Zetetics and the Art of Mathematical Enquiry

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By

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For Conny ...

Der Lehrer soll die Wissenschaft vor den Augen des Schülers entstehen lassen. Wie sie sich in dem Geiste des gereiften Denkers aus den ihm inwohnenden Grundvorstellungen entwickelt und gestaltet, so soll er sie, nur für die jugendliche Fassungskraft eingerichtet, darstellen und als ein organisch sich bildendes Produkt der Vernunftthätigkeit mittheilen. An seinem Verfahren soll der Schüler mathematisch denken lernen.

The teacher shall allow science to develop before the eyes of the students. Just as it develops out of the teacher's inherent basic ideas and takes form in the mind of the mature thinker, so shall the teacher represent it, but furnishing it for the youthful capacity to understand, and communicating its rationality as an organically formed product. It is in this way that students learn mathematics.

(Weierstrass, 1894, 329.)

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zetetick, *a*. [from ζητεω.] Proceeding by enquiry.

(Johnson 1756.)

CONTENTS

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ACKNOWLEDGEMENTS

I acknowledge the inspiration from Dr John Loxton to read, as far as possible, the originals. I remember well the first book that he recommended, and which I still treasure: *Disquisitiones arithmeticae* by Carl Friedrich Gauß (1777–1855) (Gauß 1801). John also exhorted me to "always enjoy mathematics," and I trust that this book shows that I have fulfilled this, or that I have filled this to the fullest.

I acknowledge the wonderful work that my student Minki Kim has done to prepare many of the digital images that appear in this book. Without his help, the manuscript would lack character and colour.

I acknowledge the many students who have entered into dialogue and debate with me about mathematics education, and about mathematical concepts and how they are best developed, and who have encouraged me to make my approach to learning and teaching available to a wider audience. They transformed my transactional model for learning and teaching into a transformational model (cf. Merrotsy 2017), for which I am very grateful. They include Ethan Acott, Luke Bayens, Oscar Burke, Xiying Cheng, Nam Khuan Chua, Emma Davies, Dr Elizabeth Feutrill, Antonino Giglia, James Harris, Dr Karen Knight, Michael Loh, Xin Yan Loh, Rodney MacGregor, Alexandria McLeod, Dr Simon Moore, Steven New, Dr Duy Nguyen, Michael Richards, Rajiv Ramaprasad, Rizwan Sait, Robin Shaw, Dr Robert Sisson, Gareth Stephens, Dr Mei Yi Tan, Harold Walden, Laura Wiesenekker, Dr Rob Woodward, and Cheng Zhi Yuan.

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PROLEGOMENA

> Quanquam veteres duplicens tantum proposuerant analyticem, constitui tamen etiam tertiam speciem, qua dicatur $\zeta'\eta\tau\iota\kappa\eta$ η $\varepsilon\xi\eta\gamma\eta\tau\iota\kappa\eta$, consentaneum est. Ut sit, zetetice, qua inuenitur æqualitas. Poristice, qua ueritas examinatur. Exegetice, qua magnitude exibetur.
>
> While the ancients proposed only a double analytic, I have established also a third kind, which, it may be said, is in accordance with zetetics and exegesis. Zetetics is a way to find an equivalence. Poristics examines its truth. Exegesis reveals its magnitude.
>
> (Viète 1591, Caput I, 4, my translation).

Zetetics and the art of mathematical enquiry.

In choosing this as the title for the book, I would like to say that I am a mathematician seeking understanding. With apologies to St. Anselm of Canterbury: *Mathematicus quaerens intellectum* (cf. Anselm 1940 [1078]). The title neatly encapsulates my quest. The Greek word ζητέω (zetéo) means to seek, to enquire, to search, to investigate, to examine, to desire, and to question, especially of a philosophical nature (Liddell and Scott 1968).

The word zetetics first appeared in the European lexicon in Viète's (1591) *Zeteticorum* to describe a process of solving mathematical problems using pronumerals, that is, as a development from rhetorical and syncopated algebra towards a form of modern symbolic algebra. The French term *zététique* is quite poetic.

In English, Johnson (1756) defined zetetics to be "proceeding by enquiry." This is still the Oxford English Dictionary's meaning for the term, which also notes that the verb to enquire is used especially in the sense "to ask a question" (Onions 1978).

And as Pólya (1963b, 606) states, "Teaching is not a science, but an art."

Mathematics has a rich history from cultures around the world, which can extend and enrich the appreciation and learning of mathematical concepts for both young and old alike. This book aims to provide inspiration for mathematics educators, and extension and enrichment for able students of mathematics, through an exploration of mathematical concepts from the perspective of their cultural and historical development. Others with an interest in mathematics will also enjoy reading it.

The book presents several new mathematical "discoveries and inventions," and offers a re-interpretation of traditional approaches to a range of mathematical problems. Along the way, it provides new and at times unusual perspectives on the cultural context and historical development of mathematical concepts and topics, and does so in a rigorous way. Each chapter uses historical sources to introduce a mathematical concept and to inform the exploration that follows. And each chapter contains original material and new approaches to proving and understanding mathematical results.

