# The General Theory of Particle Mechanics

# The General Theory of Particle Mechanics:

*A Special Course*

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$SO \quad R$

$SU$

$SO \quad R \qquad SU$

$\times$

$SU$

*#refresh-your-brain*

*#gain-new-technology*

*#explore-new-world*

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# PART I

# MATHEMATICAL PRELIMINARIES

# 1.1. Author's Lecture

**Algebraic systems**

| | | |
|---|---|---|
| | | One binary operation |
| Magma (groupoid) | $a, b \in M$: | $a \bullet b \in M$ ← multiplication |
| Semi-group | $a, b, c \in S$: | $(a \bullet b) \bullet \mathrm{c} = a \bullet (b \bullet \mathrm{c}) \in S$ |
| Group | $a, b, c \in G$: | $(a \bullet b) \bullet \mathrm{c} = a \bullet (b \bullet \mathrm{c}) \in G$ |
| | $e \bullet a = a \bullet e = a$ | $a^{-1} \bullet a = a \bullet a^{-1} = e$ |
| Ring | $a, b \in R$: | $a \bullet b \in R$ ← multiplication |
| | Two binary operations | $a + b = b + a \in R$ ← addition |

**Algebra**

$x, y \in A$ $\alpha, \beta$ – scalars

structure of linear space

$$x + y = y + x \in A$$

$$\alpha(x + y) = \alpha x + \alpha y$$

$$(\alpha + \beta)\, x = \alpha x + \beta x$$

Ferdinand
Georg
Frobenius
1849 – 1917

Adolf Hurwitz
1859 – 1919

**Real numbers**

Field – Algebra – **R**

**R**:

Addition – commutative and associative

Neutral element (addition) 0

Multiplication – commutative and associative

Neutral element (multiplication) 1

Division exists

Dimension $D = 2^0 = 1$ (the only unit – scalar unit 1)

Distributive properties

$a \in \mathbf{R}$

Norm $\|a\| \to a^2$

Modulus $|a| = \sqrt{\|a\|} = +\sqrt{a^2}$

**R**

**C**

**Complex numbers** Field – Algebra – **C**

$$z = a + bi \in \mathbf{C}$$

Norm $\|z\| \to a^2 + b^2$

Modulus $|z| = \sqrt{\|z\|} = +\sqrt{a^2 + b^2}$

**C**:

Addition – commutative and associative

Neutral element (addition) 0

Multiplication – commutative and associative

Neutral element (multiplication) 1

Division exists

<u>Dimension</u> $D = 2^1 = 2$ (two scalar units – 1 and *i* )

Distributive properties

**C**

**Q**

**H**

## Quaternion numbers

Ring – Algebra – **Q**

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbf{Q}$$

Norm $\|q\| \to a^2 + b^2 + c^2 + d^2$

Modulus

$$|q| = \sqrt{\|q\|} = +\sqrt{a^2 + b^2 + c^2 + d^2}$$

**Q**:

Addition – commutative, associative

Neutral element (addition) 0

Multiplication – associative, non-commutative

Neutral element (multiplication) 1

Division exists

Dimension $D = 2^2 = 4$ four units: 1 (scalar) and ***i***, ***j***, ***k*** (vectors)

Distributive properties

**R**

**Q**

**C**

**Q**

**O**

**Octonions** Algebra – **O**


Sir Arthur Cayley
1821 - 1895

**O**:

Addition – commutative, associative

Neutral element (addition) 0

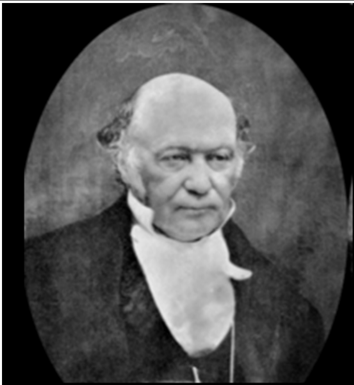
<u>Multiplication</u> – non-commutative, non-associative (“alternative”)

Neutral element (multiplication) 1

Division exists

<u>Dimension</u> $D = 2^3 = 8$ eight units: 1 (scalar) and ***7*** (vectors)

