Understanding Ocean Acoustics

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By

Jens M. Hovem and Hefeng Dong

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CHAPTER 1

AN INTRODUCTION TO UNDERSTANDING OCEAN ACOUSTICS

Understanding Ocean Acoustics is a modern textbook on the science and physics of ocean acoustics emphasizing applications and issues relevant to the ocean environment and aquatic life. Therefore, the focus is on low frequencies which are the most relevant to fish and sea mammals.

Central to the objectives is the use of theory and acoustic modelling. Although there are many models for propagation in the deep ocean environment, these are not appropriate in shallow waters with extensive bottom interactions. At low frequencies, the geoacoustic properties of the bottom cannot be ignored since the propagation loss is often dominated by shear wave conversion and requires knowledge also about wave propagation in solids. The book presents new and updated models for low frequency modelling of sound pressure and particle motions for range-independent and range-dependent scenarios where the water depth is a function of range.

The format and topics make the book suitable for use as a textbook for undergraduate and graduate students. The book is also useful for engineers, consultants, environmental scientists, regulators, and everybody else with a need and desire to know and understand sound propagation in the oceans.

The book is an interdisciplinary university-level textbook, written with the hope and ambition to bridge the gap between environmentalists, marine biologists, and acousticians, and thereby contribute to the advancement of ocean science.

1.1. Structure and organization

Understanding Ocean Acoustics gives a broad, condensed, and updated introduction to the science and physics of ocean acoustics emphasizing application and issues relevant to the ocean environment and aquatic life. Therefore, the focus is on low frequencies and shallow waters, most relevant to fish and sea mammals. This focus makes the book different from other books with similar titles.

The book has about 200 pages divided into 10 chapters; the first part provides the basic physics and mathematics of acoustic and elastic waves in fluids and solids. Elastic waves are included because shear wave conversion is important for low-frequency propagation, and man-made noise may severely impact all aquatic animals. The initial part summarizes the oceanographic and physical properties needed as input parameters to the propagation models used later. This book also introduces an advanced but practical and versatile model for a layered bottom with a fluid sediment layer over a solid half-space.

Propagation modelling is important for understanding the environmental effects on sound propagation in the oceans. The book introduces and discusses two approaches for acoustic propagation modelling. The first is ray theory and specifically the ray model PlaneRay, which has been developed over the years by Hovem and coworkers (2002). The model uses a unique sorting and interpolation algorithm to find all eigenrays that connect rays from a source to a target or receiver point at some distance and depth. The second approach is the wavenumber integration technique frequently used in acoustic modeling, especially in scenarios involving shear waves and shear wave conversion. Ray theory is essentially a high frequency approximation, and a central issue is the validity and accuracy at low frequencies. This is explored with the important conclusion that the PlaneRay implementation can be used at frequencies as low as 50 Hz, which is much lower than commonly believed and within the frequency range of interest in marine biology.

Acousticians are traditionally concerned with sound pressure. However, marine biologists have found that many species are more sensitive to particle motion than sound pressure. This is of significant interest for studies of how fish react to sounds. The book discusses the full complexity of the problem and analyzes the question of where in the wave field to expect to find high horizontal and vertical components of particle velocities.

1.2. Summary of chapter contents

Chapter 2 Acoustic waves in fluid and elastic media

This chapter provides the essential physics and mathematics of acoustic and elastic waves in fluids and solids, thereby laying the foundation for the understanding of the modelling developed in the book. Elastic waves are included because shear wave conversion is important for low frequency propagation, such as in passive sonar applications. Low frequency acoustics are used by fish and sea mammals for communication, and anthropogenic noise may severely impact all aquatic animals. The time and frequency domain solutions are developed in different coordinate systems and the Helmholtz equation is applied to solve wave equation with a source term.

Chapter 3 Oceanographic and physical properties

This chapter gives a summary on sound speed variations as a function of seasons, depth, and location. Examples of sound speed profiles for different regions and seasons are presented. The mechanisms for absorption in fresh water and saltwater are given by relaxation processes, and these are explained in this chapter.

