From a Heuristic Point of View

From a Heuristic Point of View: Essays in Honour of Carlo Cellucci

Edited by

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INTRODUCTION

Strange as it may seem, some people believe that logic is a branch of mathematics which has little to do with philosophy. Carlo Cellucci's work is a refutation of this bizarre prejudice. For a long time Cellucci has been a distinguished logician in Italy. His most significant contributions to mathematical logic concern proof-theory. *Teoria della dimostrazione*, published in 1978, was the first book on proof-theory to appear in Italian. But Cellucci's work in logic has always been connected with a concern for philosophical problems. One problem, in particular – that of discovery – became prominent. It can be summarized in the question: how is it possible that we acquire new knowledge? Cellucci gradually came to the conviction that logic can play a «relevant role in mathematics, science and even philosophy» (Cellucci 2013, p. 18) only if it helps us to solve this problem. In this respect mathematical logic, the paradigm of logic initiated by Gottlob Frege, is wholly inadequate. It is a logic of justification, not a logic of discovery. Its aim is to provide a secure foundation for mathematics, and Gödel's incompleteness theorems show that it cannot fulfill this purpose. Thus, Cellucci has gradually developed a critique of mathematical logic and, on the other hand, the project for a new logic, framed in the context of an original naturalistic conception of knowledge and philosophy. The most recent and articulated statement of these views can be found in his books *Perché ancora la filosofia* (2008) and *Rethinking Logic* (2013). According to Cellucci's naturalistic conception «logic is a continuation of the problem solving processes with which biological evolution has endowed all organisms» (2013, p. 18).

The aforementioned path from logic to a naturalistic conception of knowledge and philosophy partly explains the title *From a heuristic point of view* chosen by the editors of this volume in honour of Carlo Cellucci. It partly explains the title because it suggests an analogy with the author of the celebrated collection *From a logical point of view*, Willard Van Orman Quine, the initiator of contemporary naturalized epistemology. Another part of the explanation, of course, lies in the word "heuristic", which emphasizes the most characteristic trait of Cellucci's thought and the main difference between him and Quine: the emphasis on discovery. Both philosophers maintain that knowledge is fallible and corrigible. Fallibilism is thus a further idea that they share. But Quine's picture of human knowledge embodies a principle of conservatism, Quine's maxim of minimum mutilation: when we encounter recalcitrant experience, we make the smallest possible number of changes to our beliefs and do not alter our overall theory more than necessary. According to the maxim of minimum mutilation classical logic, though in principle not immune to revision, will not be revised, because, Quine says, losing its convenience, simplicity and beauty would be too high a price. Quine's picture of knowledge and logic thus appears almost static, compared with Cellucci's insistence on discovery methods and on the continuous growth of knowledge. Quine's doctrine that no statement, not even the logical law of the excluded middle, is in principle immune to revision appears a rather lukewarm revisionism, when compared with Cellucci's harsh critique, not only of classical logic, but of all the logics developed within the paradigm of mathematical logic.

The essays collected in this book were written by some of Carlo's colleagues and two former students (the editors) who have shared his interests and investigated topics that he dealt with. Each contributor in her or his way, has to varying degrees studied, collaborated, discussed, agreed or disagreed with Cellucci. The variety of these different attitudes is reflected in the essays. A first group of essays, collected in **Part I**, more directly concern **mathematical logic, mathematical knowledge and truth**. Cellucci's critique of mathematical logic is part of a more general critique of the axiomatic method and of the idea of providing a secure foundation for mathematics. The picture of mathematical knowledge as based on the axiomatic method and the attempt to establish the absolute certainty of mathematics by providing a firm foundation for axiomatic theories characterize the foundational schools of logicism and Hilbertian formalism in the twentieth century, which used mathematical logic as a tool for their philosophical programmes. In her paper *'Regelfolgen' in Wittgenstein's Philosophy of Mathematics and Mind* **Rosaria Egidi** examines some central concepts of Ludwig Wittgenstein's writings. She reminds us of the philosopher's severe judgment in the *Remarks on the Foundations of Mathematics*: «Mathematical logic has completely deformed the thinking of mathematicians and of philosophers». Though Cellucci usually dissociates himself from Wittgenstein, Egidi expresses the conviction that there is «a point of agreement» between Cellucci's arguments against foundational strategies and axiomatic method and Wittgenstein's idea that «mathematics is embedded in the natural history of mankind and bound up with the social and anthropological forms of life». Cellucci's arguments in *Le Ragioni della logica* (1998), *Filosofia e matematica* (2002) and *La filosofia della matematica del Novecento*

(2007) are summarized by **Donald Gillies** in the first part of his essay: Cellucci maintains that the logicist programme of Frege and Russell and Hilbert's formalist programme ended in failure because of the limitative results of Gödel and Tarski. Gillies agrees that «Gödel's first incompleteness theorem gives a fatal blow to logicism» and that Gödel's second incompleteness theorem «shows the impossibility of carrying out Hilbert's programme». He thus concludes that Cellucci's criticisms of mathematical logic show that «mathematical logic did not succeed in establishing a foundation of mathematics of the kind that the inventors of mathematical logic had hoped to create». However, he remarks, mathematical logic «has proved to be a very useful tool for computer science». This idea lies behind the title of Gillies' essay *Serendipity and Mathematical Logic*. He defines serendipity as «looking for one thing and finding another». The concept can be applied to mathematical logic because the pioneers of mathematical logic were looking for something they did not find: a foundation for mathematics that would make mathematical statements certain; but they found something other than what they were looking for: «invaluable, perhaps indeed essential, tools for the development of computers». In the last part of his essay Gillies explains how this came about.

