Microwave Theory and Techniques

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To my late wife

The book presents a detailed description of linear microwave networks, including their analysis methods. The first two chapters of the book describe the theory of microwave circuits. They consider the transition from electromagnetics to a circuit approach and introduce equivalent voltage and current concepts. Reflection coefficient, voltage standing wave ratio, input impedance and admittance are also introduced. Properties of Smith chart, scattering and other wave matrices are considered. The book describes methods of complicated network analysis, based on decomposition and recomposition algorithms.

The subsequent chapters consider basic linear microwave circuits, beginning from one-port devices, two-port matching circuits and filters, power dividers and summators, directional couplers, and ending with multiport junctions. The book describes ferrites and ferroelectric properties, including spin-wave and magnetostatic waves in solids and thin films. Networks based on ferrites and ferroelectrics are also considered.

This book will be of interest to researchers and engineers working in the microwave field, and to students and post-graduate students teaching this field of knowledge.

CONTENTS

Preface

The modern world cannot exist without electromagnetic fields of the microwave band. No doubt most people know about radars and their role in World War II. Radioastronomy, telecommunications techniques, including 5G and 6G mobile systems and microwave technology applications play an essential role in modern society. These achievements were realised due to the development of new types of microwave transmission lines, resonators, filters, power dividers and summators, directional couplers, and new computer simulation and optimisation methods. New materials for substrates with high thermal conductivity and new ferrites and ferroelectrics compounds were created.

The design and construction of microwave networks are impossible without the mathematical apparatus of microwave techniques – such concepts as the reflection factor, the voltage standing wave ratio (VSWR), the Smith diagram, and characteristic matrices of multiport networks. These concepts need transferring from "differential" electromagnetic values – from field intensities and flux densities to integral ones – equivalent voltages and currents.

The complex structure of modern microwave systems requires unique methods for their analysis. One of these methods involves decomposing networks into simpler elements – the so-called autonomous blocks (AB) – evaluating the parameters of these blocks (descriptors) and then uniting these descriptors in the whole structure (recomposition). These procedures are now accomplished by modern computer simulation methods, which help to save computational resources significantly.

What is especially worth noting is the appearance of new types and constructions of microwave networks for high-frequency microwave and terahertz bands.

This book is based on lectures delivered by the author at Saint Petersburg Electrotechnical University (LETI). Founded in 1886, it is the oldest specialised electrical engineering high education institute in Europe.

The author hopes that this book is helpful for students and post-graduates learning microwave science, and for engineers and researchers in the microwave and neighbouring fields researching and developing new, more perfect types of microwave networks and devices.

The author expresses his sincere gratitude to all persons who helped him write this book – his colleagues, his publishers, and his wife.

The author

MAIN NOTATIONS

The SI international system of units is used in this book. Scalar values are denoted by Latin letters in italics (*A*, *a*) or Greek letters (α, Ψ). Mathematical constants are denoted by upright font Greek or Latin letters (e, π). Characters denoting vectors, tensors, and matrices are upright bold (**a**, **A**). When necessary, matrix notations, including row and column vectors, are enclosed in straight brackets (|**A**|, |**a**|) and tensors are overlined twice ($\overline{\overline{\varepsilon}}, \overline{\overline{\mu}}$). The equations and matrices determinants and norms are enclosed in double straight brackets (||**A**||). Complex amplitudes (phasors) are denoted by a point above the symbol ($\dot{E}, \dot{u}$).

B – magnetic flux density, V·s/m^2
B – reactive electrical conductance, S
c = 2.9979·10^8 m/s – speed of light in vacuum
D – electric flux density, A·s/m
e = 1.602·10^{-19} C – absolute value of the electron charge
E – electric field intensity, V/m
f – frequency, Hz
h = 6.626176·10^{-34} J·s – Plank constant
$\hbar$ – h/(2π) – reduced Plank constant
H – magnetic field intensity, A/m
i = $\sqrt{-1}$ – imaginary unit
I, *i* – electric current, A
J – electric current density, A/m^2
k, **k** – wave number, wave vector, m^{-1}
k_B = 1.38·10^{-23} J/K – Boltzmann constant
M – magnetisation, A/m
n – refractive index of the medium
P – power, W
P – electric polarisation (electric momentum), A·s/m^2
Q – resonator quality factor
q – electric charge, C
R – active electrical resistance, Ohm
U, u – Voltage, V

W – energy, J
X – reactance, Ohm
Y – complex admittance, S
Z – complex impedance, Ohm
α – attenuation constant, m^{-1}
β – phase constant, m^{-1}
$\gamma=\beta-i\alpha$ – propagation constant, m^{-1}
ε – permittivity, A·s/(V·m)
$\varepsilon_0 = 10^7/(4\pi c^2) = 8.854\cdot10^{-12}$ A·s/(V·m) – dielectric constant
$\eta_0 = 120\pi = 377$ Ohm – intrinsic impedance of the vacuum
λ – wavelength in vacuum, m
μ – permeability, V·s/(A·m)
$\mu_0 = 4\pi\cdot10^{-7} = 1.256\cdot10^{-6}$ V·s/(A·m) – magnetic constant
Φ – magnetic flax, V·s
χ – magnetic susceptibility
ω – angular frequency, s^{-1}

INTRODUCTION

Microwaves came into our lives in the first 30 years of the 20th century. The role of microwave radars in World War II is hard to overestimate. As General Douglas MacArthur said, "The atom bomb ended the war, but the radar won it". Nowadays, microwave networks and devices are widely used in military and civil radar systems. Microwaves are everywhere, from cellular phones, which everyone carries in their pocket, and microwave ovens to radio vision, radio spectroscopy, and safety control systems in public places. The role of radio astronomy, based on the analysis of cosmic objects' microwave radiation, in increasing our knowledge of the universe is invaluable. 5G and 6G mobile telecommunication systems work in various frequencies, including a millimetre band. Data transfer from all satellites and other artificial cosmic objects uses the microwave frequency band. Material processing, including heating, sintering, plasma heating, and accelerating charged particles, also uses microwaves. Hence, modern engineers and researchers must understand the main principles of microwave circuits and devices.