Chapter 4 Reflection and transmission at ocean boundaries

The equations for reflection and transmission between a fluid and solid bottom are developed and important features such as critical angles are discussed. This chapter also introduces an advanced, but practical and versatile model for a layered bottom with a fluid sediment layer over a solid half-space.

Chapter 5 Acoustic transducers for sending and receiving

All practical use of acoustics requires sources and receivers of one sort or another. This chapter gives a short introduction to piezoelectric transducers used as receivers (hydrophones) and as projectors. The chapter describes basic beamforming and array processing methods used to achieve beam widths and side lobe levels to satisfy given requirements for power output and directivity.

Chapter 6 Acoustic propagation modelling

Propagation modelling is an important issue for understanding the environmental effects on sound propagation in the oceans. This chapter introduces and discusses two approaches for acoustic propagation modelling. The first model is PlaneRay, a ray model developed over the years by Hovem and coworkers (2002). The model finds and calculates all eigenrays connecting rays from a source to a target or receiver point at some distance and depth. Because of a unique sorting and interpolation algorithm, the eigenray determination of PlaneRay is arguably more accurate and effective than any other models described in the literature. The model is especially designed for eigenrays and eigenray analysis in range-dependent scenarios with any given sound speed profile in ocean waveguides with bottom reflecting and absorbing boundaries. The second approach considered in this chapter is the wavenumber integration technique, which is frequently used in acoustic modelling, especially in scenarios involving shear waves and shear wave conversion. Ray theory is essentially a high frequency approximation, and a central issue is the validity and accuracy at low frequencies. This is explored in this chapter with the important conclusion that the PlaneRay implementation can be used at frequencies much lower than commonly believed and within the frequency range of interest to marine biology.

Chapter 7 Eigenray analysis and particle motions

Traditionally acousticians are mostly concerned with sound pressure. But marine biologists have found that many species are more sensitive to particle motions in addition to sound pressure. This is of significant interest for studies of how fish react to sound. See Popper and Hawkins (2019) for a summary and overview. There is also a growing interest in particle motion driven by new developments and applications of sensor technologies. In this chapter, we first demonstrate that PlaneRay can produce plots and information of horizontal and vertical particle velocities. Secondly, we take a more thorough look at the problem of sound propagation near an elastic bottom using the wavenumber integration technique. We demonstrate the existence of Scholte waves propagating at both sides of the water/bottom interface. These waves are evanescent and characterized by strong particle movements in both horizontal and vertical directions. These waves are often ignored but could be harmful to aquatic life near or on the bottom.

Chapter 8 Sonar systems and signal processing

This chapter considers the principal components of sonar systems and discusses some limitations in performance that may occur due to environmental conditions. All relevant factors can be described by what is called sonar equations. The chapter presents the fundamentals of sonar signal processing for detecting echo signals in a background of noise and reverberation. This covers statistical parameters such as the probabilities of detection (*PD*) and false alarm (*PF*) and how these are connected in a *ROC* (Receiver Operating Characteristic) curve and the detection index. The discussion also includes elements of the signal processing technique using modulated pulses and matched filter processing to enhance the detection of weak echoes in noise.

Chapter 9 Measurement and analysis of acoustic noise from vessels

Radiated underwater sound from vessels is increasingly becoming a concern because acoustic noise affects the marine environment. This chapter discusses how measurements of radiated noise from vessels are conducted and analyzed. The first part describes modern spectral analyses based on the use of the Fast Fourier Transform (FFT). This may be a trivial issue, but there are important problems related to normalization and notation that are frequently overlooked in the literature. Third-octave frequency band analysis is used in nearly all reports concerning noise measurement, and narrow-band analysis is more useful and suited for resolving hidden spectral lines. Secondly, the chapter presents the results of noise measurements of a group of ocean-going trawlers, which demonstrate that spectral lines caused by resonances in the ship hulls are in the same low frequency bands that some fish have the highest sensitivity to. Also evident is the Lloyd mirror effect which is important for the propagation and can easily be confused with directivity.