To make the most of what the book has to offer, I recommend active engagement with the mathematics by building, buying or begging for copies of the concrete material that I have used. Children, adolescents and adults alike learn best by developing knowledge and skills through concrete, hands-on learning experiences. The cultural enrichment, the historical background, the replicas of cultural and historical artefacts, and the myriad manipulatives, all combine to provide a robust foundation for how best to learn mathematics, and to model for the mathematics educator an engaging framework for the teaching of mathematics.

Topics explored in the book include numeracy, the abacus, Mesopotamian mathematics, music and mathematics, public-key cryptography, Pythagoras' theorem, the holistic nature of trigonometry, anthyphairesis, and an introduction to integral calculus by discovering π, and determining the area of a circle and the volume of a sphere. Throughout, I have reflected my enthusiastic style of teaching and my entertaining approach to learning and teaching mathematics, which highlight active engagement with significant mathematical problems and hands-on modelling to build deep understanding of the concepts.

Before proceeding further, I must provide a trigger warning:

ἀγεωμέτρητος μηδείς εἰσίτω

Despisers of mathematics may not enter.

This inscription was purportedly written above the entrance to Plato's Academy, 387 BCE, but it was certainly included on the frontispiece of Copernicus (1543, my translation). Also written on the frontispiece may be found, *Igitur eme, lege, fruere* (Therefore buy, read, enjoy).

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SECTION I

LESSONS FROM HISTORY

On method

My philosophical framework for learning and teaching is informed and influenced by several education theorists and mathematics educators. From Johann Heinrich Pestalozzi (1746–1827), for example, comes the holistic notion of learning by the head, the heart, and the hands (Pestalozzi 1787). Again, from John Dewey (1859–1952) comes the model that learning is transformational (what I would term a transformation of perspective and of the mind, or *metanoia*), and actively developed by doing and modelling through real-world, problem-based learning, discovery learning (the "project method"), and enquiry predicated on community. Learning is learning how to think (Dewey 1933). And from Jerome Bruner (1915–2016) comes a host of pedagogical concepts, including active learning, discovery learning, the spiral curriculum, optimal structure, scaffolding, and enactive, iconic, and symbolic modes of representation (Bruner 1961, 1966).

My own predilections strongly suggest that cultural and historical perspectives of mathematics are important for mathematics learning and teaching, about which I make three assumptions.

First, cultural and historical perspectives of mathematics are important in and of themselves. So too are the mathematicians who have discovered, invented and constructed mathematical concepts and developed mathematics as a discipline.

Second, cultural, historical and biographical perspectives may be used to add depth to our understanding of mathematical concepts.

And third, cultural, historical and biographical perspectives can provide engaging and enriching learning experiences, and make mathematics a vibrant, living and relevant subject.

There are many ways in which cultural and historical perspectives of mathematics may be explored. Here are six approaches.

1. Follow news and current affairs. Be sceptical, and delve more deeply into news that is of interest to you. For example, a recent newspaper article claimed an early date for the writing of the Bakhshali manuscript. The mathematics recorded on the birch bark is interesting in itself. But does the manuscript contain the first appearance of zero as a number (as opposed to a place marker)? Again, another recent newspaper article claimed the discovery that the Babylonian clay tablet Plimpton 322 presents "exact sexagesimal trigonometry" (Mansfield and Wildberger 2017). For the details and for the debate, see Chapter Eight.

2. Regularly read journal articles. Be sceptical, and delve more deeply into articles that are of interest to you. For example, McCaffery and Weisberg (2017) revisited the story of John Snow and the waterborne spread of cholera. Like most commentators on John Snow, the authors accepted the received wisdom that Snow's work was momentous and seminal. Yet some more careful backtracking (I searched for earlier maps) would bring to the surface the work of Thomas Shapter (1809–1902), who recorded and mapped the cholera epidemic in Exeter in 1832, and who did so in a way that foreshadowed the techniques used by Snow. To whet your appetite, perhaps Chadwick, Brigham, Seaman, and Lancisi (amongst others) would provide even more backstory ... .

3. Seek out specific "snippets" of information. For example, why is h commonly used as a pronumeral for increment in differential calculus? It appears that Sylvestre François Lacroix (1765–1843) was the first to do so: in the first volume of Lacroix (1797) k was used and by the third volume (1800) this had changed to h, presumably in both cases to avoid confusion and conflicts of notation. After the translation of Lacroix's treatise into

English, *h* was popularised by Martin Ohm (1792–1872), the younger brother of Georg Ohm (1789–1854).

4. Pursue specific topics that are of interest to you. For example, the history of the trebuchet and the study of projectile motion both say much about the inter-relationship between mathematics and its applications, as well as about economic and cultural exchange (and appropriation). From the 5th century BCE Asian invention of the crouching tiger trebuchet, the war machine developed into the Roman mangonel, the Greek ballista, the Roman onager, the Greek cheiromangana, the Chinese whirlwind trebuchet, and the French couillard (Rossi, Russo and Russo 2009). Sixteenth century illustrations of projectile motion, such as those by Niccolò Fontana (~1499–1557), known as Tartaglia (Italian = the stutterer), and Thomas Harriot (1560–1621) provide a fascinating backdrop to Galileo Galilei's (1564–1642) demonstration of parabolic motion (Galilei 1638; cf. Pólya 1963a).