In his essay *What Mathematical Logic Says about the Foundations of Mathematics* **Claudio Bernardi** grants that «perhaps in the nineteenth century logic was regarded as a way to guarantee the certainty of mathematics» and that nowadays «it seems naïve, and perhaps even futile, to hope for a definitive, proven certainty of mathematics». Nevertheless, Bernardi thinks that mathematical logic offers «a fruitful model» of mathematical activity. Though it is not a faithful description of how a mathematician works, mathematical logic provides us with a theoretical framework (model theory, proof theory, recursion theory) in which various implicit features of mathematical practice are made explicit, «a precise definition of proof» is given, «rigorous methods and procedures to develop mathematical theories» are suggested. Bernardi emphasizes the fact that the theoretical framework of mathematical logic yields significant results: «mathematical logic, exactly like mathematical analysis, is justified by its results», which clarify «the sense and the limits of mathematical activity». Cellucci maintains that the founders of mathematical logic identified the mathematical method with the axiomatic method. The axiomatic method is the main object of study of mathematical logic. Mathematical logic is based on the concept of a formal system as a closed system. A closed system is : «a system based on principles that are given once for all and cannot change, and whose development consists in deducing conclusions from them. Therefore, the development of the system requires no input from the outside» (Cellucci 2013, p 178, cfr. 1993). Against the picture of mathematical knowledge as consisting of closed systems Cellucci proposes the view that all knowledge, including mathematical knowledge, consists of open systems: «an open system is a system which initially consists only of the problem to be solved, and possibly other data already available, and whose development consists in obtaining more and more hypotheses for solving the problem from the problem itself, and possibly other data already available, by non-deductive rules. The hypotheses are then checked by means of the plausibility test procedure. The other data already available from which hypotheses are possibly obtained come from other open systems, therefore the development of an open system involves interactions with other open systems» (Cellucci 2013, p. 294). Bernardi agrees with Cellucci that open systems can better account for the fact that «the mathematical community searches continuously for new axioms, which are deeper or more general, or more suited for some purpose, trying to give a more comprehensive explanation of a subject», but he thinks that both open systems and axiomatic systems «reflect mathematical activities». Bernardi's defence of the axiomatic method and of mathematical logic is strengthened by some final remarks concerning their educational value and the role of mathematical logic in the contribution of computer science to mathematics.

As Gillies explains in his essay, Gödel's incompleteness theorems are the linchpin of Cellucci's verdict that the basic assumptions of mathematical logic are untenable. A completely different interpretation of the significance of incompleteness is proposed by **Michele Abrusci** in his paper, *Dedekind, Hilbert, Gödel: the Comparison Between Logical Sentences and Arithmetical Sentences*. The equivalence of arithmetical and logical sentences established by Dedekind in *Was sind und was sollen die Zahlen* and Gödel's completeness theorem for first order logic combine to give a picture of the relation between logic and arithmetic according to which «every existential arithmetical first-order sentence belonging to Σ_1^0 is equivalent to a universal logical sentence belonging to Π^1 , and vice versa every universal logical sentence belonging to Π^1 is equivalent to an existential arithmetical first-order sentence belonging to Σ_1^0». Moreover «every universal arithmetical first-order sentence belonging to Π_1^0 is equivalent to an existential logical sentence belonging to Σ^1, and vice versa every existential logical sentence belonging to Σ^1 is equivalent to an universal arithmetical first-order sentence belonging to Π_1^0 ». Abrusci comments that these results reveal «strong mutual dependencies between logic and arithmetic» and «the need […] to seek *simultaneous* foundations of both logic and arithmetic». He remarks that «given this correspondence between logic and arithmetic, a universal arithmetical quantifier corresponds to an existential logical quantifier, and an existential arithmetical quantifier corresponds to a universal logical quantifier. Perhaps, this exchange of quantifiers – when we go from an arithmetical sentence to a logical sentence, and vice versa – is the most important discovery provided by these results, surely an unexpected result». The completeness theorem for first order logic can be reformulated as the principle that «every logical sentence belonging to Π^1 is equivalent to its logical provability». Abrusci calls this «a weak form of the completeness of logic». The simplest extension of this principle would be «every logical sentence belonging to Π^1 or belonging to Σ^1 is equivalent to its logical provability». But if we accept the hypothesis that «every logical sentence belonging to Σ^1 is equivalent to its logical provability», we reach «rather paradoxical» consequences: «every existential logical sentence belonging to Σ^1 is equivalent to a universal logical sentence belonging to Π^1» and «every universal arithmetical sentence belonging to Π_1^0 is equivalent to an existential arithmetical sentence belonging to Σ_0^1». This is «a kind of collapse of universal and existential quantifiers both in logic and in arithmetic». But the Incompleteness Theorem proved by Gödel shows that the extension of the weak form of completeness does not hold, because «not every logical sentence belonging to Σ^1 is equivalent to its logical provability». Thus, «Gödel's Incompleteness Theorem avoids the collapse of quantifiers, in arithmetic and in logic! So, it is not a disaster for logic or for arithmetic: rather, it saves logic and arithmetic from a collapse of quantifiers!».

