Entropy for Smart Kids
and
their Curious Parents

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By

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This book is dedicated to all the kids, smart or not, in the world

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LIST OF ABBREVIATIONS

ABN	Arieh Ben Naim
1D	One dimensional
3D	Three dimensions
Pr	Probability
SMI	Shannon's Measure of Information
2^{nd} Law	The Second Law of Thermodynamics

PREFACE

Entropy is one of the most interesting concepts in physics. Although it is a well-defined concept, it is still perceived by even well-known scientists as a concept cloaked in mystery. It is also the most misused and often abused concept in physics. Some scientists believe that entropy will forever remain a mysterious quantity, and for them demystifying entropy remains as elusive as ever.

This book's title "Entropy for Smart Kids" might be misleading. What it actually means is that even "Smart Kids" can understand the concept of entropy. The prerequisites for understanding entropy are:

1. You need to have a probability-sense (I will show you in Chapter 1 that you already have).

2. You need to have an information-sense (I will explain in Chapter 2, and show you that you already have).

3. You need common-sense (which I hope you have). You will need that in order to understand Chapter 3.

My aim in writing this book is to show you that if you have even a rudimentary sense of probability and of information, and if you are willing to use your common-sense, then you can understand what entropy is.

It is my conviction that in order to understand *entropy*, one needs to understand Shannon's measure of information (SMI), and in order to understand SMI one must be familiar with some basic concepts of probability.

Therefore, this book consists of three chapters. Chapter 1 discusses *probability*. You will find out in this chapter that you already know what probability is. Once you know what probability is, you can also understand what SMI means. This is discussed in Chapter 2. This knowledge will lead you to a straightforward understanding of entropy. You will see in Chapter 3 that entropy is nothing but a special case of SMI. Please memorize this acronym. It will appear many times in this book. A simple way of memorizing the meaning of this term is to think of a twenty-question (20Q) game. In this game there is always a minimal number of binary questions one needs to ask in order to find out one out of *N* possibilities.

In Chapter 3 we also briefly discuss the Second Law of Thermodynamics (2^{nd} Law). We shall see that the 2^{nd} Law is nothing but a law of probability. We shall also see under what conditions the 2nd Law is related to the concept of entropy. We shall conclude this book by mentioning a few misuses and misapplications of entropy and the Second Law. You will learn how, and why entropy, and the 2nd Law became so mysterious as a result of these very same misuses and misapplications, as well as the gross-exaggeration of the "power" of entropy.

You do not need to know any mathematics in order to understand this book. It is helpful to know though what a logarithm is, but in case you have no clue as to what it is, you can still understand both entropy and the Second Law. In Appendix A you will find a simple, qualitative discussion of logarithm. If you do not know what the symbol $\log_2\left(\frac{1}{2}\right)$ means you can simply look at the relevant graph in Appendix A.

I urge you to read this book slowly, carefully, and critically. You also need to do some exercises in order to test your comprehension. Once you do all these, I guarantee that upon reaching the end of the book, you would know what entropy is, you will understand the 2nd Law, and you will also understand why entropy has become such a mysterious concept in physics.

Furthermore, I promise with confidence, that once you read this book your understanding of entropy will surpass the understanding of entropy by many authors who write about entropy.

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The cover designed is based on a picture taken by the author at the Villa Escudero Plantations and Resort located in Tiaong, Quezon Philippines, with permission from one of the resort owners, Mr. Aaron Escudero.

CHAPTER 1

PROBABILITY AND PROBABILITY DISTRIBUTIONS

In this chapter we discuss the concept of probability. We will start by playing some very simple games. These games were designed in such a way that playing them will prove to you that you already know what probability means. To put it another way, you will be convinced that you already possess a probability-sense; you have an intuitive understanding of the concept of probability although you do not know how to define it.

Next, we shall briefly learn how the concept of probability evolved from gambling games into a well-respected branch of mathematics. This will lead us to attempt to define the concept of probability. As we shall see, all definitions of probability are circular, i.e. they use the concept of probability (or an equivalent one, like "chances," or "likelihood") in order to define probability.

***Next, we will further train ourselves with some simple probability problems which will be important to understanding the concept of Shannon's measure of information discussed in Chapter 2, and the concept of entropy discussed in Chapter* 3.**

Finally, we will discuss a few probability distributions, and learn how to calculate average quantities. We shall use the concept of average to create a measure of an average uncertainty about the outcomes of an experiment. This average will have a similar meaning to Shannon's measure of information. It will also be indispensable for understanding entropy.

1.1 Your probability-sense

In this section, we shall play a few simple games. These games were so designed in such a way that while playing, you will either consciously or subconsciously be using probabilistic reasoning even before knowing what it is, or how it is defined. We shall go back to discussing the "definitions" of probability later on.

Let us start with a very simple game.

Figure 1.1(see color centerfold) shows seven different dice. Each die has six faces. I tell you that the dice are "fair." You might ask "what does a fair die mean?" For the moment I will tell you that all the dice we will play with are perfect cubes, i.e. all faces have the same area, and all the edges have the same length. I also tell you that the density of the material of which the dice are made of (plastic, metal, or any other material) is the same at each point within the dice which means that there is no unsymmetrical mass distribution within the dice. If I throw the dice into the air, there is no *preferred* face on which the dice will land.