5. Explore the lives and the times of your "heroes". For example, my mathematical hero is Ramanujan (1887–1920), and on the occasion of the centenary of his death I wrote an article about his life, his times, and his mathematical creativity (Merrotsy 2020), which makes compelling reading.

6. Find and treasure ethically responsible repositories of source material. None of the sources that appear in the References has been cited via a secondary source. In each case I have accessed and read the original, in some form or other. Most are accessible on the Internet; many are free to download. Those not accessible on the Internet either include books published during the past 100 years, or are facsimile copies of older books and manuscripts, and these are part of my own library.

I recommend the following websites as good places to start looking for sources of historical publications and documents: Digital Mathematics Library, Internet Archive, University of Adelaide ebooks, Apollo, Early Modern Texts, Science Direct, B-ok, and Google Books.

On manipulatives

Let us here return to Bruner's notion that cognitive development occurs from learned action patterns, to images that stand for events, to a symbol system, that is, in his terms, through the transitions from enactive to iconic to symbolic modes of representation (Bruner, 1966). The fundamental idea of course is well known through the work of constructivists such as Jean William Fritz Piaget (1896–1980) and Lev Semyonovich Vygotsky (1896–1934) and their disciples, although its roots should perhaps better be seen in Charles Sanders Peirce's (1839–1914) pragmatic theory of signs (signification, representation, reference, and meaning: Peirce 1977).

Rather than this process of cognitive development being developmental in an age-related sense, John Pegg and David Tall interpreted the theoretical and research models to reveal a fundamental cycle that underlies the construction of any mathematical concept. Their saw-tooth model of cyclical long-term cognitive growth applies when each new mathematical concept is met (Pegg and Tall, 2005). While they do not claim that this is the only way in which people learn, they do conclude that interacting with and manipulating concrete, hands-on objects, and seeing these objects from multiple perspectives, are very important for constructing mathematical concepts. The concrete manipulative builds manipulable concepts or mathematical "objects" that are expressed in iconic and symbolic ways.

Another very good reason for creating and using manipulatives is that they can (and should) reduce cognitive load (Sweller, Ayres and Kalyuga 2011). One neat way to express this is to say that concrete mathematical objects are used by us to think with: they scaffold thinking; they drive forward cognition; and they are used by us to communicate. In this vein, Graeme Wilson (2018, 141) poses the question, "Where does the mind stop and the rest of the world begin?" Using the game of chess as an example, Wilson suggests that the relationship between the mind and the chess board and chess pieces is quite complex. While the game does take place in the mind, the pieces and the board are tools to think with: they are used as a form of documentation to record the progress of the game; and the movement of the pieces is part of the player's cognitive engagement with the world. "The movement and interaction with objects provides [*sic*] ways for humans to think and act that *could not otherwise be achieved*" (Wilson, 2018, 31, emphasis in original).

This understanding of manipulative is analogous, in a certain sense, to the Socratic method. In the dialogue Theaetetus (Plato 1921 [360 BCE]), Socrates' approach to learning and teaching is described by Plato as maieutics (μαιευτικός, maieutikós, classical Greek = obstetric). Similarly, manipulatives are an obstetric aid.

As you read the book, I highly recommend that one way or another you "get your hands on" the concrete material, get your hands dirty, and manipulate these objects as you proceed with each enquiry. Some are available commercially (for example, centicubes, Section II); or easily replicated (for example, counting boards, Chapter One); or a bit fiddly to replicate but the time and effort are well rewarded (for example, 3D "prints" of the Liu Hui solids, Chapter Fourteen); or would find a worthy place on your coffee table (for example, YBC 7289, see Chapter Three).

In any case, manipulatives are simply a fun way to engage with the mathematical concepts, and to see the concepts from different perspectives.

On modelling

Learning is an active process, and as Dewey (1933) points out, active learning means learning by doing and modelling. Albert Bandura (1925–2021) has also highlighted the central role of modelling in learning, enacted in vicarious experience, and symbolic and self-regulatory processes (Bandura 1986). Modelling here has several layers of meaning. People learn from others, through observation of modelled attitudes and modelled behaviour, as well as observation of the outcome of this behaviour. People model by imitating, replicating, practising, and reinforcing modelled behaviour, and they do so first in a group and then individually. And people model what has been learnt to others.

In order to model how an enquiry might proceed, I take a topic suggested to me by a student whose career before mathematics education was as a brewer: beer, and whisky. If you have nothing to do with either or both of these, then the beer can be replaced by non-alcoholic ginger beer, and carbonated lemonade and ice-cream, and the whisky by perfume, and lemon juice preserved with sugar and vitamin C. The following ideas worthy of exploration (but explored in moderation) arose in our conversations.

1. Fermentation of cereals to make beer appears to have occurred at about the same time as the domestication of cereals, that is, towards the end of the last ice-age about 12,000 years ago when cultures were developing from hunter-and-gathering to agriculturally based societies. Distillation of alcoholic drinks to increase the concentration of alcohol appears to have been discovered towards the end of the bronze age, that is, about 3,200 years ago. Both processes are found in many cultures around the world. The related history provides insight into scientific, technological and economic development of cultures, and the exchange of knowledge between cultures.