The essays by Abrusci, Bernardi and Gillies concern mathematical logic. Cellucci's criticisms, however, are not levelled only at mathematical logic, they are also aimed at a deeper target: the role of deduction itself in mathematics. The axiomatic method is deductive. But the method of obtaining hypotheses for solving problems that Cellucci calls «the analytic method» is non-deductive. Cellucci rejects the common view that mathematical knowledge is obtained by deductive inference. In his essay *The status of mathematical knowledge* **Dag Prawitz** intends to defend the common view. Prawitz agrees that «after Gödel's incompleteness theorem, one cannot think that mathematics is rightly characterized by saying that it aims at proving theorems in given axiomatic systems». But he maintains that «the fall of the axiomatic method does not affect the view that mathematical knowledge is acquired by deductive proofs from obvious truths, because this view is not tied to the idea that one can specify once and for all a set of axioms from which all deductive proofs are to start. For instance, in arithmetic a deductive proof of an assertion can start from reflective principles that are not given in advance but are formulated in the context of the assertion in question and are then seen to be obviously true». Cellucci rejects the common view for two reasons: 1) the process by which one finds a hypothesis that is capable of solving a mathematical problem is a proof process, but not a deductive one; 2) there is no rational way of knowing whether primitive premisses are true; the initial premisses of a deductive proof are only more or less plausible hypotheses. Prawitz agrees that there is «a heuristic phase in the solution of a mathematical problem in which guesses are made and different strategies are tried. The search for suitable premisses from which it would be possible to deduce an answer can to some extent be described as a rule-governed process, and we may choose to call it proving. But there are good arguments against stretching the term proof or inference that far». So, (1) above is «partly, at least, a question of terminology» and «the main argument for the inadequacy of the deductive method» is (2). Prawitz thinks, however, that argument (2) «goes too far». He contends that we often do know that the premises of deductive inferences are true (and not only plausible). We can know the truth of the premises in three ways. A «fallible method […] for getting to know that the initial premisses are true [is] the analytic method as Cellucci describes it, or just careful arguments pro and con». But Prawitz thinks that we can also find «conclusive grounds for the initial premises […] in virtue of what one takes the involved concepts to mean». A third way of knowing that certain premisses are true is by means of deductive inference. The conclusion of one deductive inference can become the premiss of another deductive inference. Prawitz is aware that, on the basis of «a philosophical dictum […] that the content of the conclusion of a deductive inference is already contained in the content of the premisses», some philosophers, including Cellucci, claim that deductive inference cannot generate new knowledge. Prawitz thinks that the notion of a deductively valid inference ought to be analysed in a new way, not in terms of model-theoretic logical consequence. He has proposed a new analysis of this notion in terms of conclusive grounds. According to Prawitz's analysis, if an inference is deductively valid, it can provide new knowledge: it yields a conclusive ground for the conclusion when conclusive grounds for the premises are given. The general idea that mathematical knowledge is obtained deductively, Prawitz writes, can be vindicated by showing that «for all deductive inferences used in mathematics there are operations that transform conclusive grounds for the premises to a conclusive ground for the conclusion». This is a project of whose realization, Prawitz admits, he has given here only a hint. In the closing sentences he remarks, however, that it is an open question whether the project outlined can be carried through and that he sees «Cellucci's criticism as a stimulating challenge». The status of mathematical knowledge is also the topic of *Empiricism and Experimental Mathematics*, by **Gabriele Lolli**. Cellucci's fallibilist conception of mathematical knowledge might perhaps be seen as sharing some features of a general trend in the philosophy of mathematics: neo-empiricism. After dealing with neo-empiricism and its relation with so-called experimental mathematics, Lolli argues that the limitations of neo-empiricism result from considering mathematics only from the point of view of procedures. Lolli criticizes the idea that mathematics can be characterised by its method: «the limit of the empiricist vision is that of restricting the focus only on method of validation, thus ending in the stale antithesis between deduction and induction, between logic and experiment. Even Cellucci, although he is not an empiricist, suffers from this type of limit; in many writings (e.g. 1998) he opposes the method of logic to different methods, in particular the method of analysis, or the open systems. The discussion of the method is inevitably shallow; knowledge is not given by the reasoning method, but by the models of reality one builds, and the concepts one uses. I am sure Cellucci would agree». In the last part of his essay, Lolli proposes the ideas of the Russian thinker Pavel A. Florenskij as a model for a philosophy of experimental mathematics.

A discussion of mathematical knowledge and of knowledge in general often leads to the problem of truth. For example, in his essay Dag Prawitz implicitly endorses the widespread view that knowledge and truth are strictly connected notions. Analytical epistemologists think that "x knows that p" implies "it is true that p". Cellucci, however, proposes a completely different conception of knowledge: «knowledge does not aim at truth, it aims at plausibility» (Cellucci 2008, p. 177). Plausibility is a key notion in Cellucci's heuristic view of knowledge. Knowledge consists of plausible hypotheses: «a hypothesis is plausible if, and only if, it is compatible with the existing data» (p. 177). By «compatibility with the existing data», Cellucci means that «if we compare the arguments for and against the hypothesis based on the existing data, the arguments in favour of the hypothesis outweigh the arguments against it» (pp. 177-8). Truth does not play any role in science: it is only a chimera that prevents us «from adequately understanding the character of knowledge» and therefore «must be disposed of». The question *Is Truth a Chimera?* provides the title for **Cesare Cozzo**'s contribution to this volume. Cozzo summarizes Cellucci's arguments for the thesis that truth is a chimera and then raises four objections to Cellucci's views on truth. He argues that, Cellucci's arguments notwithstanding, a notion of truth is necessary for the human activity of problem solving and therefore for an adequate understanding of the phenomenon of knowledge.

In the essays hitherto considered the contributors agree with some of Cellucci's ideas, but argue in favour of some author, some notion, some distinction, some standard view, or scientific programme that Cellucci criticizes, rejects or at least seems to put aside in pursuit of his programme for a heuristic conception of knowledge. Other essays develop lines of thought that parallel and sometimes follow Cellucci's suggestions for research and some of the main features of his programme. The latter essays, collected in **Part II**, generally concern **the project for a logic of discovery**. In particular they deal with the nature of mathematical objects, the role of ampliative inference and axiomatization, the continuity between mathematics and the empirical sciences and naturalistic epistemology.