Can you explain why such a die is called a fair die?[1]

All the dice in Figure 1.1 are "fair, but instead of a regular die with different numbers of dots (as in Figure 1.2a (see color centerfold) on its face, we have different colors. The dice in Figure 1.1 are colored as follows:

die *a* has six blue faces

die *b* has one red, and five blue faces

die *c* has two red, and four blue faces die *d* has three red, and three blue faces

die *e* has four red, and two blue faces

die *f* has five red and one blue face

die *g* has six red faces

Altogether, we have seven different dice. All of these dice are *fair*, which means that any one of the six faces of any die has the same likelihood of appearing when I throw it.

Here are the rules of the game. Read them carefully before you accept, or refuse to play this game.

I choose a die from the seven dice in Figure 1.1. I tell you that it is fair, and that I will throw it high into the air in such a way that it will roll over several times before it falls on the ground, Figure 1.3. I ask you to please trust me, at least for this particular game – that I have no control on the outcome of this throw. In other words, I cannot affect, nor do I know on which face the die will land on the ground. If you do not trust me (why should you?), then just imagine that a machine or a robot will be throwing the die.

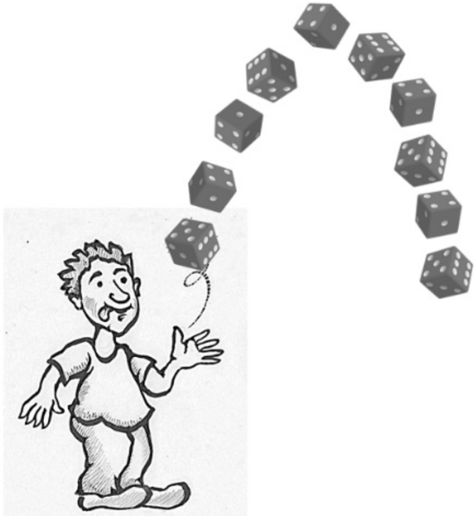

Figure 1.3. A die whirls in the air

You look at the die, count the number of blue and red faces, and see that the die is a perfect cube having no discernible defects, cuts, or irregularities.

Now, I offer you to play the following game:

I will choose one of the dice in Figure 1.1. You examine it, and choose either a blue or a red die.

Then I throw the die 100 times. Whenever the color you chose appears on the upper face of the die, you will get $1.00. If the color of the upper face is not the one you chose, you pay $1.00.

If you understand the rules of the game, repeat them before we continue playing this game.

First game:

I choose the die *a,* from Figure 1.1. Which color will you choose?

Before you decide on the color, refresh your memory about the rules of the game (I chose the die, you chose the color after examining the die. I toss the die 100 times, and every time the outcome is the color you chose, you get $1.00, otherwise you pay $1.00). Are you ready?

Obviously, presuming that you understood the rules of the game, you will choose the color blue. (One important aspect which I would like to tell you is that I implicitly presume that in playing any of these games, your goal is to maximize your earnings).

Clearly, in case of die *a* you will choose the color *blue*, as this choice will ensure you that on each toss you will get $1.00. Altogether, you shall have earned $100.00 after I toss the die 100 times.

What an easy game! You do not have to think hard in order to choose the color of the die. You also do not have to use your *probability-sense.* You only need plain common-sense.

Can you use the word “probability” to explain why you chose the color blue? [2]

Second game:

I choose die *b*. You examine it, refresh your memory with the rules of the game, and choose a color. If you forgot the rules, read them again before you proceed to make a choice.

Which color will you choose?

I am sure you will choose blue again. Why am I so sure? In the previous game, I was sure you’d choose blue because in case *a,* because you were certain to win $1.00 on each toss. Now, I am also *sure* that you will *choose* blue. But, I am *not sure* that you *will win* on each toss. In fact, I cannot even guarantee that you will win any money after 100 tosses.

Can you explain why you chose blue?

Here, unlike the previous game you must use your probability-sense to make the “right” choice. You see that die *b* has five blue faces, and one red. Your judgement tells you that there are "more chances" that the outcome blue will occur, therefore, it is in your interest to choose the color blue.

Can you explain why you chose blue by using the term probability?[3]

Note, at each toss of this die there is a chance that you will have to pay $1.00. However, your probability-sense tells you that if you play this game many times you will, "on average," earn money. It is not *certain* that you will always earn, but it is *very likely*, or *highly probable*.

Although we did not define the concept of an average, I believe that you have a qualitative estimate of the average (or the expected, as mathematicians refer to it) earnings after 100 tosses?[4]

Remember that in this game there are some chances that you will earn $100.00. There are also some chances that you will lose $100.00. Your probability-sense tells you that the former is more likely than the latter. Knowing probability theory allows you to make a more precise statement on the probabilities of these two extreme events.[5]

Third game:

I chose die *c* from Figure 1.1, you examine it, count the number of faces having different colors, and choose a color.

Which color will you choose?