2. Models of fermentation and distillation provide unusual data for analysis. Factors for consideration include water content, nitrogen content, temperature, starch content, sugar content, method of warming (cf. peat), and length of time for each stage. The instruments invented to measure the various factors are also worth investigating and replicating (for example, a hydrometer). The various units of measurement tell their own story (for example, those for specific gravity). Data (for example, related to health and taxes) are important.

3. The chemical equation for the conversion of sugar into alcohol is

$$C_6H_{12}O_6 + \text{Zymase} \rightarrow 2\,C_2H_5OH + 2\,CO_2$$

where zymase is an enzyme used by the yeast to digest the sugar. It is very important to know the difference between ethanol and methanol, and how to quickly determine which is which.

4. There are many mathematical equations related to brewing and fermenting. For example, a formula for alcohol content is

$$\mathrm{ABV} \approx {}^{(\mathrm{OG} - \mathrm{FG})}/_{7.46} + 0.5$$

This formula is important (for several reasons) because it calculates the approximate alcohol content (ABV) of my home brewed beer, given the original gravity (OG) and the final gravity (FG) of the brew.

5. The Apollonian gasket (named after Apollonius of Perga, ~240 to ~190 BCE) is a fractal that provides a good model for beer foam, froth or head. The collapse of the beer foam is well modelled by exponential decay. To investigate beer foam collapse, you could well experiment using several different kinds of beer (lager, ale, stout) poured at a range of temperatures, and replicate the experiment many times (over a long period of time).

6. Any measurements in your investigations are prone to errors. Galileo first noted that errors of measurement are symmetrical about a mean (his illustrations measuring the movement of the moons of Jupiter are accessible online). de Moivre noted that the distribution of errors of measurement is binomial, and he showed that this approaches an exponential distribution. Gauß determined the constant involved.

7. Talking of statistics and errors of measurement, Student (well known for the Student t test) researched ways to improve the quality of beer and whisky by improving each step of the production process. His work had ramifications far beyond brewing and distilling: as well as improving the quality of barley, his work led to wide-ranging changes to the farming of barley and to agriculture in general. His real-life name was William Sealy Gosset (1876–1937), and he published his statistical research under the name Student so that other brewers would not realise the nature of his research for Guinness brewery in Dublin, Ireland.

8. Imagine a bottle of whisky, and consider the unconscionable event of a small amount being removed and replaced by water. If this was to occur many, many times, what would happen to the concentration of alcohol in the whisky? In theory, this may be expressed mathematically as

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = \frac{1}{e} \approx 0.37$$

In theory, there is no difference between theory and practice. In practice, ...

9. Finally, the size and shape of packaging are both mathematical and psychological, both aspects of which are worthy of investigation.

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CHAPTER ONE

THE ABACUS: COUNTING BOARDS AND ABACI

Fig. 1-1: Etruscan cameo, or abacus gem, 12.5 mm × 16.0 mm. Bibliothèque nationale de France, Paris, CII 2578 ter. [Image: BnF, with permission.]

Introduction

To what extent should hand calculators be used to assist us with our arithmetic, calculations, and problem solving? I will outline my thoughts on this vexed question in a round-about way by drawing a hand-waving picture of the 5,000-year development of calculating devices up to the 17th century.

For the first part of my story, I rely on Ifrah (1998), Lang (1957), Lewis and Short (1975), and Liddell and Scott (1968).

The story really does begin much earlier than 5,000 years ago. The use of body parts for counting is clearly ancient, and remarkable examples can be found in cultures around the world (my favourite is the Base 47 Foe counting system from Lake Kutubu in the New Guinea Southern Highlands). There are examples of marks on bone used as an aid for counting, such as a lunar calendar dated at approximately 32,000 years ago, and the Ishango bone dated at approximately 20,000 years ago. The use of pebbles to aid counting must certainly be at least as ancient as these examples, and some cultures have continued to use this method in the ethnographic present. In Babylon, and several other cultures, these pebbles were made from soft stone or baked clay, and some included symbols to denote various meanings or numerical values. The formal word for these pebbles is calculi (of course, much later the singular *calculus* takes on another well-known meaning), from the Latin word *calx*, meaning pebble, small stone, gravel stone, limestone, kidney stone, and even counter (in the sense of a counter used in draughts or other games).

My narrative of this story begins shortly after the invention of writing approximately 3,400 BCE, and at a time when arithmetical calculations were needed to determine the value of tributes and the amount of tax payable to a ruler, and to plan engineering projects and large public works. This cultural shift appears to have begun about 3,000 BCE, or about 5,000 years ago. The sand or dust, on which calculations were made outside in the marketplace, was brought inside the treasury and placed on a table or in a thin box, both called a sand board. Most commentators suggest that this shift first occurred in Mesopotamia, where "The hand" calculating board had four or five sexagesimal levels, but very little is really known about the computational algorithms used.

Use of the sand board then spread to other administrative and scientific centres. Perhaps first, it appeared in Vedic India, where medieval Sanskrit texts refer to *dhūlīkarman* (dust computation) carried out on a *pāṭīgaṇita* (computation board); hints at the computational strategies used were orally transmitted in texts such as the *Śulbasūtra* (cf. Chapter Two).