The idea of a logic of discovery is central to the papers by Reuben Hersh and Emily Grosholz. In his *To establish new mathematics, we use our mental models and build on established mathematics* **Reuben Hersh** maintains that mathematical knowledge grows by means of two tools: established mathematics and heuristic reasoning, in particular mental models of mathematical entities. At the beginning of his paper Hersh rejoices at the fact that, thanks in part to Cellucci's work, the topic of mathematical practice «arrived as a legitimate theme of philosophical investigation». The question is «What do real mathematicians really do?». A mathematician's proof is not an axiomatic proof. To illustrate his point Hersh considers Andrew Wiles' proof of Fermat's Last Theorem (FLT). A mathematician's proof, like Wiles' proof, starts from established mathematics and establishes new mathematics by means of mental models. Established mathematics is «the body of mathematics that is accepted as the basis for mathematicians' proofs» and every mathematician has complete confidence in it, since «confidence in established mathematics is for a mathematician as indispensable as confidence in the mechanics of a piano for a piano virtuoso, or confidence in the properties of baseballs and bats for a big league baseball player. If you're worried about that, you aren't even in the game». Hersh argues that established mathematics is a historical product, and does not need a foundation, because it is what any mathematician is trying to build on. «The status of established mathematics is not absolute truth, but rather, warranted assertibility». So Hersh partly shares Cellucci's criticism of the notion of truth, arguing that «questions of 'truth' versus 'fiction' are irrelevant to practice» and that «truth in the sense of perfect certainty is unattainable». For Hersh, in mathematics the "true" should be understood as "well justified" or "firmly established", in the sense of John Dewey's "warranted assertibility". The second essential ingredient of mathematical knowledge, mental models of mathematical objects, are constituted by the properties and capabilities effectively employed in using these objects in proofs and in general in mathematical practice. Mental models are «socially regulated»: they are acquired and developed in the course of a social practice of interaction with other mathematicians and not simply because the single mathematician has «memorized a formal definition». In this sense, a number theorist «knows what a Galois representation is, [and] knows what a semistable elliptic curve is»: he (or she) knows how to use these objects effectively because his (or her) mathematical practice «has shaped them and molded them to be congruent or matching to the models possessed by other experts [in the same field]». Mental models are the «candidates for the new semantics» of mathematics. In this sense Hersh characterises mathematical reasoning as essentially semantic.

The FLT proof is also the starting point of *Fermat's Last Theorem and the Logicians* by **Emily Grosholz**, who employs it as an exemplary case study to show the interplay of two essential tasks in scientific discovery, namely analysis and reference, which usually generate «internally differentiated texts because thinking requires us to carry out two distinct though closely related tasks in tandem». Analysis requires the abstract, «more discursive project of theorizing», the search for conditions of intelligibility of problematic objects or solvability of objective problems, while reference requires «the choice of 'good' predicates, durable taxonomy, and useful notation and icons». The central claim of her paper is that productive mathematical discourse must carry out these two distinct tasks in parallel: «more abstract discourse that promotes analysis, and more concrete discourse (often involving computation or iconic representations) that enables reference, are typically not the same», so that «the resultant composite text characteristic of successful mathematical research will thus be heterogeneous and multivalent, a fact that has been missed by philosophers who begin from the point of view of logic». According to Grosholz, the integration of the various discourses into a rational relationship generates the growth of knowledge, and she gives tthe proof of FLT, as well as some of its proposed logical reconstructions, as examples. In this sense, the FLT proof suggests that in the first stage, «modular forms are investigated as the objects of reference, and treated 'geometrically' as holomorphic differentials on a certain Riemann surface, while elliptic curves are treated as instruments of analysis; and conversely in the second stage, Wiles' proof, elliptic curves serve initially as objects of reference, while modular forms become the instruments of analysis». Sharing two crucial theses of Cellucci's heuristic conception of the nature of mathematical objects and the axiomatic method, Grosholz argues that in real mathematics, the discovery, identification, classification and epistemic stability of objects is controversial and that «it is not generally true that what we know about a mathematical domain can be adequately expressed by an axiomatized theory in a formal language, nor that the objects of a mathematical domain can be mustered in a philosophical courtyard, assigned labels, and treated as a universe of discourse».

A core feature of Cellucci's heuristic conception, that is the detection of inferential means of discovery, is the central theme of *Christiaan Huygens's 'On reckoning in games of a chance': a case study on Cellucci's heuristic conception of mathematics* by **Daniel G. Campos.** He offers an insightful discussion of the effectiveness of Cellucci's account of scientific discovery by examining a case study in probability theory. Campos endorses Cellucci's philosophical conception of mathematics as «an open-ended, heuristic practice as opposed to the *foundationalist* view of mathematics as a closed-ended body of knowledge that is completely determined by self-evident axioms» – though he also briefly mentions a few examples from the history of mathematics to raise some questions «about Cellucci's strong claim that axioms never have a heuristic function or cannot be regarded as hypotheses». In his essay, he examines the case of Christiaan Huygens's *On Reckoning in Games of Chance* (1657) to show that Cellucci's heuristic conception provides an insightful way to account for Huygens's approach to solving mathematical problems in probability theory. In particular he shows that Huygens's practice consists «in problem-solving that can be described by the analytic method and its ampliative inferences to search for hypotheses». More specifically, he argues that Huygens uses three rules—of those explicitly treated by Cellucci—to generate hypotheses, as Huygens «employs the heuristic methods of particularization, generalization, and reduction to solve one of the main problems in his *Reckoning*».

In Cellucci's programme, the rules for finding hypotheses (such as those discussed by Campos) are correlated to his characterization of the naturalistic approach to heuristics, a theme that is the topic of *Natural Mathematics and Natural Logic* by **Lorenzo Magnani.** In this paper Magnani builds on Cellucci's version of the naturalistic approach, and in particular on the distinction between natural and artificial logic and mathematics. Magnani aims to provide new insights into *distributed cognition*, especially into «the role of logical models as forms of cognitive externalizations of preexistent in-formal human reasoning performances».

He argues that the ideal of a formal logical deduction is an «optical illusion» and that the growth of scientific knowledge relies on the interplay between internal and external semantic, pragmatic and non-demonstrative representations. As a consequence, he endorses Cellucci's «skeptical conclusion about the superiority of demonstrative over non-demonstrative reasoning», since «to know whether an argument is demonstrative one must know whether its premises are true. But knowing whether they are true is generally impossible», as Gödel's theorems show. As a matter of fact, premises in demonstrative arguments «have the same status of the premises of non-demonstrative reasoning» and «demonstrative reasoning cannot be more cogent than the premises from which it starts; the justification of deductive inferences in any absolute sense is impossible, they can be justified as much, or as little, as non-deductive – ampliative – inferences».