Your intuition, or your probability-sense tells you that it will be advantageous to choose blue. Can you explain why you chose blue in the term probability? [6]

Clearly, in this particular case the chances of earning in 100 tosses is less than in case *b*, but it is still to your advantage to choose the color blue. The argument favoring the color blue is not as powerful as in the previous case (*b*), and certainly less powerful than in case *a*, yet it is still a good choice. Can you estimate your average net earnings in 100 tosses? [7]

Remember that in this case, you might earn $100.00. You might also lose $100.00 in 100 tosses. Can you estimate the probabilities of these two extreme events? [8]

Fourth game:

Next, I choose die *d*. You look at the faces, count how many reds, and blues there are. Refresh your memory about the "rules of the game" before you choose a color.

Which color are you going to choose?

This is the most "difficult" game. It is difficult because you are clueless as to the preference of occurrence of either the red, or the blue. In Chapter 2, we shall see that in this case you are "given" the minimal "*information*" on how to make a choice. However, without knowing information theory, and without knowing even probability theory, your probability-sense tells you that there is no preferred color. In other words, you can choose either blue or red; there is no advantage in any of these choices.

Can you repeat the argument on the lack of advantage for any particular choice in terms of probabilities? [9]

Your intuition – or probability-sense tells you that whichever color you choose, and that no matter how many times you play this particular game, your expected "gain" is the same as your expected "loss," and therefore your expected net gain is $0.0. This is another way of characterizing a fair die which has three red, and three blue faces.

Note again that even with this die (*d*), there is a chance that you will earn $100.00, but there is also a chance that you will lose $100.00. However, these two extreme cases are extremely improbable, and their probabilities of occurrence are equal. [10]

You can easily calculate that your average earnings in this case is $0.00. However, we do not need to calculate this average as we can trust our intuition, or our probability sense.

Although this particular case is relatively simple, it is instructive to pause, and calculate a few more probabilities. This will be important for understanding the Shannon measure of information in Chapter 2, and very important as well for understanding the Second Law in Chapter 3.

We saw in case *d* that the two outcomes; blue and red have equal probabilities; ½. We also saw that in 100 throws there is a small probability (very small but finite) that all of the outcomes will be red (and the same that all of the outcomes will be blue).

However, there is also a probability that any possible sequence of blues and reds will occur. For simplicity (to minimize on writing), suppose that we toss the die *d* ten times. A possible outcome in 10 tosses could be:

B, B, R, B, B, R, R, B, B, R

$(\mathrm{B} = Blue\ and\ \mathrm{R} = Red)$. This is referred to as a *specific* sequence of 10 outcomes. By *specific*, we mean that we know which color occurred at which throw; first B, second B, third

R, and so forth. In this specific sequence, there are 6Bs, and 4Rs. We shall later learn that all *specific* sequence with 6Bs, and 4Rs have the *same probability* as the sequence of all Rs, or all Bs.

Sometimes we are interested in a *non-specific* sequence which we call a *generic* sequence. This means that we toss the die *d* ten times, and got, say 6Bs, and 4Rs, but we do not care about the *order* of the occurrence of the Bs, and the Rs, For instance, the sequences:

B, B, B, R, R, R, B, R, B, B

R, R, B, R, R, B, B, B, B, B

These two sequences have 6Bs, and 4Rs, therefore, they are two *different specific* sequences, but they are of the *same generic* sequence. I should mention at this point that the distinction between a *specific* and a *generic* sequence is essential for understanding the Second Law. As an exercise, write down all possible sequences of four tosses of die *d*. Can you tell why a generic sequence of Bs and Rs will always have larger probabilities compared with a *specific* sequence, except for the extreme case of all Bs, or all Rs?[11]

Table 1.1: All possible sequences of four outcomes of blues and reds

Sequence	Probability of the specific sequence	Probability of the generic sequence
BBBB	1/16	1/16
BBBR BBRB BRBB RBBB	1/16	$4 \times \frac{1}{16} = \frac{1}{4}$
BBRR BRBR RBBR RBRB RRBB BRRB	1/16	$6 \times \frac{1}{16} = \frac{3}{8}$
RRRB RRBR RBRR BRRR	1/16	$4 \times \frac{1}{16} = \frac{1}{4}$
RRRR	1/16	1/16
	sum = 1	sum =1

Note that the probability of each specific sequence is the same 1/16.

At this point, I mentioned the distinction between a *specific*, and a *generic* sequence only in the context of probability. As you can see from Table 1.1 (Note 11), except for the two extreme cases, the probability of the generic sequence is always larger than the probability of a specific sequence. Can you explain why?

Note also that each specific sequence has the same probability.[12]

Fifth game:

Next, I choose die *e*. You examine it, and as in the previous game, recall the rules of the game, and choose a color.

You realize that this case is "similar" to the case of die *c*. Which color will you choose now, and why? Can you estimate your net earnings in this case? What if you have chosen blue?

Sixth game:

Next, I choose die *f*, and you have to choose a color in order to play the same game with exactly the same rules.

Which color would you choose? Is this game similar to any of the previous games?

Seventh game:

I choose die *g*. By remembering the rules of the game, you should not have any problem in choosing the color which will