Microwave networks are one of the most critical parts of microwave systems. They are used to transfer electromagnetic energy and distribute it between several consumers and for the summation of the power of several sources. In the recent past, new materials for microwave circuits, new schematics, and new approaches for microwave circuits were elaborated. Unique methods oriented around the wide use of computer simulation were developed to analyse and synthesise microwave networks.

According to the International Telecommunication Union (ITU), the microwave frequency band occupies a part of the electromagnetic oscillation spectrum, lying between frequencies $f = 3\cdot10^8$ and $3\cdot10^{11}$ Hz, which corresponds to wavelengths in free space $\lambda = 1$ m…1 mm. The microwave frequency band includes ultra-high frequency (UHF) ($f = 300$ MHz…3 GHz), super-high frequency (SHF) ($f = 3$ GHz…30 GHz), and extremely high-frequency (EHF) ($f = 30$ GHz…300 GHz) sub-bands. Separate parts of the microwave band are usually noted by symbols (see Appendix F).

The microwave frequency band is positioned on the frequency axis between the radio band (30 kHz…300 MHz) and the terahertz band (0.3…3 THz). Microwave band wavelengths are comparable to the dimensions of most objects. This fact limits the methods of scattering problem analysis.

There is a stable tendency to increase telecommunication and radar systems' working frequencies. For example, the maximum working frequency of telecommunication systems has increased a million times in the last 100 years to reach 1 THz. Hence, this book gives sufficient attention to the terahertz circuits and devices.

J. Southworth, W. W. Hansen, A. Ginston, R. Fano, and other researchers lay the foundations of the microwave technique in the '30s and '40s. C. Kittel, P. C. Fletcher, B. Lax, K. J. Button, A. G. Gurevich, O. G. Vendik, and others went on to develop the theory and applications of ferrites and ferroelectrics for microwaves.

This book considers linear microwave circuits, which do not contain such elements as *p-i-n* diodes, varactors, and ferrite and ferroelectric devices, working in the nonlinear regime. Microwave antennas are also beyond the scope of this book.

The book's first chapters describe the transfer from "differential" values used in electrodynamics – field intensities and flux densities – to "integral" values – the equivalent voltages and currents defined in a specific cross-section of a transmission line.

On this basis, the reflection factor, the voltage standing wave ratio (VSWR), the input, and load impedances are introduced. The book also describes the Smith diagram formation principles. Scattering, impedance, admittance, and the transmission matrices of multiport microwave devices are introduced. Their basic properties and the relationships between them are deducted. The decomposition process – dividing the complex system into parts (autonomous blocks, AB), the analysis of each AB with its descriptor, and the subsequent integration of all descriptors in one – determining the circuit as a whole (recomposition) is described in detail. The book describes the methods of decomposition, formation of AB's descriptors, and recomposition.

The subsequent chapters of the book detail the basic types of microwave devices and networks, their parameters, and their characteristics, beginning with one-port devices – matched loads, reactive and resonant one-port devices; two-port devices, including attenuators, phases shifters, matching circuits, and filters; three-port dividers and summators; and ending with four-, five-, and multiport devices.

The book also describes the chemical composition and properties of ferrites and the microwave devices based on them, magnetisation processes, and electromagnetic wave propagation in a magnetised ferrite. Ferrite filters and spin-wave devices are also described in the book. The main ferroelectric properties and the devices based on them are also considered, including electrically tunable capacitors, phase shifters, and delay lines.

CHAPTER 1

BASICS CONCEPTS OF MICROWAVE CIRCUIT THEORY

1.1. Equivalent voltages and currents in a transmission line

The main goal of complex microwave network analysis is, as a rule, finding the distribution of power between various parts of the network. The electrodynamics approach for solving this problem deals with so-called differential values – electric and magnetic field intensities and flux densities – defined at each point of the computational region. Hence, we need to solve Maxwell's partial differential equations (PDE) with appropriate initial and boundary conditions to find the electric and magnetic fields in the system. After that, we use integration techniques to calculate the power and wave propagation parameters.

Conversely, in the 1880s, O. Heaviside formulated his *telegraph equations* for "long transmission lines" in which the voltage between two wires and the current flowing along the wire are unknown values, depending only on the coordinate along the line and time. For a lossless TL, the telegraph equations have the form

$$\frac{\partial U}{\partial z} = -L_1 \frac{\partial I}{\partial t}; \quad \frac{\partial I}{dz} = -C_1 \frac{\partial U}{\partial t},$$

where U, I are the voltage and current, respectively, in the TL, and L_1 and C_1 are the specific inductance and capacity of the TL.

These equations are much simpler than Maxwell's equations but are valid only for two-wire transmission lines (TL) with TEM mode propagating in them. Hence, we need to generalise Heaviside's TL theory to an arbitrary transmission line with an arbitrary mode propagating in it, transferring differential values into integral ones, such as voltage and current.

Transmission lines are divided into regular and irregular. A regular TL has an area of the cross-section, its form, and properties of the filling medium constant along the TL or changing according to the periodic law. These parameters change according to an arbitrary law in irregular TLs. The following text deals, as a rule, with regular TLs.

We know from electromagnetic analysis that an infinite number of modes can propagate in each regular TL. Each mode has its own lower cut-off frequency f_c and field distribution pattern. When a given mode frequency is less than the cut-off frequency, the mode does not propagate along the line, and its field attenuates exponentially. Some modes in periodic TLs have, along with the lower, an upper cut-off frequency.

Different modes are orthogonal in lines without losses; that is, they propagate independently of each other. This property is also approximately applicable for analysis TLs with low losses. The subsequent analysis is based on the preposition of mode orthogonality, which permits the independent analysis of each mode. In this case, the power transmitted along the TL equals the sum of powers transmitted by each propagating mode.