Chapter 10 Case studies and propagation simulations

PlaneRay, with the unique sorting and interpolation usage, is ideally suited for range-dependent situations where the water depth changes with range. This chapter uses PlaneRay to illustrate sound propagation in the oceans with examples from experiments, field studies, and simulations. The first group of cases deals with sonars for surveillance, for object detection and with underwater communication using sonars frequencies. The second group of cases is relevant for low frequency applications, typically for seismic exploration and studies of propagation of man-made noise. Several examples are used to illustrate propagation over sloping bottoms and over seamounts. This chapter also presents a new geometrical spreading law combining spherical nearfield with a gradual transition to cylindrical spreading for range-independent scenarios and variation in water depth for range-dependent bathymetry.

CHAPTER 2

ACOUSTIC WAVES IN FLUID AND ELASTIC MEDIA

2.1. Introduction

This chapter is an introductory chapter on the derivation of wave equations for acoustic waves in fluids and solids. This material can also be found in many other books; especially recommended are the books by Kinsler et al. (2000), Pierce (2019), and Medwin and Clay (1998). The book by Duda and Pierce (2008) is an interesting description of the history of the development of environmental acoustics from 1960 to 2000. However, none of these books cover wave propagation in solids. Instead, this issue is covered in the book *Computational Ocean Acoustics* (2011) by Jensen et al., which is the main reference for this chapter and this book.

Acoustic waves are mechanical vibrations. When an acoustic wave passes through a medium, it causes local density changes that are related to local displacements of mass about the rest positions of the particles in the medium. This displacement leads to the formation of forces that act to restore the density to the equilibrium state and move the particles back to their rest positions. The medium may be a gas, liquid, or solid material.

Stress is the ratio of the average internal forces within a deformable body to the area over which that force is distributed. It is expressed in units of newtons/m^2. Strain is the relative change in the dimensions or shape of a body when subjected to stress. It is expressed as a ratio of the change in dimension to the original dimension. There are several types of strain. For example, tensile strain is expressed as the ratio of the change in length to its original length when subjected to stress. Volume strain is defined as the ratio of change in volume to the original volume when subjected to stress. Shear strain is defined as a change in angular deformation measured in radians. The main difference between fluid and solid media is that, in the former, only longitudinal waves can propagate, but the latter can support both longitudinal and transverse waves. In a longitudinal or compressional wave, particles of the medium are displaced only in the direction of the wave propagation, but in a solid medium, shear waves with particle displacement perpendicular to the direction of propagation can also exist. It should be noted that in seismic literature, solid media are often referred to as elastic media.

2.2. Stress and strains in isotropic fluid and solid media

Consider a small cubical element that is exposed to external forces or stresses. Stress is defined as force per unit of surface area and is a vector with components in the three directions *x, y,* and *z*. Furthermore, stresses can act in all three planes; therefore, there are nine different components of stress acting on an elementary element. These components are illustrated in Figure 2.1. For solid media, the derivation is similar but slightly more complicated. However, by making a few assumptions, which are nearly always valid for practical applications, the discussions are considerably simplified. The first assumption is of isotropy, i.e., that the elastic parameters are independent of directions. The second assumption is that of vertical polarization. The strains resulting from a stress, or a force per unit of surface area, on a small cubical volume element *dV* with sides of Δx, Δy, and Δz are shown in Figure 2.1.

In rectangular coordinates, the particle displacement **u**, a vector, has the components u_x, u_y, and u_z. The volume's relative expansions or compressions are expressed as

$$
\begin{aligned}
e_{xx} &= \frac{\partial u_x}{\partial x} \\
e_{yy} &= \frac{\partial u_y}{\partial y} \\
e_{zz} &= \frac{\partial u_z}{\partial z}
\end{aligned}
\tag{2.1}
$$

The relative change of volume, called dilatation, is denoted as Δ and defined as

$$\Delta = \frac{dV}{V} = e_{xx} + e_{yy} + e_{zz} \tag{2.2}$$

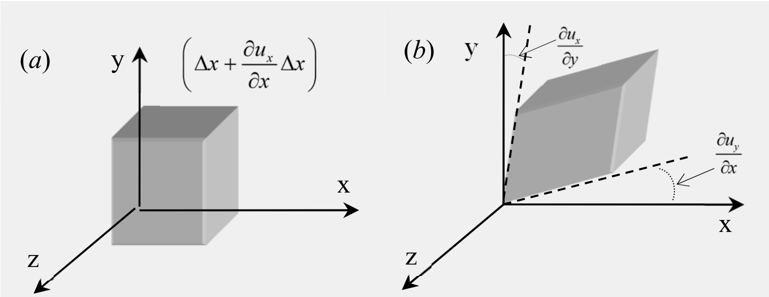

Figure 2.1. Strains, or the deforming effects of stress, in a cubical element: (*a*) volume strain caused by compression in the *x*-direction; (*b*) an example of shear strains without volume change.