A cornerstone of Cellucci's work is his contrast between a bottom-up and a top-down approach to mathematics. Cellucci (2013, p. 10) writes: «most mathematicians follow the top-down approach to mathematics, which has been the mathematics paradigm for the past one and a half centuries. According to the top-down approach: 1) a mathematics field is developed from above, that is, from general principles concerning that field; 2) it is developed by the axiomatic method, which is a downward path from principles to conclusions derived deductively from them». Against the top-down approach, Cellucci argues in favour of a bottom-up approach, according to which «1) A mathematics field is developed from below, that is, from problems of that field or from problems of other mathematics, natural science or social science fields. 2) It is developed by the analytic method, which is an upward path from problems to hypotheses derived non-deductively from them » (2013, p. 11). The aim of **Maria Carla Galavotti**'s essay, *For A Bottom-Up Approach To The Philosophy Of Science*, is «to extend Cellucci's bottom-up approach […] to the philosophy of science at large». Galavotti is convinced that a bottom-up approach is capable of promoting a better understanding of the nature of scientific knowledge. In her paper she expounds some lines of thought concerning the philosophy of science and statistical methodology that can contribute to this extension of the bottom-up approach. In the first place, Patrick Suppes' probabilistic empiricism «can be deemed bottom-up, although this is not the terminology he uses». Suppes holds «that the relation between empirical theories and data "calls for a hierarchy of models" characterized by different degrees of abstraction, where there is a continuous interplay between theoretical and observational model components». The hierarchy is developed «from bottom to top because given a model of the data exhibiting a certain statistical structure of some phenomenon under investigation a fitting theoretical model is sought». Suppes rejects a clear-cut distinction between theories and data: «depending on the desired level of abstraction different pieces of information […] will count as data, and what qualifies as "relevant" will inevitably depend on a cluster of context-dependent elements». Suppes' approach is contextual and pluralist: «scientific structures» admit of a multiplicity of representations, the choice of which depends on contextual factors. For Suppes scientific activity is a kind of perpetual problem-solving and in this respect too it is clear that «probabilistic empiricism has much in common with Cellucci's bottom up approach». In his book *Probabilistic Metaphysics* Suppes rejects the «chimeras» of a traditional conception of science: certainty, determinism, the idea of the unity of science, the idea that science is converging towards some fixed result that will give us complete knowledge of the universe. Galavotti compares Suppes' views with Cellucci's theses on the chimeras that have prevented philosophy from adequately understanding the character of knowledge (cf. Cellucci 2008, pp. 77-8). She concludes that «Suppes and Cellucci can be seen as complementary to the extent that they develop different aspects of the bottom-up approach». A thesis emphasized by Galavotti is that in the bottom-up movement each step from problem to model depends on the context in which the problem arises. Therefore, «context constitutes the bedrock on which the bottom-up approach to the philosophy of science rests». This thesis is exemplified by Galavotti's description of Christian Hennig's approach to statistics «that can be regarded as an expansion of Cellucci's approach to mathematics and Suppes' view of philosophy of science» and is further illustrated by her outline of the literature on the statistical methodology for assessing causal relations. In the last section of her essay Galavotti points out that any account of the notion of context has to include (i) «the disciplinary framework in which some problem originates, and more specifically its conceptual reference setting, compounded by the body of theoretical and methodological knowledge shared by the scientific community addressing the problem in question», (ii) the nature and amount of the available evidence (in the case of statistical data the composition and size of the population from which they are obtained), and (iii) the aims of a given investigation (explanation or prediction).

The bottom-up approach and the heuristic perspective challenge the received view on the characterization of mathematical objects and the nature of mathematical modelling. This point is a central theme of *Mathematizing risk. A heuristic point of view* by **Emiliano Ippoliti**. In his paper, Ippoliti endorses Cellucci's view on heuristics, examining its consequences for the mathematization of several phenomena—the applicability and the effectiveness of mathematics. Using the notion of risk as an example, Ippoliti shows how the heuristic view accounts for it and its mathematical treatment. To this end he investigates the main approaches to risk, namely the probabilistic, the psychological, the fractal and the evolutionary (not to mention the co-evolutionary), and shows that the lack of success of the various approaches to the treatment of risk is due to the ways in which they conceptualize and mathematize it. Then, taking as a starting point Cellucci's characterization of mathematical objects (namely, «mathematical objects are hypotheses tentatively introduced to solve problems») he sets out to show that the heuristic point of view can offer a better characterization of risk and can improve the other approaches, but this requires a different conceptual path, bottom-up, local and oriented towards problem-solving. He argues that risk is a complex object from a mathematical point of view, whose identity is continuously open to new determinations and properties and in this sense, it shows the extent to which mathematical objects are hypotheses tentatively introduced to solve problems and that «the hypotheses through which mathematical objects are introduced characterize their identity. The identity of a mathematical object can be characterized differently by different hypotheses», and this implies that «hypotheses do not characterize the identity of mathematical objects completely and conclusively, but only partially and provisionally. For the identity of mathematical objects is always open to receiving new determinations through interactions between hypotheses and experience» (Cellucci 2013, p. 104).

This characterization of mathematical objects shapes Cellucci's approach to two long-standing issues in the philosophy of mathematics, namely the ontology of mathematical entities and the relations between mathematics and empirical science. In particular, the heuristic view argues both for continuity between mathematics and empirical science and for the dismissal of the relevance of the ontological issue of mathematical entities. These two tenets are the starting point for *Reflections on the objectivity of mathematics* by **Robert Thomas**. As regards the first tenet, Thomas employs the notion of «assimilation» to argue for the objectivity of mathematics. His concept of assimilation is «roughly that of Piaget in which new things we meet are assimilated to notions we already have, which notions in their turn are accommodated to accept the new arrivals». The assimilation process makes mathematics similar to empirical science, in the way that new concepts are acquired and made to fit with other concepts, classes and the *corpus* of knowledge we already have. As for the second tenet, Thomas argues that objectivity does not imply a dependence on an ontology. More specifically, he maintains that «objectivity is achieved in mathematics by public agreement (including agreements to differ) on styles of inference and definitions in terms of relations». Moreover «mathematics is about relations and not about mathematical objects», as «the mathematical objects are just things that we wish on the relations we want to talk about in order to be able to do the talking»: the existence or otherwise of the mathematical entity is completely irrelevant to the construction of mathematics and its inquiry. Thomas also clarifies this point by introducing an evolutionary characterization of the role of mathematics and reasoning in general. He argues that «since we must be able to reason as dependably about what does not exist—even in a mathematical sense—as about what does, for instance in *reductio* proofs, whether some things exist or not is not of any practical importance. It is not just in mathematics that we need to be able to reason effectively about what does not exist; it seems to me that the evolutionary advantage to our reasoning ability is primarily our capacity for reasoning about the future».