Fields of the wave propagating in a TL in the positive direction (along the z-axis) are described by the expressions

$$\dot{\mathbf{E}}_\perp(x_1,x_2 z)=\dot{\mathbf{E}}_\perp(x_1,x_2,0)\,\mathrm{e}^{-\mathrm{i}\gamma z}. \tag{1.1.1}$$

$$\dot{\mathbf{H}}_\perp(x_1,x_2 z)=\dot{\mathbf{H}}_\perp(x_1,x_2,0)\,\mathrm{e}^{-\mathrm{i}\gamma z}. \tag{1.1.2}$$

where $\dot{\mathbf{E}}_\perp,\dot{\mathbf{H}}_\perp$ – complex amplitudes of transverse components of electric and magnetic field intensities (phasors), $\gamma=\beta-\mathrm{i}\,\alpha$ – propagation constant, α – attenuation constant, and β – phase constant. We use the coordinate system (x_1, x_2, z) and the z-axis directed along the TL axis here. A *characteristic impedance* Z_c couples the transverse components of field intensities in a TL:

$$\dot{\mathbf{E}}_\perp(x_1,x_2,z)=Z_c\left[\dot{\mathbf{H}}_\perp(x_1,x_2,z)\times\mathbf{e}_z\right], \tag{1.1.3}$$

where x_1,x_2 – point coordinates in the TL cross-section z = Const. We see that transverse components of electric and magnetic fields and propagation unit vectors form the right-hand triple at any point of the TL cross-section, and their modulus ratio is constant at any point of the cross-section and equals Z_c.

One of the essential wave parameters is transmitted power. It is equal to the Poynting vector flux through the TL cross-section. For the field, harmonically changing in time, power transmitted by the given mode

$$P=\frac{1}{2}\mathrm{Re}\int_S\left(\dot{\mathbf{E}}_\perp\times\dot{\mathbf{H}}_\perp^*\right)\mathbf{e}_z dS, \tag{1.1.4}$$

where $\mathbf{e}_z$ – a unit vector of the TL axis, and S – TL cross-section. The upper index * means complex conjugate value.

Modes in TLs are distinguished by the presence of longitudinal components of electric and magnetic fields:

- TEMs (transverse electric and magnetic), or T- modes, have no longitudinal electric and magnetic field intensity components. For them:

$$Z_c^{TEM} = \sqrt{\mu_0 / \varepsilon_0}\sqrt{\mu_r / \varepsilon_r} = Z_0\sqrt{\mu_r / \varepsilon_r}. \quad (1.1.5)$$

where ε_r, μ_r – relative permittivity and permeability of the medium filling the TL, $Z_0 = 120\pi \approx 377$ Ohm is the characteristic impedance of the free space.

- TMs (transverse magnetic), or E-modes, have the longitudinal component of electric field intensity. The characteristic impedance of these modes is

$$Z_c^{TM} = Z_c^{TEM}\sqrt{1-(f_c / f)^2}, \quad (1.1.6)$$

 where f_c – cut-off frequency of the mode; f – mode frequency.

- TEs (transverse electric), or H-modes, have the longitudinal component of magnetic field intensity. Their characteristic impedance is

$$Z_c^{TE} = Z_c^{TEM} / \sqrt{1-(f_c / f)^2}. \quad (1.1.7)$$

- Hybrid or mixed modes have non-zero components of both electric and magnetic field intensities. They are denoted as EH or HE modes, depending on whether the E_z or H_z field component is prevalent. Hybrid modes can be considered as the superposition of TE- and TM-modes. The characteristic impedance of these modes depends on frequency by complex law.

Coupling between transverse field intensities makes it possible to express the transmitted power with the formulas:

$$P = \frac{1}{2}\int_S \left(\dot{\mathbf{E}}_\perp \times \dot{\mathbf{H}}_\perp^*\right) d\mathbf{S} = \frac{1}{2\,\mathrm{Re}\, Z_c}\int_S \left|\dot{\mathbf{E}}_\perp\right|^2 dS = \frac{1}{2}\mathrm{Re}\, Z_c \int_S \left|\dot{\mathbf{H}}_\perp\right|^2 dS.$$

(1.1.8)

We can see from (1.1.3), (1.1.6) and (1.1.7) that when $f > f_c$, the transverse components of the electric and magnetic field intensities are in-phase and transmitted power is $P>0$. When $f < f_c$, these components are out-of-phase, and transmitted power equals zero[1].

---

[1] *For TLs without losses ($\alpha = 0$).*

The theory of long transmission lines, developed by Heaviside, defines power transmitted along the TL by voltage phasor $\dot{U}(z)$ between conductors, and current phasor $\dot{I}$ flowing along one of the conductors:

$$P = \frac{1}{2}\dot{U}\dot{I}^* = \frac{1}{2Z_g}\left|\dot{U}\right|^2 = \frac{1}{2}Z_g\left|\dot{I}\right|^2. \tag{1.1.9}$$

Here, $Z_g = \dot{U}(z)/\dot{I}(z)$ is the wave impedance of the long line (do not confuse it with the characteristic impedance).

Expressions (1.1.9) are much simpler than formulas (1.1.8). Hence, it is expedient to introduce analogues of voltage and current for an arbitrary mode in an arbitrary regular TL. For this

1) Let us introduce normalised field intensities, which are proportional to the real values of the fields:

$$\mathbf{E}_\perp^n(x_1,x_2) = A\dot{\mathbf{E}}_\perp(x_1,x_2,0); \quad \mathbf{H}_\perp^n(x_1,x_2) = B\dot{\mathbf{H}}_\perp(x_1x_2,0). \tag{1.1.10}$$

The following expressions define normalised field intensities as:

$$\int_S\left|\mathbf{E}_\perp^n\right|^2 dS = 1, \quad \int_S\left|\mathbf{H}_\perp^n\right|^2 dS = 1. \tag{1.1.11}$$