And then, perhaps second, references to the sand board were recorded in classical Greek literature. The name for the Greek sand board appears to have been derived from a semitic word, perhaps Phoenician *abq* meaning sand, or perhaps Hebrew אבק *abaq* meaning dust, which in biblical Hebrew not only meant to wrestle (that is, to get dusty) but also to drift away (as in very fine dust), and which later in medieval Hebrew meant sand. In any case, the root word seems to have morphed into the Greek word ἄβαξ *abax* (and, much later, resurfaced in the medieval Arabic word *gubar* meaning dust, and in the Arabic term *gubar* numerals). Here, ἄβαξ *abax* refers to a slab or board, sprinkled with sand or dust, that is used for carrying out calculations and for drawing geometrical diagrams.

Of course, the sand board was not the only medium used for writing, and other media evolved. Papyrus was invented also circa 3,000 BCE, and the best-known mathematical examples that have survived the ravages of time include the Rhind Papyrus and the Moscow Papyrus, but the Lahun or Kahun Papyrus is also worth getting to know (it concerns Egyptian fractions, and was written approximately 1825 BCE). Stone was used by several cultures for formal documents, perhaps the most famous being the Code of Law of Hammurapi, engraved circa 1,750 BCE. Interestingly, slate was used in Roman times for building, but appears to have been used for writing only much later, perhaps from the 14th century CE, with a wet sponge used as an eraser. The great Indian mathematician Ramanujan (1887–1920) is renowned for using a slate to produce many of his brilliant results, and to use his elbow as the eraser. However, a piece of stone or timber covered in wax was used as a wax tablet for writing and calculations from about 1,400 BCE, until, surprisingly, the 19th century CE. The wax tablet was accompanied by an iron or bronze stylus, pointed for writing at one end, and flattened like a spatula for erasing at the other end. The wax could also be melted at a low temperature and the wax surface smoothed by the stylus to create a "clean tablet" or *tabula rasa*.

Even with these technological improvements, the sand board was used by many cultures continually from about 3,000 BCE until the 13th century CE, and perhaps even until a wee bit later. But along the way, and perhaps quickly, the sand board *abq*, *abaq* or *abax* also metamorphised into a board-without-sand, or a counting board: the sand disappeared from the sand board, to be replaced by lines and counters, but the name *abacus* remained in its various forms.

The earliest record of a counting board seems to be in Greek literature from the 5th century BCE, in works by Aeschylus (~523 BCE to ~456 BCE), and Herodotus (~484 to ~425 BCE). The Greek term for the counting board was still ἄβαξ *abax*, along with its diminutive ἀβάκιον *abakion* (although this term from Lysias circa 400 BCE was not commonly used, perhaps because it is similar to the word ἀβάκής *abakés*, meaning speechless). On the counting board, the person who calculated (λογιστής *logistés*) was able to count, reckon, or calculate, to carry out a numerical calculation and to audit the accounts of another person (λογίζομαι *logizomai*). It should be no surprise that there is a related word λογεία *logeia* meaning the collection of taxes. The computation and reckoning (λόγου *logou*) and counting and calculation (λογισμός *logismos*) were carried out using pebbles (ψῆφοι *psephoi*, singular ψῆφος *psephos*) that were used specifically for reckoning or calculating, to arrive at a sum, sum total, or total amount (κεφάλαιος *kephalaios*).

In his public oration *On the crown* delivered in Athens in 330 BCE, Demosthenes (384–322 BCE) used the analogy of comparing the public enquiry into the war with Philip II of Macedon with the examination of a balance-sheet in order to convince his audience (his "auditors" or "accountants") to accept the conclusion of his argument (to logically reckon) by calculating as with counters until they are all cancelled out (the sum is completed) and balanced (Demosthenes 1903 [330 BCE], Oration 18, §§ 227, 229, 231). Here, the language used is decidedly that of the counting-board-with-counters and not of the sand-board-with-actual-sand. Similarly, Polybius (~200 to ~118 BCE; 1904, §5:26) also made it clear that these terms are not referring to the sand board but to the counting board by using an analogy comparing people to the ψῆφοι *psephoi* on an ἀβάκιον *abakion*. According to where the pebbles or counters are placed on the board, their position may signify either a talent (τάλαντον *talanton*, the gold equivalent of the value of an ox or a cow), or a chalcus (χαλκός, a copper coin of low value). Interestingly, this analogy is also attributed to Solon (~630 to ~560 BCE).

In an amusing story, the long-lived comic poet Alexis (~375 to ~275 BCE) related the splitting of a bill for a shared meal. One of a group of friends undertook to organise a dinner, and the wine for drinking afterwards, while the others engaged to pay their shares of the cost after the dinner. A slave used an ἀβάκιον *abakion* and ψῆφον *psephon* (meaning the plural ψῆφοι *psephoi*) to do the calculation. Boiled cabbage was recommended as a remedy against a hangover. (Arnott, 1997, 88, 'Απεγλαυκωμενος, Fragment 15.) In a similar story, a chef who had arranged a meal for a group of people had to calculate the individual bills (λογισασθαι, to reckon up), for which he could have used ψηφοι *psephoi*, with or without a counting board, or he could have used his fingers (Arnott 1997, 560).