The shear strains, which are also called shear deformations, are defined as

$$\begin{aligned} e_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ e_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ e_{xz} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \end{aligned} \tag{2.3}$$

The shear strain e_{xy} is the rotation of the cubic element in the *x-y* plane; e_{yz} and e_{xz} are the rotations in the *y-z* plane and the *x-z* plane, respectively. In these equations, the first index of each component indicates the direction in which the stress acts, while the second index gives the orientation of the plane in which the stress acts.

Assuming linear elasticity, the stresses σ_{ij} are related to the strains ε_{ij}, and because equilibrium requires that the shear strains have symmetric properties, $\sigma_{xy} = \sigma_{yx}$, $\sigma_{zy} = \sigma_{yz}$, and $\sigma_{xz} = \sigma_{zx}$. Therefore, a homogeneous, isotropic solid has six independent stress components.

In solid media, stresses have linear relationships to strains. If, in addition, the material is also isotropic (that is, the material has the same elastic properties in all directions), the elastic properties are completely described by just two independent elastic coefficients that relate a stress to the resultant strains. Commonly used elasticity coefficients are the Lamé coefficients λ and μ, Young's modulus E, the bulk modulus K, and the Poisson ratio ν. When the Lamé coefficients are used to describe these relations, the stress and its resultant strains may be defined

$$\begin{aligned}
\sigma_{xx} &= (\lambda + 2\mu)e_{xx} + \lambda e_{yy} + \lambda e_{zz} \\
\sigma_{yy} &= \lambda e_{xx} + (\lambda + 2\mu)e_{yy} + \lambda e_{zz} \\
\sigma_{zz} &= \lambda e_{xx} + \lambda e_{yy} + (\lambda + 2\mu)e_{zz} \\
\sigma_{xy} &= 2\mu e_{xy} \\
\sigma_{yz} &= 2\mu e_{yz} \\
\sigma_{zx} &= 2\mu e_{zx}
\end{aligned} \tag{2.4}$$

The connections between the different elasticity coefficients can be understood better by considering a thin, long, square rod subjected to an axial stress σ_{xx}. When all other stresses are equal to zero, equation (2.4) yields

$$\begin{aligned}
\sigma_{xx} &= (\lambda + 2\mu)e_{xx} + \lambda e_{yy} + \lambda e_{zz} \\
0 &= \lambda e_{xx} + (\lambda + 2\mu)e_{yy} + \lambda e_{zz} \\
0 &= \lambda e_{xx} + \lambda e_{yy} + (\lambda + 2\mu)e_{zz}
\end{aligned} \tag{2.5}$$

Young's modulus E is defined as the ratio between the axial stress and the relative axial extension. After solving equation (2.5) Young's modulus becomes

$$E = \frac{\sigma_{xx}}{e_{xx}} = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)} \tag{2.6}$$

The Poisson ratio ν is defined as the ratio between the transverse strains, where $e_{yy} = e_{zz}$, and the axial strain e_{xx}, so that

$$\nu = -\frac{e_{yy}}{e_{xx}} = \frac{\lambda}{2(\lambda + \mu)} \tag{2.7}$$

By definition, a fluid has $\mu = 0$. Consequently, the Poisson ratio for a fluid is $\nu = 0.5$, which can be seen from equation (2.7).