Here the editors end their introduction. We thank all the authors who have made this volume possible by kindly accepting our request to participate in the project for a book in honour of Carlo Cellucci and we thank the Department of Philosophy of Sapienza University of Rome for financial support. We hope that this book will allow the reader to form an opinion of the variety of perspectives inspired or challenged by Carlo's work. Thank you, Carlo, for your contribution to philosophy and to our personal development.

Cesare Cozzo, Emiliano Ippoliti

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SECTION I

MATHEMATICAL LOGIC, MATHEMATICAL KNOWLEDGE AND TRUTH

CHAPTER ONE

'REGELFOLGEN' IN WITTGENSTEIN'S PHILOSOPHY OF MATHEMATICS AND MIND

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Der mathematische Satz hat die Würde einer Regel. Das ist wahr daran, daß Mathematik Logik ist: sie bewegt sich in den Regeln unserer Sprache. Und das gibt ihr ihre besondere Festigkeit, ihre abgesonderte und unangreifbare Stellung.

Remarks I, §165

SUMMARY The existence of mutual influences between the philosophy of mathematics and philosophical psychology was widely documented from Wittgenstein's 1930s writings onwards. In particular, his notes on mathematics in the "Proto-Investigations" (1937), later included in Part I of the *Remarks on the Foundations of Mathematics*, continued the analysis dedicated to "following-a-rule". The paper aims, firstly, to highlight the Wittgensteinian project for a general examination of normative concepts, characteristic of mathematics and of rule-governed activities; and secondly, to exemplify the affinities between mathematical and psychological methods in the critique of the conceptual confusions lurking in their languages and in the treatment of key concepts such as rule and proof.

KEYWORDS Wittgenstein, following-a-rule, mathematics, mind, normativity, proof.

1. The project for a general investigation of normative contexts

In a recent paper, Carlo Cellucci (2008) returned to one of his favourite themes: the opposition between the notions of axiomatic and analytic proof, employing fresh arguments to defend the superiority of the latter. Though Cellucci's notion of "analytic" differs significantly from the "normative" concept Wittgenstein assigns to the mathematical proof and though the vindication of the biological and evolutionary bases of mathematical procedures is far removed from Wittgenstein's anti-Darwinian attitude, I think that his idea that mathematics is embedded in the natural history of mankind and bound up with the social and anthropological forms of life, is a point of agreement for all the arguments used in the refutation of foundationalist strategies as well as for those exemplified in axiomatic methods. Prompted by this conviction, I present here a short examination of some central concepts of Wittgenstein's writings on the foundations of mathematics.

The existence of mutual influences and parallel methods between the philosophy of mathematics and philosophical psychology is widely documented in Wittgenstein's writings from the 1930s onwards. One of the most important documents in this sense is taken from the second half of the so-called "Proto-Investigations" of 1937, which examines the philosophical problems of mathematics and follows the first half, devoted to issues of language, meaning, understanding and following rules. Taken together, the two parts, along with the 1938 Preface, make up the "Frühfassung" of the last version (the "Spätfassung" composed in 1945-46) of the *Philosophische Untersuchungen*,[1] published posthumously in 1953 with an English translation and with the title *Philosophical Investigations* (hereafter *Investigations*).[2]

The closeness, present in the "Proto-Investigations", of the theme of "following rules" (*Regelfolgen*) and verbs related to the investigation of the "foundations of mathematics", in the special non-logicist meaning Wittgenstein assigns to the term, is in no way accidental. It is, as we shall see, integral to Wittgenstein's "second philosophy" and to the task of rejecting the hegemonic model of the name-object relation and supporting

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[1] See the critical-genetic edition including the five known versions of the *Philosophische Untersuchungen* in Wittgenstein (2002). On their composition and for the tables of correspondence between them, see J. Schulte's "Einleitung" and "Anhang". While Wittgenstein included the first half of the "Proto-Investigations" in §§1-190 of the *Philosophische Untersuchungen*, with a few variations, the second half appeared as Part I in his *Remarks on the Foundations of Mathematics* (hereafter *Remarks*). This work contains a selection made by the editors of Wittgenstein's writings on the philosophy of mathematics composed between 1937 and 1944, the year in which he stopped writing about this topic and turned his attention exclusively to the philosophy of mind and related issues up until his death in 1951.

[2] With the reference to *Investigations* I mean only Part I of the work.

the plurality of language-games, hence avoiding the drawbacks of the «one-sided diet» which in his opinion was the main cause of the «philosophical disease» (*Investigations*, §593). Mathematics or rather mathematical logic is also plagued, according to Wittgenstein, by the same disease, so that the therapeutic method he suggests for philosophy will also be applied to mathematics. His diagnosis is concisely expressed in a passage of the *Remarks* that is worth recalling:

> 'Mathematical logic' has completely deformed the thinking of mathematicians and of philosophers, by setting up a superficial interpretation of the forms of our everyday language as an analysis of the structures of the facts. Of course in this it has only continued to build on the Aristotelian logic (V, §48).