The development of the abacus in Greece was mirrored at about the same time in Roman culture. At first, in Latin the term for ἄβαξ was *abax*, which then became Latinised as the word *abacus*. Initially, and similar to the Greek story, *abacus* meant a square tablet covered in sand or dust, and used for arithmetic and for mathematical calculations. When the sand on the tablet was replaced by a permanent grid, the tablet retained the name *abacus*. Over the centuries, the word *abacus* also assumed other meanings: a counting table; a reckoning-board for counting votes; a piece of dining room furniture also known as a sideboard; a dice-board or gaming board divided into compartments for playing with counters or dice; a wooden tray (Cato the Elder, ~200 BCE); a painted panel in the wall or ceiling; and a flat square slab of stone on top of column (Vitruvius, ~30 BCE).

The sand board was clearly in the form of a tray (in order to hold the sand or dust). The abacus in the form of a counting board was flat and made from a slab of stone or timber, or made from cloth: lines were used to denote place value; and unmarked pebbles or counters (ψῆφοι *psephoi* or *calculi*) were placed between lines (and later, in medieval times, placed on lines) to denote numerals. That is, the abacus was a counting board, although not a very portable one, and as the need for a portable calculating aid grew (perhaps, for example, from outdoor surveying and quantity surveying work), a hand-held device was invented (cf. The Roman Abacus, below).

There are many variations on the abacus. In the following sections, I will provide a range of examples of both kinds of abacus: six examples of "counting boards", and four examples of "abaci". For the details of the various models that follow, I rely on Barnard (1916), Cullen (2004), Ifrah (1998), Kangshen, Crossley and Lun (1999), Lam and Ang (2004), Keyser (1988), Menninger (1969), Russian State Technological University (2006), Volkov and Freiman (2018), and Yoke (1985) as my main sources.

"Counting boards"

The Greek counting board

Although mentioned in literature written about 500 BCE, the earliest dated extant example of a Greek counting board is the Salamis Tablet dated from about 300 BCE. It was uncovered in 1846 on the island of Salamis near Athens, and is made from a slab of marble measuring approximately 149 cm × 75 cm × 4·5 cm. Counters in the form of small flat pebbles would have been placed on the tablet to represent numerals.

In the image of my replica of the Salamis Tablet (Fig. 1-2), you will see two groups of parallel lines, one with 11 parallel lines, and the other with 5 parallel lines, and each of these groups is bisected by a perpendicular line. The smaller group is for fractions, and the larger group is for whole numbers: indeed, the semicircular lines are radix characters, representing a shift from integer parts of a number to fractional parts of a number. You will also see three similar groups of Greek (acrophonic Attic) numerical symbols representing the place value assigned to the columns (see Table 1-1). The base for the place value system is quasi-Base 10, accompanied by kind-of-sort-of Base 5 and Base 6 systems; that is, it is a bi-quinary or dual-quinary system superimposed on a senary (or heximal, or seximal) system.

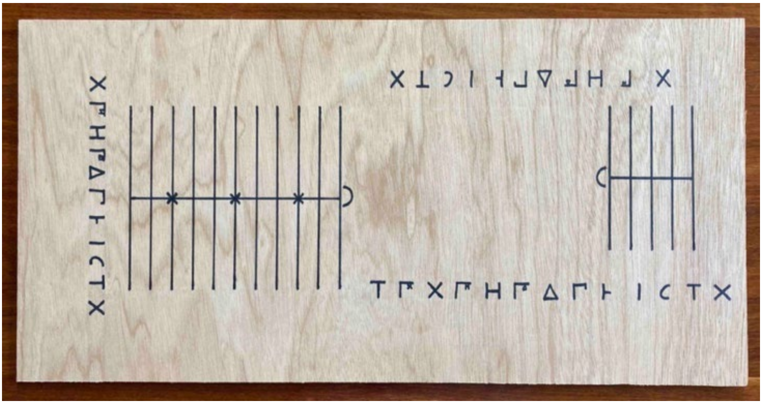

Fig. 1-2: Replica of Salamis Tablet on timber slab. [Image: Peter.]

Table 1-1: Classical Greek monetary numerical symbols

Numeral	**Meaning**	**Value**
Τ	Talanton	1 talent = 6,000 drachma
𐅇	Pentachilioi	5 chilioi = 5,000 drachma
Χ	Chilioi	1,000 drachma
𐅆	Pentahekaton	5 hekaton = 500 drachma
Η	Hekaton	100 drachma
𐅅	Pentadeka	50 drachma
Δ	Deka	10 drachma
𐅃	Penta	5 drachma
⊢	[Symbol for 1]	1 drachma
I	Obol	$^1/_6$ drachma
C	[Half of O]	$^1/_2$ obol
T	Tetartemorion	$^1/_4$ obol
X	Chalcus	$^1/_8$ obol

There is debate about how numbers would be represented on a Greek counting board such as the Salamis Tablet. There are four suggestions.

1. Two numbers may be represented either side of the dividing lines before they are added or multiplied.

2. The dividing lines allow for positive numbers on one side, and the equivalent of negative numbers on the other (this could be not just for subtractions, but also for multiple representations of numbers, for example 9 = 10–1, similar to how some Roman numbers are represented).

3. Multiples of 5 are placed "above" the dividing lines (depending on your orientation), so that the place values for the whole numbers range from 1 drachma (in the first of the 10 columns, closest to the centre) to 10,000 talents (in the 10^{th} column, furthest from the centre).