The bulk modulus K, or the volume stiffness, is defined by the relative change of volume when a material is exposed to hydrostatic stresses, which means that $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$. These values are inserted into equation (2.4), giving

$$\sigma = \left(\lambda + \frac{2}{3}\mu\right)\left(e_{xx} + e_{yy} + e_{zz}\right) \tag{2.8}$$

It then follows that

$$\sigma = \left(\lambda + \frac{2}{3}\mu\right)\Delta \tag{2.9}$$

Consequently, the bulk modulus K may be expressed as

$$K = \frac{\sigma}{\Delta} = \lambda + \frac{2}{3}\mu \tag{2.10}$$

The plane wave modulus H may be defined either as

$$H = \lambda + 2\mu \tag{2.11}$$

or as

$$H = K + \frac{4}{3}\mu \tag{2.12}$$

2.3. Derivation of the wave equations

The wave equation describing acoustic waves in fluids is based on three simple principles:

- The momentum equation, also known as Euler′s equation
- The continuity equation, or the conservation of mass
- The equation of state, i.e., the relationship between changes in pressure and density or volume

The detailed derivation for the fluid case can be found in most textbooks and shall not be repeated here, but the result is that the acoustic wave equation expressed for sound pressure is

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{2.13}$$

In equation (2.13), p is the sound pressure, i.e. the deviation of pressure around the ambient static pressure, ∇^2 is the Laplacian operator, and the sound speed c_0 – the square root of the ratio between volume stiffness and density – is given by

$$c_0 = \sqrt{\frac{K_0}{\rho_0}} \tag{2.14}$$

Equation (2.13) is the acoustic wave equation for sound pressure, but it is also common to express the wave equation in terms of a particle velocity potential ϕ defined by

$$\mathbf{v} = \nabla \phi \tag{2.15}$$

This results in the wave equation for the velocity potential

$$\nabla^2 \phi - \frac{1}{c_0^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{2.16}$$

such that the sound pressure p is given by

$$p = -\rho_0 \frac{\partial \phi}{\partial t} \tag{2.17}$$

The derivation of the wave equation for solids follows the same principle but is slightly more complicated since there are also shear components. It is often convenient to express the particle displacement vector as two potential functions, a scalar potential ϕ and a vector potential $\boldsymbol{\psi}$**,** so that the particle displacement vector is expressed as

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\Psi} \tag{2.18}$$

The waves are expressed with two wave equations

$$\rho \frac{\partial^2 \phi}{\partial t^2} = \left(\lambda + 2\mu\right) \nabla^2 \phi \tag{2.19}$$

$$\rho \frac{\partial^2 \mathbf{\Psi}}{\partial t^2} = \mu \nabla^2 \mathbf{\Psi} \tag{2.20}$$

Equations (2.19) and (2.20) *for* propagation of mechanical waves correspond to Maxwell's equations for electromagnetic waves.

From equations (2.19) and (2.20), we conclude that the scalar potential ϕ propagates at the compressional wave speed c_p, defined as

$$c_p = \sqrt{\frac{\left(\lambda + 2\mu\right)}{\rho}} = \sqrt{\frac{H}{\rho}} \tag{2.21}$$

and that the vector potential $\mathbf{\Psi}$ propagates with the shear wave speed c_s, defined as

$$c_s = \sqrt{\frac{\mu}{\rho}} \tag{2.22}$$

Inserting the respective wave speeds into equations (2.19) and (2.20), respectively, gives

$$\nabla^2 \phi - \frac{1}{c_p^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{2.23}$$

$$\nabla^2 \mathbf{\Psi} - \frac{1}{c_s^2} \frac{\partial^2 \mathbf{\Psi}}{\partial t^2} = 0 \tag{2.24}$$

Using the Poisson relationship, the ration between the two wave speeds defined by equations (2.21) and (2.22) is expressed by

$$\frac{c_s}{c_p} = \sqrt{\frac{1 - 2\nu}{2\left(1 - \nu\right)}} \tag{2.25}$$

Equations (2.19) and (2.20) are the two wave equations relevant to acousto-elastic wave propagation in an isotropic elastic medium. Note that these equations are mutually independent – compressional waves do not affect the shear waves, and *vice versa*. However, this independence pertains only to waves in homogeneous media. In an inhomogeneous medium with space-dependent parameters (at the interface between two different media, for instance) conversions between compressional waves and shear waves take place.

In two-dimensional cases, the particle motions are in the *x-z* plane and there is no *y*-plane dependence. Shear waves that are polarized with the particle motion in the *x-z* plane are called vertically polarized shear waves or *SV* waves.