In particular, the second half of the "Proto-Investigations" reveals the point at which Wittgenstein's philosophy of mathematics is grafted onto the trunk of his analysis of propositions containing verbs such as "can", "to be able", "to understand" and finally "to follow-a-rule". This shows that these two lines of research belong to the same categorical domain, in that they share the status of normative constructs and do not describe, as in the case of empirical propositions, states of affairs or mental states (in other words: physical, psychological or abstract objects), but are rules or norms of description, consisting in grammatical stipulations about the meaning to be attributed to the words comprising them. In this way, from the mid-1930s onwards, Wittgenstein confirms the distinction between descriptive propositions and rules of description, i.e. the different use (empirical and normative) of propositions that he would gradually develop until it became canonical in *On Certainty* (§167).

The second half of the "Proto-Investigations" is also interesting because from it emerge traces of the perplexities that in 1937 marked Wittgenstein's analysis of *Regelfolgen* and momentarily blocked its continuation, prompting him to shift his attention to mathematical propositions and procedures. As a matter of fact, this "deviation" was to last seven years, during which Wittgenstein devoted himself almost exclusively to the philosophy of mathematics[3], only returning in 1944-45

[3] In addition to the hundreds of manuscript pages, most of which we find in the *Remarks*, Wittgenstein's activity in this field is testified to by the lectures he gave in English in 1939. These were collected by his pupils, later edited by Cora Diamond in 1975 and entitled *Lectures on the Foundations of Mathematics* (hereafter *Lectures*).

to work with more assured self-awareness on the theme of rules. But where did what I have called the "blocking" of Wittgenstein's investigation of "following-a-rule" occur?

In the "Proto-Investigations", after devoting a detailed series of remarks to the problems arising from the particular language-game of the pupil asked by the teacher to follow the rule of adding 2 after 1000 in the succession of natural numbers, i.e. to apply the mathematical formula [' +n'], Wittgenstein appears to stumble upon a question asked by his imaginary interlocutor: «But then the steps [from the formula 'add n' to its application] are *not* determined by the mathematical formula?» and replies that «there is an error in this question» ("Proto-Investigations", §§167-168; *Investigations*, §189). Although Wittgenstein reveals the error, he does not apply himself, as is the case later in the celebrated §§191-242 of the *Investigations*, to explaining the reasons why it is a mistake to ask how the rule given by the teacher determines the steps in advance, i.e. the pupil's subsequent applications of the formula, as if the connection between rule and application existed even before the application takes place. In the next two remarks Wittgenstein confines himself to reiterating the argument – already adopted in the transition writings and in particular in the *Big Typescript* about expecting, wishing, intending, thinking, believing[4] and later extended to following rules, inferring and proofing – according to which these procedures are not the result of mental processes that «magically» contain all the steps already preformed, in the same way as in inferring the conclusions are already logically present in the premises. In the *Remarks* this argument will be confirmed when Wittgenstein denies that the result of a calculation exists even before the proof has been established and affirms, recalling an old sentence of the *Tractatus*, that «process and result are equivalent» (I, §82). In normative contexts, and in particular in mathematical procedures, the connections between process and result do not pre-exist, the steps are not already taken, are not predetermined (I, §22), but are in some way created *ex novo*. The proof «makes new connexions, and it creates the concepts of these connexions (it does not establish that they are there; they do not exist until it makes them» (III, §31). Here Wittgenstein only mentions the basic idea that the answer to the problem of clarifying what is *the criterion* for establishing *how* a rule is followed and applied should not be looked for in underlying mental processes, but in the fact that people are trained in such a way that when ordered to follow a given rule or apply a given formula

[4] Cf. Wittgenstein (2005, 76-84, pp. 354-399).

they always take the same step at the same point. It is therefore on the basis of our received education and learning that we follow certain rules, and the criterion for establishing how the rule is applied lies in the way in which it was taught and how we learnt the technique for using it (“Proto-Investigations”, §168; *Investigations*, §§189-90).

At this point of the dialogue, Wittgenstein’s reflections on *Regelfolgen* appear to encounter a setback. From §170 of the “Proto-Investigations”, his analysis of understanding and following-a-rule breaks off and his attention turns to issues of an apparently different nature, concerning the inexorability of counting and calculating, the difference between calculation and experiment, the procedures for inferring and proofing, logical compulsion and the role of invention in mathematics. The impression that the argument has been abruptly changed appears to be confirmed by the fact that in the later versions of the *Investigations*, composed from 1944 onwards, Wittgenstein deleted the sections devoted in the “Proto-Investigations” to the philosophy of mathematics, replacing them with his more recent reflections on *Regelfolgen* at the point at which he had broken them off.[5] But on closer inspection, the impression is deceptive and the brusque transition to other subjects only apparent. In tackling the mathematical themes indicated above, Wittgenstein actually completes his analysis of the constructs representing language-games whose propositions express rule-governed activities.[6]

We can say that the “Proto-Investigations” give a germinal idea of the grand design pursued by Wittgenstein of a systematic inquiry into language-games in which sentences occur whose meaning is not given by reference to physical or mental states but depends on the rules of the grammar of our language. The inquiry starts from the analysis of the propositions we call “normative”, including, according to Wittgenstein,

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[5] See in Wittgenstein (2002) the intermediate version (“Zwischenfassung”), §190 and the final version (“Spätfassung“), §191. In these two versions the treatment of *Regelfolgen* is followed by a series of remarks on the critique of private language which is incorporated into the theme of the normative contexts – insofar as, in my opinion, it should be read as a detailed objection to those who claim that language can be built on entirely subjective and therefore “private” bases. This would contradict the assumption Wittgenstein has just made that «what people say» does not depend on the agreement of individual opinions but on shared practices and interpersonal conventions, in other words on what Wittgenstein comprises under the concept of «form of life» (*Investigations*, §241).