4. The multiples of 5 each have their own separate column, so that the place values for the whole numbers range from 1 drachma (in the first of the 10 columns, closest to the centre) to 5 talents (in the 10^{th} column, furthest from the centre).

I would support this fourth suggestion, because 10,000 talents, the equivalent in gold of about 10,000 oxen, is a huge amount; and because there is no Μ for Murioi (tens of thousands) indicated amongst the numerals written on the Salamis Tablet. It is also possible that the place value of columns was determined by the nature of the sum involved. In any case, the Salamis Tablet was large enough for several people to stand around, one to carry out the calculation, and the others to audit and record the calculation.

The Roman counting board

There appears to be no extant example of a Roman counting board. They certainly did exist, as indicated clearly by the words used in the bountiful literature, mainly from the 1^{st} century BCE and the 1^{st} century CE, but the earliest reference I could find in Lewis and Short (1975) was by Cato the Elder (234–149 BCE), the first author to write history in Latin. In the Bibliothèque nationale de France, there is an Etruscan cameo clearly showing a scribe using a counting board to carry out a calculation (and possibly holding a wax tablet, clearly showing numerical symbols, in his right hand), and this image might date from before the decline of Etruscan culture following the Roman-Etruscan wars which ended about 350 BCE (see Fig. 1-1 at the head of this chapter).

Reconstructions of Roman counting boards assume that they are Base 10, but bi-quinary, and typically have eight parallel lines that form seven columns (see Fig. 1-3). The columns are divided into three sections by two perpendicular lines, and from right to left along the middle row place value is indicated by the numerals representing increasing powers of 10, from units to hundreds of thousands:

⋈̄ , (((|))) , ((|)) , (|) , C , **X** , **I** .

Splitting several of these numerals in half will also show the source of the numerals for 5 V, 50 ⩛ (V with vertical line, which later morphed into L), and 500 |) (which later became D). Multiples of five for each place value are placed in the top of its respective column: that is, five counters in the bottom section of a column are replaced by one counter in the top section of that column; two counters in the top section of a column are replaced by one counter in the bottom section of the next column to the left.

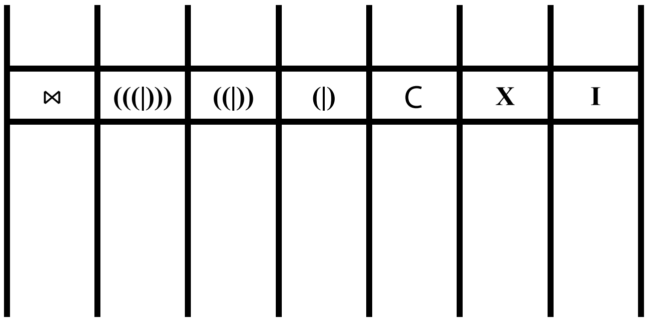

Fig. 1-3: A stylised depiction of a Roman counting board.

Chinese counting boards

An internet search for images related to early Chinese mathematics will yield very few if any results, and any images found will tend to be neither Chinese nor ancient. Yet great works such as *Zhoubi Suanjing* (Zhou Dynasty Canon of Gnomonic Computations, written and compiled between the 11^{th} and 3^{rd} centuries BCE), and *Jiuzhang suanshu* (*The nine chapters on the mathematical art*, written and compiled between the 10^{th} and 2^{nd} centuries BCE), attest to the wondrous and wonderfully rich and complex nature of Chinese mathematics. At first when Chinese counting boards were developed, often claimed to have been before Confucius (551–478 BCE) but probably from circa 400 BCE, counting rods made from bamboo, bone, ivory, metal, or jade were used to represent Chinese *chousuan* rod numerals in a Base 10 system.

The bamboo counting rods were about 7 cm to 15 cm long, and for several centuries were round before non-rolling bamboo rods with a triangular or square cross-section were introduced. Perhaps inspired by the *yin* and *yang* of Taoist philosophy, bamboo counting rods were marked either with a red dot, representing positive numbers (red is a symbol for good luck), or with a black dot, representing negative numbers (black is a symbol for bad luck), a remarkable development in the history of mathematics.

The main variation of a counting board using bamboo counting rods comprises a simple grid (in Japanese art from the 18th century, this grid is shown as a 5 × 5 array of squares). The grid may have been imagined; otherwise, it was painted onto a flat surface or onto a piece of cloth. *Chousuan* rod numerals are formed in two ways (see Table 1-2).

Table 1-2: *Zong* and *Heng* Chinese rod numerals.

Place value	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Zong Units, 100s, 10,000s, …	𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
Heng 10s, 1,000s, 100,000s, …	𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

A blank space on the counting board represents zero. After the year 690 CE, a circle was used to indicate the blank space, but at that time it did not have the meaning of zero as a number.

Variations of the traditional Chinese counting board appear in the literature from the 3rd Century CE. In some examples, the bamboo rods are replaced by counters or beads, which are placed in a *chu pan* bead tray. The grid on the bead tray could be formed by painted lines, or from raised pieces of timber creating compartments in which the beads are placed.