The dimension of both values is m^{-1}. Normalised field intensities do not depend on the *z*-coordinate. We obtain from (1.1.10) and (1.1.11) that:

$$\mathbf{E}_\perp^n = \frac{\mathbf{E}_{\perp 0}}{\sqrt{\int_S\left|\mathbf{E}_{\perp 0}\right|^2 dS}}; \quad \mathbf{H}_\perp^n = \frac{\mathbf{H}_{\perp 0}}{\sqrt{\int_S\left|\mathbf{H}_{\perp 0}\right|^2 dS}}.$$

2) Normalised field intensities relate to real ones by the expressions:

$$\dot{\mathbf{E}}_\perp(x_1,x_2,z) = a_c\dot{u}(z)\mathbf{E}_\perp^n(x_1,x_2); \tag{1.1.12}$$

$$\dot{\mathbf{H}}_\perp(x_1,x_2,z) = b_c\dot{i}(z)\mathbf{H}_\perp^n(x_1,x_2); \tag{1.1.13}$$

where $\dot{u},\dot{i}$ – equivalent voltage and current, and a_c, b_c – calibration factors which are supposed to be real values. These factors are introduced to get an additional degree of freedom in defining equivalent voltage and current.

After the substitution of (1.1.12) and (1.1.13) into (1.1.9), we get:

$$P = \frac{1}{2}a_c b_c\dot{u}\dot{i}^* = \frac{a_c^2\left|\dot{u}\right|^2}{2\operatorname{Re}Z_c} = \frac{1}{2}b_c^2\operatorname{Re}Z_c\left|\dot{u}\right|^2. \tag{1.1.14}$$

Let us introduce the *wave impedances* of a given mode in TL, defined by power and voltage Z_g^{Pu}, power and current Z_g^{Pi}, and voltage and current Z_g^{ui}:

$$Z_g^{Pu} = Z_c / a_c^2, \quad Z_g^{PI} = b_c^2 Z_c; \quad Z_g^{ui} = (b_c / a_c) Z_c = \sqrt{Z_g^{Pu} Z_g^{Pi}}. \tag{1.1.15}$$

Formula (1.1.14) now can be rewritten as:

$$P = \frac{1}{2\,\mathrm{Re}\, Z_g^{Pu}} |\dot{u}|^2 = \frac{1}{2} \mathrm{Re}\, Z_g^{pi} |\dot{i}|^2 = \frac{1}{2} \dot{u}\dot{i}^*. \tag{1.1.16}$$

These expressions have almost the same form as formulas for long TLs (1.1.9).

3) A further simplification of (1.1.14) is achieved by the proper choice of the calibration factors. There are two definitions of these factors.

- We define the values of calibration factors as

$$a_c^2 = Z_c; \quad b_c^2 = 1 / Z_c. \tag{1.1.17}$$

Obviously, under such a choice, wave impedances of any mode in any TL are equal to one. The voltage and current acquired with such a choice are called normalised and denoted as $\dot{u}^n, \dot{i}^n$. The expressions for transmitted power get the form

$$P = \frac{1}{2} \dot{u}^n \dot{i}^{n*} = \frac{1}{2} |\dot{u}^n|^2 = \frac{1}{2} |\dot{i}^n|^2. \tag{1.1.18}$$

The dimensions of normalised voltage and current are $W^{1/2}$. It is convenient to use normalised voltage and current when we analyse a system containing several types of TLs because the wave impedances of all of them are equal.

- Another choice is to define the voltage and current of the given mode as integrals from field intensities along some lines (Fig. 1.1):

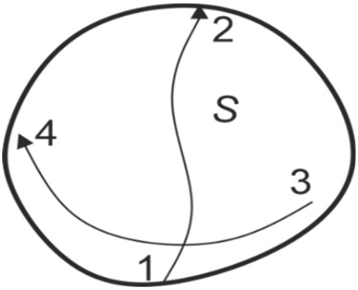

Fig. 1-1. TL cross-section with lines of voltage and current definition

$$\dot{u}^e(z) = \int_1^2 \dot{\mathbf{E}}_\perp (x_1, x_2, z) d\mathbf{l}; \quad \dot{i}^e(z) = \int_3^4 \dot{\mathbf{H}}_\perp (x_1, x_2, z) d\mathbf{l}. \tag{1.1.19}$$

The voltage and current defined by these expressions are called equivalent. The dimensions of equivalent voltage and equivalent current are volt and ampere.

From (1.1.19), we can find the values of the calibration factors:

$$a_c = \left(\int_1^2 \mathbf{E}_\perp^n d\mathbf{l} \right)^{-1}; \quad b_c = \left(\int_3^4 \mathbf{H}_\perp^n d\mathbf{l} \right)^{-1}. \tag{1.1.20}$$

Hence, we have transmitted power expressed through the equivalent voltage and equivalent current:

$$P = \frac{1}{2Z_g^{Pu}}\left|\dot{u}^e\right|^2 = \frac{1}{2}Z_g^{Pi}\left|\dot{i}^e\right|^2 = \frac{1}{2}\mathrm{Re}(\dot{u}^e\dot{i}^{e*}) \qquad (1.1.21)$$

In practice, we can use both normalised and equivalent values. Hence, the upper indexes in the notations of voltage and current are usually omitted if they are not needed.

Both equivalent and normalised voltage and current satisfy the telegraph equations. For a lossless TL and values that change harmonically in time

$$\frac{d\dot{u}}{dz} - \mathrm{i}\,\omega L_1\dot{i} = 0; \quad \frac{d\dot{i}}{dz} - \mathrm{i}\,\omega C_1\dot{u} = 0.$$

Let us consider two examples.

TEM-mode in a coaxial line. The following formulas describe the electric and magnetic field intensities of this mode in the cylindrical coordinates r,θ,z:

$$\dot{E}_r^n = \frac{C}{r}\mathrm{e}^{-\mathrm{i}\gamma z}; \quad \dot{H}_\theta^n = \frac{C}{Z_c r}\mathrm{e}^{-\mathrm{i}\gamma z}, \qquad (1.1.22)$$

To find normalised field intensities, we must calculate the C constant from the integral:

$$\int_S |C\dot{E}_r^n(r,\theta,0)|^2\, dS = |C|^2\, 2\pi\ln(a/b) = 1.$$

From this:

$$E_r^n = \frac{1}{r\sqrt{2\pi\ln(a/b)}}; \quad H_\theta^n = \frac{1}{r\sqrt{2\pi\ln(a/b)}}, \qquad (1.1.23)$$

where a and b – outer and inner TL conductors' radii.

