined the concept of number, Frege proposes to demonstrate its usefulness in *Grundlagen* by proving from it the fundamental properties of numbers such as, for example, the infinitude of the natural number series, but actually he only informally proves *Hume's principle* from this definition. Having established this principle he doesn't use the extensions of concepts anymore. We can conclude then, that in *Grundlagen* Frege informally proves the fundamental properties of the natural number series from *Hume's principle* and second-order logic.

1.5 The Infinitude of the Natural Number Series

The definition of number (Num) to which we have referred in the previous section is a pure cardinal definition of the concept of number, which should not be confused with the definition of natural number, since the number *n* is defined from the expression $Nx : Fx$ for some concept *F*, which allows us to count or list the individuals or objects falling under the above concept. It follows from this definition that, in order to obtain the finite cardinal numbers, or, as Frege says, the *individual numbers* [*einzelne Zahlen*], it is only necessary to find a concept *F* appropriate for each case. For example, Frege defines the number 0 as the number that corresponds to the concept "not equal to itself," i.e., $0 = Nx : x \neq x$, the number 1 as the number that corresponds to the concept "equal to 0," i.e., $1 = Nx : x = 0$, $2 = Nx : x = 0 \vee x = 1$, $3 = Nx : x = 0 \vee x = 1 \vee x = 2$, and so on. However, although in this way you can get a series of numbers that could be identified intuitively with the natural numbers, the truth is that Frege has not yet defined the concept of natural number and, therefore, he does not have a concept that applies to all finite cardinal numbers and only to them.

To define the concept of natural number, Frege first defines the successor relation "*n* follows immediately *m*" (Smn or, in the usual mathematical notation, nSm) as follows:

> There is a concept *F* and an object *x* that falls under it, such that the number that corresponds to the concept *F* is *n* and the number that corresponds to the concept "falls under *F* but is not equal to *x*" is *m*.[8]

In symbols:

$$\exists F \exists x \exists G \left(Fx \wedge Nx : Fx = n \wedge \forall y \left(Gy \leftrightarrow Fy \wedge y \neq x \right) \wedge Ny : Gy = m \right).$$

We know that *x* belongs to the extension of the concept "belongs to the *Q* series ending with *y*" if it is a weak ancestral of *y* in the *Q* series, that is, if *y* follows *x* or is equal to *x* in the *Q* series. In particular, if *Q* is the successor relation, we speak of the "series of successors" instead of the *Q* series and we have, therefore, that "*x* belongs to the series of successors ending with *n*" or, equivalently, "*n* belongs to the series of successors beginning with *x*," if *x* is a weak ancestral of *n* in the series of successors, i.e., if $S_{=}^{*}xn$. Now Frege can define the natural numbers as the numbers that belong to the series of successors starting with 0. In symbols:

$$\mathbb{N}n \equiv S_{=}^{*}0n . \qquad (1)$$

Actually, this definition implies the next, much more manageable:

$$\mathbb{N}n \equiv \forall F \left(F0 \wedge \mathrm{Her}\left(F\right) \rightarrow Fn \right), \qquad (2)$$

that is, *n* is a natural number if, and only if, it has all the properties inherited from 0 (The proof that (1) implies (2) is very easy given the definition of the ancestral: if $F0$ and $\mathrm{Her}\left(F\right)$, then $F1$. Now, since 1 is the only successor of 0, $\mathrm{In}\left(0, F\right)$, and hence, from the hypothesis $S_{=}^{*}0n$ and the definition of ancestral, we have finally *Fn*).

Having defined the set of natural numbers, Frege proves in *Grundlagen* its most important properties, particularly, its *infinitude*. Frege's basic idea for showing that every natural number has a successor is the observation that for the series of finite cardinals we have $Nx : x \leq 0 S 0$, $Nx : x \leq 1 S 1$, $Nx : x \leq 2 S 2$, etc. For example, $Nx : x \leq 2$ immediately follows after 2, because there are exactly three numbers that are lower than or equal to 2, namely: 0, 1 and 2. Thus, Frege's aim is to show from the definition of natural number that if *n* is a natural number,

[8] Ibid., § 76, 89.

then the number that corresponds to the concept "*x* belongs to the series of natural numbers ending with *n*" immediately follows after *n*. In symbols:

$$\forall n\left(\mathbb{N}n \rightarrow Nx : x \leq n\, S\, n\right). \qquad (3)$$

Obviously Frege also needs to prove that the successor of every natural number is a natural number. In symbols:

$$\forall n\left(\mathbb{N}n \wedge Snm \rightarrow \mathbb{N}m\right). \qquad (4)$$

Now, from (3) and (4) the theorem below immediately follows:

$$\forall n\left(\mathbb{N}n \rightarrow \exists m\left(\mathbb{N}m \wedge Snm\right)\right), \qquad (5)$$

that is, every natural number has a successor which is also a natural number or, which is the same, the infinitude of the series of natural numbers.

1.6 The Basic Laws of Arithmetic

In the first lines of *Grundgesetze*, Frege takes on the logicist program of *Grundlagen* as follows:

> In my *Grundlagen der Arithmetik* I tried to make plausible that arithmetic is a branch of logic and doesn't need to draw any argument from experience or intuition. This will be proved in this book, deducing the most basic laws of numbers with logical means alone.[9]

Surely by the "most basic laws of numbers" Frege meant those laws from which the remaining laws of arithmetic can be deduced. In fact, it is clear that if Frege wanted to show that "arithmetic is a branch of logic" then he must show that all the laws of arithmetic can be derived from logic alone and this is only possible if we isolate some principles, the basic laws of arithmetic, from which we can derive the other laws of arithmetic, and prove these basic laws only by logical means. For this purpose, in *Grundgesetze* Frege first derives *Hume's principle* from the explicit definition of number and then derives from that principle the basic laws of arithmetic that give title to the work. This procedure is the same as that he had followed in *Grundlagen*, but now Frege will use the extensions not only in the proof of *Hume's principle* from the explicit definition of

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[9] Frege (1893) 1962, 1.