When $u_y = 0$ and the derivatives of all quantities with respect to *y* are zero (which we denote by the shorthand $\partial/\partial y = 0$), the vector potential $\boldsymbol{\Psi}$ has only a *y*-plane component Ψ_y, which, for notational simplicity, we will denote as Ψ. The two components of the particle displacement in equation (2.18) then become

$$\begin{aligned} u_x &= \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \\ u_z &= \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \end{aligned} \tag{2.26}$$

Equation (2.4) defines the stresses. When applied to the equations in (2.26), the stresses can be expressed by the potentials

$$\begin{aligned} \sigma_{xx} &= \rho \frac{\partial^2 \phi}{\partial t^2} - 2\mu \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right) \\ \sigma_{zz} &= \rho \frac{\partial^2 \phi}{\partial t^2} - 2\mu \left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial z} \right) \\ \sigma_{xz} &= \rho \frac{\partial^2 \psi}{\partial t^2} + 2\mu \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial z^2} \right) \\ &= -\rho \frac{\partial^2 \psi}{\partial t^2} + 2\mu \left(\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} \right) \end{aligned} \tag{2.27}$$

In the following, equations (2.26) are used for displacements and equations (2.27) for the stresses.

Sources of underwater sound are produced by the artificial or natural injection of mass. This can be accounted for by adding a source term to the wave equation (2.16)

$$\nabla^2\phi - \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} = -f(\mathbf{r},t) \tag{2.28}$$

Transformation to the frequency domain using equation (2.28) results in an inhomogeneous wave equation in the frequency domain expressed as

$$\left[\nabla^2 + \kappa^2(\mathbf{r})\right]\Phi(\mathbf{r},\omega) = -F(\mathbf{r},\omega) \tag{2.29}$$

where $F(\mathbf{r},\omega)$ is the Fourier transform of the source function $f(\mathbf{r},t)$ in the time domain and the wave number $\kappa(\mathbf{r})$ is defined as

$$\kappa(\mathbf{r}) = \frac{\omega}{c_0(\mathbf{r})} \tag{2.30}$$

Equation (2.29) is the wave equation in the frequency domain and is also referred to as the inhomogeneous Helmholtz equation, which is often easier to solve than the corresponding wave equation in the time domain, something we will make extensive use of later. In equation (2.29), we have included a source term and allowed for the possibility that the sound speed is dependent on the co-ordinates **r**.

2.4. Solutions in time and frequency domains

The wave equations take on different forms depending on the problem and the coordinates used. The Laplace operator in rectangular coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.31}$$

It can be shown that the general solution for a plane wave traveling in an arbitrary direction given by **r** is

$$\phi(\mathbf{r},t) = \phi_+\left(t - \frac{|\mathbf{r}|}{c_0}\right) + \phi_-\left(t + \frac{|\mathbf{r}|}{c_0}\right) \tag{2.32}$$

The first part of equation (2.32) represents a wave propagating with speed c_0 in the positive **r** direction. The second part of the equation represents a wave in the negative **r** direction. Both waves propagate without any change of amplitude or form, and the wave equation imposes no conditions upon their shapes and forms. Since the waves are plane, a coordinate system with directions concurring with one of the axes, for instance the *x*-axis, is used. In this case, equation (2.32) becomes

$$\phi(x,t) = \phi_+\left(t - \frac{x}{c_0}\right) + \phi_-\left(t + \frac{x}{c_0}\right) \tag{2.33}$$

The sound pressure and particle velocity in the *x* direction are correspondingly

$$\begin{aligned} p(x,t) &= -\rho_0 \frac{\partial \phi(x,t)}{\partial t} = -\rho_0 \left[\phi_+'\left(t - \frac{x}{c_0}\right) + \phi_-'\left(t + \frac{x}{c_0}\right)\right] \\ v_x(x,t) &= \nabla_x\left[\phi(x,t)\right] = -\frac{1}{c_0}\left[\phi_+'\left(t - \frac{x}{c_0}\right) - \phi_-'\left(t + \frac{x}{c_0}\right)\right] \end{aligned} \tag{2.34}$$

Here, the ′ symbol stands for the derivative with respect to time *t.* Notice the minus sign in the expression for the particle velocity for the wave component traveling in the negative direction.