The prototype of the *chu pan* bead tray comprised a grid with ten rows; the number of columns would depend on the place values required for typical calculations that would be carried out on this version of the counting board. Numerical values from 1 to 9,999,999 can be represented, the placement of each counter indicating its numeral value and its place value. In the following stylised example (Fig. 1-4), the number 74,920 is represented.

A refinement of the *chu pan* bead tray comprised a grid with six rows; again, the number of columns would depend on the place values required for typical calculations that would be carried out on this version of the counting board. The difference here is that two different coloured counters are used, one to represent numerals from 1 to 4, and the other to represent numerals from 5 to 9 (see Fig. 1-5). The colour and placement and of each counter will indicate its numeral value and its place value. Note that there are several ways in which the rows of the grid may be organised. In the stylised example shown in Fig. 1-5, o is used for the numerals 1 to 4, and • is used for the numerals 5 to 9; the number 74,925 is represented.

	10^6	**10^5**	**10^4**	**10^3**	**10^2**	**10**	**1**
9					o		
8							
7			o				
6							
5							
4				o			
3							
2						o	
1							

Fig. 1-4: A stylised depiction of a *chu pan* bead tray, showing the number 74, 920.

	10^6	**10^5**	**10^4**	**10^3**	**10^2**	**10**	**1**
4				o	•		
3							
2			•			o	
1							
							•

Fig. 1-5: A stylised depiction of an alternative form of *chu pan* bead tray, showing the number 74, 920.

The counting board of Gerbert d'Aurillac

Gerbert of Aurillac (~945–1003 CE) was a French mathematician, teacher, philosopher, and theologian. He was elected Pope in 999 CE, taking the name Pope Sylvester (or Silvester) II. Several interesting if not gory myths surround his later life and his death. Gerbert's contribution to mathematics had the potential to bring about a paradigm shift in the western conception of number, how numbers are used, and how calculations are carried out. Following a visit to Spain, where he worked with Islamic scholars, he attempted to introduce the Hindu-Arabic numerals 1 to 9 to Europe north of the Iberian Peninsula. Sadly, zero was missing from Gerbert's repertoire, and so too was the Indian arithmetic that should accompany the Hindu-Arabic numerals. More sadly, and perhaps through fear of the unknown, or fear of supposedly pagan or heathen cultures from the "east", or unfailing belief and confidence in the superiority of the received wisdom from classical Roman culture and tradition, the significance of Gerbert's mysterious innovation was not appreciated, and it was actively resisted and decidedly not adopted.

The counting board or reckoning board of Gerbert of Aurillac, the "monastic abacus", was a simple row of columns: at the top of each column was an arched section (similar to the top of a paddle pop stick) in which the place value for the column was indicated by Roman numerals (see Fig. 1-6)

$\mathbf{\bar{C}}$, $\mathbf{\bar{X}}$, $\mathbf{\bar{I}}$, **C**, **X**, and **I**.

The calculi or counters that Gerbert used were cone-shaped (and hence called *apices*), and each apex was numbered. Placement of a numbered apex in a column would then represent a particular numerical value.

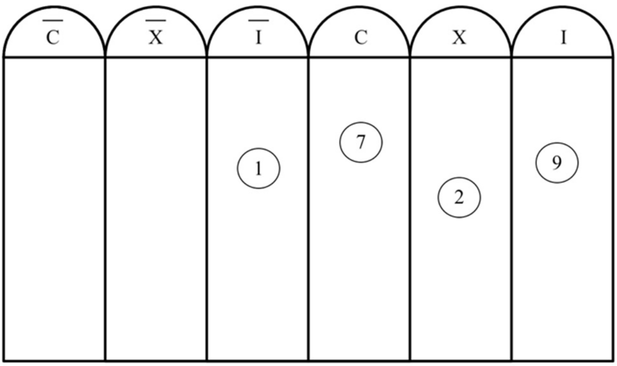

Fig. 1-6: A stylised depiction of Gerbert's reckoning board, showing the number 1,729.

A Late Medieval coin board

Over the centuries, counting boards were naturally and commonly used for calculations related to commerce and taxation. Because of the historical way in which monetary systems developed (for example, using weights or volumes of particular grains or metals), they seldom had place value amounts with a common base. Hence counting boards were constructed specific to purpose.

In the following example (see Fig. 1-7) of a 15th century or late medieval coin board, the units are libra, solidus, and denarius (cf. £ pounds, s shillings, and d pence), with £1 = 20s, and 1s = 12d. Medieval letters for X, C, and M add a place value for 10, 100, and 1,000 libras. Note that the counting board now has an orientation different from the previous ones shown, and that the letters are sometimes written on the left-hand side and sometimes on the right-hand side of the grid.

M		
C		
X		
Lib		
s		
d		

Fig. 1-7: A stylised depiction of a late medieval coin board.

A Late Medieval line board or counter-cloth

During the late medieval period, from the middle of the 13th century, a counting board or a counting cloth became a necessary and highly valued item in wealthy households. Then, after 1439 and the invention of the printing press with movable type by Johannes Gutenberg, came textbooks on arithmetic, the most famous written by Adam Ries (also spelt Ris, Reyeß, Rise, Ryse, Riese, and Risen). When adding up simple numbers, my wife, who is German, uses the phrase “nach Adam Riese” (according to Adam Ries) as proof that the sum must be correct, accompanied by the joke, “oder