Distributive properties



William Rowan Hamilton
1806–1865


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for quaternion multiplication
and cut it on a stone of this bridge

## Hamilton's formula for multiplication of quaternion units

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

Three different imaginary units

real unit
(neutral element)

## 16 axiomatic equalities: multiplication of quaternion units

Hamilton's notation

Four Q-units: 1; **i**, **j**, **k**

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1·1 = 1

**i · i = j · j = k · k = - 1**

**i · j = - j · i = k**

**j · k = - k · j = i**

**k · i = - i · k = j**

1· **i** = **i** ·1 = **i**

1· **j** = **j** ·1 = **j**

1· **k** = **k** · 1 = **k**

Traditional multiplication table

| 1 | **i** | **j** | **k** |
|---|---|---|---|
| **i** | -1 | **k** | -**j** |
| **j** | -**k** | -1 | ***i*** |
| **k** | **j** | -**i** | -1 |

## A quaternion number

$$q = \underbrace{a \cdot 1}_{\text{Scalar part}} + \underbrace{b\mathbf{i} + c\mathbf{j} + d\mathbf{k}}_{\text{Vector part}}$$

Scalar part

Vector part

real coefficients

## Product of two quaternions

$$q_1 = a + b\mathbf{i} + c\mathbf{j} + d\,\mathbf{k} \qquad q_2 = e + f\,\mathbf{i} + g\,\mathbf{j} + h\mathbf{k}$$

$$q_1 q_2 = (a + b\mathbf{i} + c\mathbf{j} + d\,\mathbf{k})(e + f\,\mathbf{i} + g\,\mathbf{j} + h\mathbf{k}) =$$
$$= ae - bf - cg - dh + (af + eb + ch - dg)\,\mathbf{i} + (ag + ec + df - bh)\,\mathbf{j} + (ak + ed + bg - cf)\,\mathbf{k}$$

$$q_2 q_1 = (e + f\,\mathbf{i} + g\,\mathbf{j} + h\mathbf{k})(a + b\mathbf{i} + c\mathbf{j} + d\,\mathbf{k}) =$$
$$= ae - bf - cg - dh + (af + eb - ch + dg)\,\mathbf{i} + (ag + ec - df + bh)\,\mathbf{j} + (ak + ed - bg + cf)\,\mathbf{k}$$

$\mathbf{q}$ $b\mathbf{i}$ $c\mathbf{j}$ $d\mathbf{k}$
$\mathbf{q}$ $f\mathbf{i}$ $g\mathbf{j}$ $h\mathbf{k}$
$\mathbf{q}$ $\mathbf{q}$

## Addition of quaternions

$$q_1 = a + b\mathbf{i} + c\mathbf{j} + d\,\mathbf{k} \qquad q_2 = e + f\,\mathbf{i} + g\,\mathbf{j} + h\mathbf{k}$$

$$q_1 + q_2 = a + e + (b + f)\,\mathbf{i} + (c + g)\,\mathbf{j} + (d + h)\mathbf{k}$$

$\mathbf{q}$ $\mathbf{q}$ $a + e$ $b + f$ $\mathbf{i}$ $c + g$

$\mathbf{j}$ $\mathbf{k}$

## Quaternion conjugation

$$q = a + b\mathbf{i} + c\mathbf{j} + d\,\mathbf{k} \quad \Longrightarrow \quad \overline{q} = a - b\mathbf{i} - c\mathbf{j} - d\,\mathbf{k}$$

Norm

$$q\overline{q} = (a + b\mathbf{i} + c\mathbf{j} + d\,\mathbf{k})(a - b\mathbf{i} - c\mathbf{j} - d\,\mathbf{k}) = a^2 + b^2 + c^2 + d^2 = \|q\|$$

Modulus

$$|q| = \sqrt{q\overline{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

## Conjugate product of two quaternions

$$q_1 = a + b\mathbf{i} + c\mathbf{j} + d\,\mathbf{k} \qquad q_2 = e + f\,\mathbf{i} + g\,\mathbf{j} + h\mathbf{k}$$

$$\overline{q_1 q_2} = \overline{q}_2 \overline{q}_1$$

Homework!