[6] An extensive and well-documented treatment of the theme of rules is contained in volume II of Baker & Hacker’s commentary (1985).

both sentences containing verbs such as "can", "to be able", "to understand", "to follow-a-rule", and mathematical propositions. But in the subsequent versions of the *Investigations* the field of inquiry gradually widens. For example, the 1944 intermediate version includes the analysis of language of sense data, thoughts and representations, and the 1945-46 late version extends to the multifarious forms of intentional contexts. The mere fact that in the "Proto-Investigations" he chose an arithmetical example such as the succession of natural numbers to clarify the rule-following procedure shows us the link between the two themes and clarifies Wittgenstein's original idea that they should both be addressed in the same work. This is testified to by the inclusion, in the two Prefaces to the *Investigations* of 1938 and 1945,[7] of the "foundations of mathematics" in the list of subjects that will be explored in the work. Moreover, the very fact that Wittgenstein did not complete the last version of the *Investigations* supports the hypothesis that he may not have abandoned his original idea and that, if he had had time, he would have completed the treatment of the variety of language-games and added to that of intentional contexts, contained in the last one hundred and seventy sections or so of the *Investigations*, the analysis of mathematical propositions.

2. Mathematical propositions as rules

The manuscripts of 1943-44, the bulk of which are included in Parts VI and VII of the *Remarks*, offer a mature exposition of what I have called the grafting of the philosophy of mathematics onto the trunk of *Regelfolgen*. These contain *in nuce* the arguments that will be developed in §§185-242 of the *Investigations* and illuminate a variety of aspects. In particular, Wittgenstein highlights the typical properties that mathematical concepts and procedures share with all normative contexts, conferring on them a status that preserves them from the conceptual confusion stemming from traditional views. The first of these properties is that according to which mathematical propositions do not have a descriptive status:

> There is no doubt at all – he states – that in certain language-games mathematical propositions play the part of rules of description, as opposed to descriptive propositions (*Remarks* VII, §6a).

The distinction between rules and descriptions is in fact one of the most

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[7] Cf. Wittgenstein (2002, pp. 207-9; 565-68).

important consequences of Wittgenstein’s critique, from the transition writings onwards, of the denominative theories of meaning, exemplified by the so-called Augustinian conception of language and which in the *Remarks* is extended to include the mathematical Platonism inherent in the logic of Frege-Russell. The circumstance that within this logic mathematics is regarded as «the natural history of mathematical objects», which «introduces us to the mysteries of the mathematical world» is properly «the aspect against which – Wittgenstein says – I want to give a warning» (II, §40). Even the very *mysteriousness* of a reference to such a world suggests the comparison of mathematics with alchemy:

> It is already mathematical alchemy, that mathematical propositions are regarded as statements about mathematical objects, — and mathematics as the exploration of these objects? […] All that I can do, is to shew an easy escape from this obscurity and this glitter of the concepts » (V, §16 b, e).

Wittgenstein’s critique begins with the rejection of the logistic ideal of founding mathematics on logical objects and establishes itself as an attempt to replace the task of a foundation of mathematical propositions with a «clarification of their grammar» (VII, §16a). Unlike propositions that describe objects, be they physical, psychological or abstract, mathematical propositions are not genuine propositions with an objective reference point but derive their meaning from the system of rules to which they belong; they do not have a verifiable cognitive status, and are neither true nor false but are the fruit of grammatical stipulations that do not obey the truth-functional logic but have a normative status, of a conventional nature, insofar as they are expressions of rule-governed activities.

In this sense, the attribution of a non-descriptive status to mathematical propositions appears to exhibit a special affinity in Wittgenstein’s arguments against mathematical Platonism with formalistic concepts of mathematics and in particular, as F. Mühlhölzer points out (2008; 2010, pp. 29ff, 72ff), with the mathematician Johannes Thomae, whom Frege had criticized in the *Grundgesetze der Arithmetik* (II, pp. 98ff). According to Thomae, mathematical calculations do not describe anything but are pure games of marks on paper, manipulations of symbols, devoid of any referential meaning. Wittgenstein’s normative conception of mathematics, however, differs substantially from the formalistic theory since mathematical constructions, on a par with codes, rules of law, rules of calculation, and, as we shall see, mathematical proofs, cannot be reduced to a mere manipulation of signs but have practical needs and ends and consist in the possibility of their application – a circumstance that is not contemplated in formalistic theories and for which Wittgenstein also criticises Russell for not having taken it into sufficient consideration (*Remarks* III, §29d). Despite having no connection with facts, which is by contrast the case for descriptive propositions, normative constructs have a pragmatic connotation because they refer to actions and therefore fall into the domain of the phenomena of doing (*Phänomene des Tuns*). We can therefore conclude that the pragmatic or applicative dimension of normative contexts enables us to clarify that mathematical propositions can in no sense be assimilated to mere manipulations of symbols in the style of formalistic conceptions. In this way, Wittgenstein recovers a constructive role for mathematics that links it to the creation of shared customs, to consolidated practices and techniques based on the learning and use of mathematical operations.

The pragmatic and applicative dimension that Wittgenstein attributes to normative contexts, and therefore also to mathematical propositions, finds further confirmation in the idea that all rule-guided activities can be assimilated to "creative acts" (*schöpferische Akten*) and that the most important mathematical procedures, such as proofs, are aimed at the creation of ever «new» techniques of proof. It is obvious that this Wittgensteinian conception whereby «mathematics is a motley of techniques of proof. – And upon this is based its manifold applicability and its importance» (III, §46a) shows his sharp opposition to the foundationalist theories that reduce mathematics to the unique and exclusive model of logic.[8] The well-known pronouncements of the *Remarks* lead back to this creative aspect, according to which mathematical concepts and procedures, lacking any referential meaning, do not belong to the context of discovery (*Entdeckung*) but rather to that of invention (*Erfindung*) (I, §168, Appendix II, §2; II, §38; V, §11f).

If it is true that the applicative dimension of normative contexts confers on them the dignity of procedures that are different from mere manipulations of signs with no objective content, this does not, however, seem a sufficient requisite to protect the language-games in which these contexts recur from the intrusion of subjectivist elements that could expose the connection between the expression of a rule and its application to sceptical doubt; in the case in point between a mathematical construct and the result of calculations and measurements (*Investigations*, §242). Doubt is thereby cast on the logically necessary nature of the connection, which marks «an important part of our life's activities», and by virtue of which

[8] Cf. Mühlhölzer (2010, pp. 309ff).