Using $a_c^2 = Z_c,\, b_c^2 = 1/Z_c$ and substituting (1.1.23) into (1.1.10), we get the normalised voltage and current:

$$\dot{u}^n = \dot{A}\frac{\mathrm{e}^{-\mathrm{i}\gamma z}}{\sqrt{2\pi Z_c\ln(a/b)}}, \quad \dot{i}^n = \dot{A}\frac{\mathrm{e}^{-\mathrm{i}\gamma z}}{\sqrt{2\pi Z_c\ln(a/b)}}. \qquad (1.1.24)$$

Now we define equivalent voltage and current by integrals:

$$\dot{u}^e = \int_b^a \dot{E}_r dr = \dot{A}\ln(a/b)\mathrm{e}^{-\mathrm{i}\gamma z}; \quad \dot{i}^e = \int_0^{2\pi}\dot{H}_\theta r d\theta = 2\pi\dot{A}Z_c^{-1}\,\mathrm{e}^{-\mathrm{i}\gamma z}.$$

And transmitted power as:

$$P = \frac{1}{2}\int_S \dot{E}_r\dot{H}_\theta^* dS = \frac{\pi\left|\dot{A}\right|^2}{Z_c\ln(a/b)}.$$

This expression helps us find the wave impedances:

$$Z_g^{Pu} = \frac{\left|\dot{u}^e\right|^2}{2P} = \frac{Z_c}{2\pi}\ln\frac{a}{b};\ Z_g^{Pi} = \frac{2P}{|\dot{i}^e|^2} = \frac{Z_c}{2\pi}\ln\frac{a}{b};\ Z_g^{ui} = \frac{\dot{u}^e}{\dot{i}^e} = \frac{Z_c}{2\pi}\ln\frac{a}{b}. \quad (1.1.25)$$

We see that all three wave impedances are equal. But this is true only for TEM modes in TL. In contrast to characteristic impedances, wave impedances depend clearly on the TL dimensions. We can express a coaxial TL wave impedance in a more convenient form for practice:

$$Z_g = 138\sqrt{\mu_r/\varepsilon_r}\,\log(a/b).$$

The TE10 mode in a rectangle waveguide. The transverse components of the field intensities in the Cartesian coordinate system are:

$$\dot{E}_y = \dot{A}\sin(\pi x/a)\mathrm{e}^{-\mathrm{i}\gamma z};\quad \dot{H}_x = \dot{A}Z_c^{-1}\sin(\pi x/a)\mathrm{e}^{-\mathrm{i}\gamma z},$$

where a is the size of the wide waveguide wall. We suppose that the x-axis is directed along the wide wall and the y-axis along the narrow wall.

Normalised field intensities have the following form:

$$E_y^n = \sqrt{\frac{2}{ab}}\sin\frac{\pi x}{a};\quad H_x^n = \sqrt{\frac{2}{ab}}\sin\frac{\pi x}{a},$$

where b – narrow wall size.

To calculate normalised voltage and normalised current, we set $a_c^2 = Z_c;\ b_c^2 = Z_c^{-1}$. Substituting these values into (1.1.12) and (1.1.13), we find:

$$\dot{u}^n = \frac{\dot{E}_y}{\sqrt{Z_c}E_y^n} = \dot{A}\sqrt{\frac{ab}{2Z_c}}\mathrm{e}^{-\mathrm{i}\gamma z};\quad \dot{i}^n = \frac{\sqrt{Z_c}\dot{H}_y}{H_y^n} = \dot{A}\sqrt{\frac{ab}{2Z_c}}\mathrm{e}^{-\mathrm{i}\gamma z}. \quad (1.1.26)$$

Now we define equivalent voltage and current by the expressions:

$$\dot{u}^e = \int_0^b \dot{E}_y(a/2, y, 0)dy = \dot{A}b;$$

$$\dot{i}^e = \int_0^a \dot{H}_x(x,0,0)dx = \dot{A}(2a/\pi)Z_c^{-1}.$$

To define the calibration factors, we use the formulas (1.1.20):

$$a_c = \sqrt{\frac{a}{2b}};\quad b_c = \frac{\pi}{2}\sqrt{\frac{b}{2a}}.$$

The formulas define the wave impedances of the H10 mode in a rectangular waveguide:

$$Z_g^{Pu} = \frac{Z_c}{a_c^2} = 2\frac{b}{a}Z_c;\quad Z_g^{Pi} = b_c^2 Z_c = \frac{\pi^2}{8}\frac{b}{a}Z_c;\quad Z_g^{ui} = \sqrt{Z_g^{Pi}Z_g^{Pu}} = \frac{\pi}{2}\frac{b}{a}Z_c.$$

As we see, wave impedances also clearly depend on waveguide dimensions. The values of Z_g^{Pu}, Z_g^{Pi} and Z_g^{ui} differ only by a constant multiplier. As usual, only the ratio of connected TLs wave impedances is essential; hence this difference is of no significance if the wave impedance in each TL is defined in the same way. From this, the upper indexes of wave impedance notations are usually omitted.

The wave impedances for other TLs are defined in a similar way. Note that the equality of wave impedances, not characteristic impedances, determines the least reflections from the connection of two TLs. The reflection level is the physical sense of the wave impedance despite some conventionality in its definition.

So, an equivalent two-wire TL with voltage $\dot{u}$ and current $\dot{i}$ corresponds to a real TL with a specific mode propagating in it. If several modes propagate in a real TL, a separate equivalent two-wire transmission line corresponds to each mode.

1.2. Reflection factor and voltage standing wave ratio

Every transmission line contains inhomogeneities (obstacles) caused by fabrication errors or made artificially. Fig. 1.2 shows a TL with a particular mode propagating in it. The voltage of this mode is denoted as $\dot{u}^+$. We call it *a falling wave*. The origin of the coordinate system is placed at the inhomogeneity, and the *z*-axis is directed against the falling wave (to a generator). The falling wave excites a current with density **J** in the inhomogeneity, which, in turn, excites an infinite number of modes propagating in both directions in the TL. Modes excited by the inhomogeneity are called *reflected* ($\dot{u}^-$) and *transmitted* ($\dot{u}^{tr}$). Note that an inhomogeneity placed at the end of a TL is called a *TL load*.

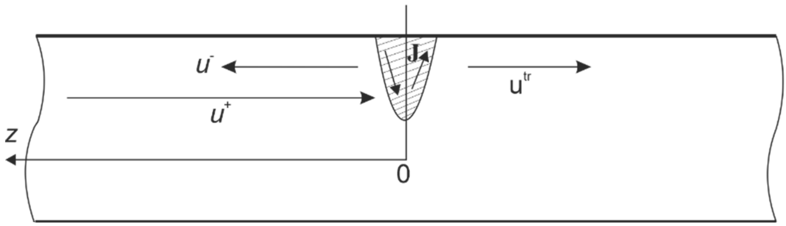

Fig. 1-1. A transmission line with an obstacle

If a TL works in the single-mode regime, higher modes attenuate quickly with the distance from the inhomogeneity. Hence, we can consider only falling, reflected, and transmitted modes of the same type at a sufficient distance from the inhomogeneity.

We can write in the given coordinate system:

$$\dot{u}^{+} = \dot{u}_0^{+}\, \mathrm{e}^{\mathrm{i}\gamma z}, \qquad (1.2.1)$$

where $\dot{u}_0^{+}$ – voltage of the falling mode in the place of the inhomogeneity, as if the inhomogeneity is absent. Similarly:

$$\dot{u}^{-}(z) = \dot{u}_0^{-}\, \mathrm{e}^{-\mathrm{i}\gamma z}. \qquad (1.2.2)$$

The ratio of the reflected wave voltage to the falling wave voltage is called the (*voltage*) *reflection factor*. This factor depends on the coordinate z:

$$\Gamma(z) = \dot{u}^{-}(z) / \dot{u}^{+}(z) = \Gamma_l\, \mathrm{e}^{-\mathrm{i}2\gamma z} = \Gamma_l\, \mathrm{e}^{-2\alpha z}\, \mathrm{e}^{-2\mathrm{i}\beta z}, \qquad (1.2.3)$$

where $\Gamma_l = \dot{u}^{-}(0) / \dot{u}^{+}(0)$ – inhomogeneity (load) reflection factor. The modulus of the reflection factor decreases as we move to the generator because the amplitude of the reflected wave decreases and the amplitude of the falling wave increases.

Along with the voltage reflection factor, we can consider the *current reflection factor*:

$$\Gamma_I(z) = \frac{\dot{i}^{-}(z)}{\dot{i}^{+}(z)} = -\frac{\dot{u}^{-}(z)}{\dot{u}^{+}(z)} = -\Gamma(z). \qquad (1.2.4)$$

The use of the minus sign in this formula shows that when a wave changes its propagation direction, the transverse electric or magnetic components of field intensities also change direction to save their right-hand orientation with the propagation unit vector.

It is convenient to consider the reflection factor as a vector on the complex plane. The modulus of the reflection factor from passive inhomogeneities cannot be greater than unity. Hence, all possible Γ values are placed inside or on the unit circle on the complex plane. The end of the Γ vector delineates a spiral on the complex plane as we move along the TL. The vector Γ rotates clockwise as we move to the generator and anti-clockwise if we move to the load (Fig. 1.3a). The reflection factor makes one turn when we move the distance $\Delta z = \lambda_g / 2$ along the TL, where $\lambda_g = 2\pi / \beta$ – wavelength in the TL. The end of the Γ vector delineates a circle if the TL is lossless ($\alpha = 0$) (Fig. 1.3b).

Fig 1.3b also shows vectors of full voltage $\dot{u} = \dot{u}^{+} + \dot{u}^{-}$ and full current $\dot{i} = \dot{i}^{+} + \dot{i}^{-}$ in the TL. Their modules and the phase shift ψ between them change along the line, but transmitted power $P = |\dot{u}|\,|\dot{i}|\cos\psi$ stays constant in any cross-section (for $\alpha = 0$).

Along with the reflection factor, we can consider the *voltage standing wave ratio (VSWR)*:

$$k_s = \frac{|\dot{u}|_{\max}}{|\dot{u}|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}. \tag{1.2.5}$$

This value is easier to measure but contains less information than Γ because it is a real number, while Γ has a modulus and phase. Sometimes, the *travelling wave ratio* $k_t = 1/k_s$ is used instead of VSWR.

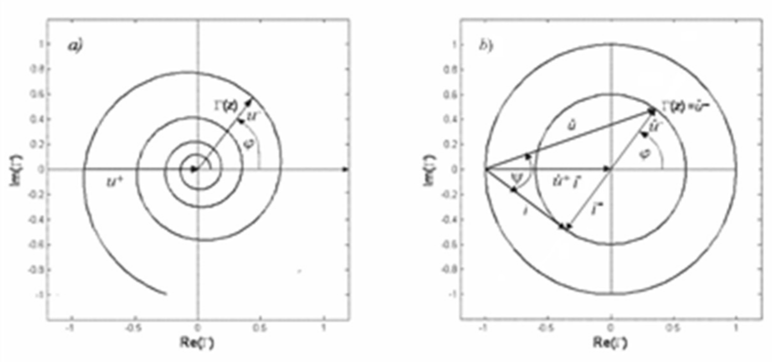

Fig. 1-3a,b. Vector of the reflection factor on the complex plane

1.3. Input impedance and load impedance

The impedance of the TL section in the cross-section *z* is defined by the expression:

$$Z(z) = \frac{\dot{u}(z)}{\dot{i}(z)} = \frac{\dot{u}^+(z)+\dot{u}^-(z)}{\dot{i}^+(z)+\dot{i}^-(z)} = Z_g\frac{1+\Gamma(z)}{1-\Gamma(z)}. \tag{1.3.1}$$

This impedance is called *load impedance* Z_l if $z=0$ and *input impedance* Z_{in} if $z=l$; *l* is the section length. The relationship of the reflection factor to the impedance is found by the formula:

$$\Gamma(z) = \frac{Z(z)-Z_g}{Z(z)+Z_g} \tag{1.3.2}$$

We can rewrite the expression (1.3.1) to find the coupling between input and load impedances:

$$Z_{in} = Z_g \frac{1+\Gamma_l\,\mathrm{e}^{-2\mathrm{i}\gamma l}}{1-\Gamma_l\,\mathrm{e}^{-2\mathrm{i}\gamma l}} = Z_g \frac{1+\dfrac{Z_l - Z_g}{Z_l + Z_g}\mathrm{e}^{-2\mathrm{i}\gamma l}}{1-\dfrac{Z_l - Z_g}{Z_l + Z_g}\mathrm{e}^{-2\mathrm{i}\gamma l}} =$$

$$= Z_g \frac{Z_l + Z_g + Z_l\,\mathrm{e}^{-\mathrm{i}2\gamma l} - Z_g\,\mathrm{e}^{-2\mathrm{i}\gamma l}}{Z_l + Z_g - Z_l\,\mathrm{e}^{-\mathrm{i}2\gamma l} + Z_g\,\mathrm{e}^{-2\mathrm{i}\gamma l}}.$$

After multiplying the numerator and denumerator of this formula by $\mathrm{e}^{\mathrm{i}\gamma l}$, we get:

$$Z_{in} = Z_g \frac{Z_l\,\mathrm{e}^{\mathrm{i}\gamma l} + Z_g\,\mathrm{e}^{\mathrm{i}\gamma l} + Z_l\,\mathrm{e}^{-\mathrm{i}\gamma l} - Z_g\,\mathrm{e}^{-\mathrm{i}\gamma l}}{Z_l\,\mathrm{e}^{\mathrm{i}\gamma l} + Z_g\,\mathrm{e}^{\mathrm{i}\gamma l} - Z_l\,\mathrm{e}^{-\mathrm{i}\gamma l} + Z_g\,\mathrm{e}^{-\mathrm{i}\gamma l}} = Z_g \frac{2Z_l\cos\gamma l + 2\,\mathrm{i}\,Z_g \sin\gamma l}{2\,\mathrm{i}\,Z_l \sin\gamma l + 2Z_g\cos\gamma l}.$$

Dividing the numerator and denumerator by $2\cos(\gamma l)$,we get the relationship:

$$Z_{in} = Z_g \frac{Z_l + \mathrm{i}\,Z_g\ \mathrm{tg}\,\gamma l}{Z_g + \mathrm{i}\,Z_l\ \mathrm{tg}\,\gamma l}. \quad (1.3.3)$$

Admittances are often used instead of impedances; that is, wave admittance $Y_g = Z_g^{-1}$, input admittance $Y_{in} = Z_{in}^{-1}$, and load admittance $Y_l = Z_l^{-1}$. We can easily deduce formulas for admittances from (1.3.1) …(1.3.3):

$$Y(z) = Y_g \frac{1-\Gamma(z)}{1+\Gamma(z)}; \quad \Gamma(z) = \frac{Y_g - Y(z)}{Y_g + Y(z)}; \qquad (1.3.4)$$

$$Y_{in} = Y_g \frac{Y_l + \mathrm{i}\,Y_g\ \mathrm{tg}\,\gamma l}{Y_g + \mathrm{i}\,Y_l\ \mathrm{tg}\,\gamma l}. \qquad (1.3.5)$$

In practice, *normalised* impedances and admittances are often used:

$$\overline{Z} = Z / Z_g; \quad \overline{Y} = YZ_g. \qquad (1.3.6)$$

These dimensionless values allow simplifying the formulas (1.3.1) …(1.3.5):

$$\Gamma = \frac{\overline{Z}-1}{\overline{Z}+1} = \frac{1-\overline{Y}}{1+\overline{Y}}; \quad \overline{Z} = \frac{1+\Gamma}{1-\Gamma}; \quad \overline{Y} = \frac{1-\Gamma}{1+\Gamma}; \quad (1.3.7)$$

$$\overline{Z}_{in} = \frac{\overline{Z}_l + \mathrm{i}\,\mathrm{tg}\,\gamma l}{1+\mathrm{i}\,\overline{Z}_l\ \mathrm{tg}\,\gamma l}; \quad \overline{Y}_{in} = \frac{\overline{Y}_l + \mathrm{i}\,\mathrm{tg}\,\gamma l}{1+\mathrm{i}\,\overline{Y}_l\ \mathrm{tg}\,\gamma l}. \qquad (1.3.8)$$

1.4. The Smith chart

It is convenient to represent the reflection factor as a vector in a complex plane (Fig 1-4a). A TL normalised impedance can also be represented on a complex plane (Fig. 1-4b). The expressions (1.3.7) couple the normalised impedance and reflection factor. We can say that these functions map a point on the Z-plane into a point on the Γ-plane and vice versa. These functions satisfy the Cauchy-Riemann conditions; hence, they are *analytic* functions. Analytic functions realise *conformal* mapping, i.e., unambiguous mutual mapping, which preserves the angles between mapped lines.

The complex plane of the reflection factor with mapped lines $\overline{R} = Const$ and $\overline{X} = Const$ is called the Smith chart after P. H. Smith, who proposed it in 1939. The lines $\overline{R} = Const$ are circles with centres on the line $\mathrm{Im}\,\Gamma = 0$ and radii $r_r = 1/(\overline{R}+1)$. These lines have a common point Γ = 1. Lines $\overline{X} = Const$ are arcs with centres lying on the straight-line L, tangent to the circle $|\Gamma| = 1$ at the point Γ = 1. The radii of these lines are $r_x = 1/\overline{X}$.

Because $\Gamma_i = -\Gamma$, this *impedance* diagram is simultaneously the diagram for admittances. Points $\overline{Z}$ and $\overline{Y} = 1/\overline{Z}$ are deposited on the same circle $\Gamma = Const$ but at opposite points. The diameter from the point Γ = 1 to the point Γ = -1 corresponds to the real values of Z and Y. $\overline{X} = \overline{B} = 0$ on this line. The arc $\overline{X} = 1, \overline{B} = -1$ passes across point D $\beta l = \pi/4$), and the arc $\overline{X} = -1, \overline{B} = 1$ passes through point E ($\beta l = 3\pi/4$). The outer circle of the diagram corresponds to $|\Gamma| = 1$. $\overline{R} = \overline{G} = 0$ on this circle.

Point A (the circle's centre) corresponds to the zero reflection factor. The circle $\overline{R} = 1$ passes through this *matching* point. Point B is the *open-circuit* point for the impedance diagram and the short-circuit point for the admittance diagram. At this point, $\overline{R} = \overline{X} = \infty$ and $\overline{G} = \overline{B} = \infty$. Point C is the short-circuit point for impedances and the open-circuit point for admittance. At this point, $\overline{R} = \overline{X} = 0$ and $\overline{G} = \overline{B} = \infty$.

Normalised distance l/λ_g and phase βl scales are usually inserted on the outer circle. Fig. 1-5 shows a typical Smith chart. The chart often contains a rotating ruler marked by values of $|\Gamma|$ and k_s fixed by one end to the chart centre.

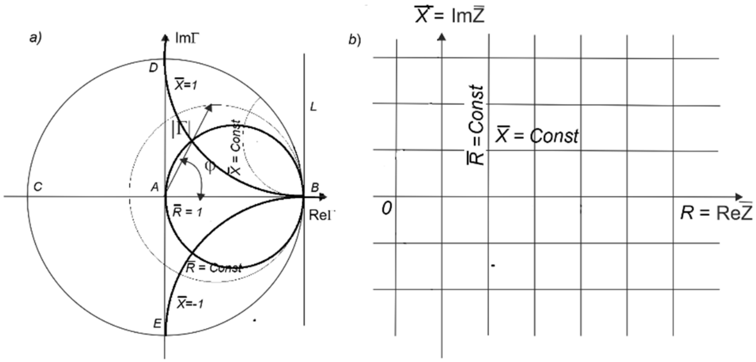

Fig. 1-4a,b. Complex planes of Γ (a) and Z (b)

The Smith chart allows the solving of various microwave technique tasks. Of course, such calculations are not very accurate; instead, they are more descriptive and straightforward. For example, if we know the load impedance of a TL section $\overline{Z}_l$, we can find its input impedance.

For this, we ought to mark the point $\overline{Z}_l$ and move along the circle $|\Gamma| = Const$ clockwise by the angle βl, where l – TL's length. The lines $\overline{R} = Const$ and $\overline{X} = Const$ passing across this point define the input impedance of the TL.

We can plot points on the Smith chart corresponding to the impedance or admittance of a device at various frequencies. Connecting these points with a line, we get a *hodograph* of a device's impedance (admittance). The hodograph is oriented from the lowest frequency to the highest.

The reader can find other examples of Smith chart usage in the next sections of this book.

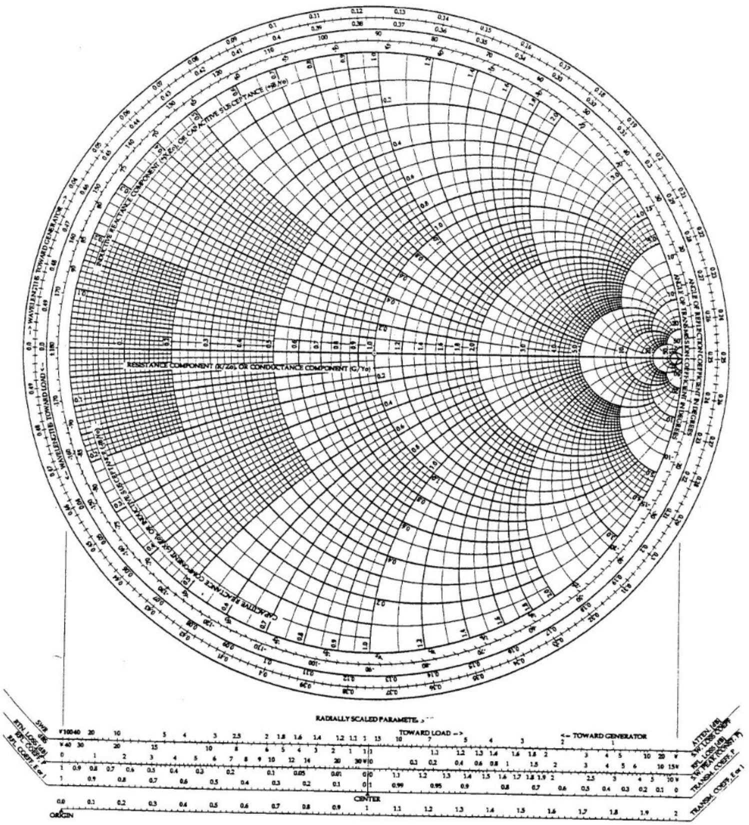

Fig. 1-5. The Smith chart

1.5. Basic working regimes of the TL

1. *Matched regime.* $Z_l = Z_g$. A TL's load impedance equals its wave impedance. The reflection factor equals zero. The reflected wave is absent. Hence, amplitudes of total voltage and current are equal to the amplitudes of the falling wave. These values do not change along the line if there are no losses (Fig. 1-6a). The reflection factor of the load equals zero, and VSWR equals unity. The input impedance is active and constant along the TL (Fig. 1-6b). Point A corresponds to this regime on the Smith chart.