The ratio between sound pressure and particle velocity (which may be complex), called the specific acoustic impedance, *Z*, is equal to the product of the medium's density and its sound speed. For a plane wave propagating in the positive *x* direction

$$Z = \frac{p(x,t)}{v_x(x,t)} = \rho_0 c_0 \tag{2.35}$$

Plane waves with components propagating in both directions can be written as

$$
\begin{aligned}
p(x,t) &= p_+\left(t-\frac{x}{c_0}\right)+p_-\left(t+\frac{x}{c_0}\right)\\
v_x(x,t) &= \frac{p_+}{Z}\left(t-\frac{x}{c_0}\right)-\frac{p_-}{Z}\left(t+\frac{x}{c_0}\right)
\end{aligned}
\tag{2.36}
$$

where p_+ is the pressure wave propagating in the positive x direction and p_- is the pressure wave propagating in the negative x direction.

The equations we have considered so far describe wave propagation in the time domain; now we extend the discussion to describing the propagation in the frequency domain, which normally is easier to solve. The transformation of a time signal $f(t)$ to a signal $F(\omega)$ in the frequency domain and back to the time domain is accomplished by using the Fourier transformation pair

$$F(\omega)=\int_{-\infty}^{\infty} f(t)\exp(i\omega t)\,dt \tag{2.37}$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(-i\omega t)\,d\omega \tag{2.38}$$

Note that this sign convention is usual in the literature on physics but is opposite to the sign convention commonly used in signal processing literature. Analysis in the frequency domain means that we assume a solution of the wave equation in the form $F(\omega)\exp(-i\omega t)$. After this solution is found, the solution in the time domain, and consequently the time response, is found by using the inverse transformation in equation (2.38).

The specific acoustic impedance for a spherical wave is

$$Z=\frac{p(r,t)}{v(r,t)}=\rho_0 c_0\frac{i\kappa r}{(i\kappa r-1)} \tag{2.39}$$

The impedance can be split into a real resistive part and an imaginary reactive part:

$$Z = \rho_0 c_0 \frac{(\kappa r)^2}{(\kappa r)^2 + 1} - i\rho_0 c_0 \frac{\kappa r}{(\kappa r)^2 + 1} \tag{2.40}$$

Close to the source, $\kappa r << 1$, and the reactive part dominates, whereas at far distances, $\kappa r >> 1$, the resistive part dominates, and the spherical wave impedance becomes the plane wave impedance $\rho_0 c_0$.

Equation (2.40) shows that the acoustic impedance of a spherical wave is a function of its distance from the source. However, we do not have to go far from the source before the wave may be considered locally plane. This is the case when $\kappa r >> 1$, that is, when the distance is considerably greater than the wavelength. Then the second term in equation (2.40) may be ignored and the impedance can be given as $Z = \rho_0 c_0$. At a distance r_0 satisfying this condition, the intensity consequently becomes

$$I(r_0) = \frac{p_0^2}{\rho_0 c_0} = v_0^2 \rho_0 c_0 \tag{2.41}$$

Note that here we have redefined p_0 and v_0 as the rms values of the amplitudes of the sound pressure and particle velocity evaluated at r_0.

The total acoustic power of the source, W, is found by integrating over the surface of the sphere, so that W is defined as

$$W = 4\pi r_0^{\,2} \frac{p_0^2}{\rho_0 c_0} = 4\pi r_0^{\,2} v_0^2 \rho_0 c_0 \tag{2.42}$$

Intensity I is defined as the time-averaged energy flux per unit surface normal to the direction of propagation and is measured in power per square meter. For plane harmonic waves and complex notation, the time-averaged intensity is given by the real parts of the quantities $p(x,t)$ and $v(x,t)$ as

$$I = \frac{1}{T}\int_0^T \mathrm{Re}\left[p(x,t)\right]\mathrm{Re}\left[v(x,t)\right]dt \tag{2.43}$$

This result can also be expressed as (* denotes complex conjugate)

$$I = \frac{1}{2}\mathrm{Re}[pv^*] \tag{2.44}$$

which gives

$$I = \frac{1}{2} p_0 v_0 \tag{2.45}$$

The intensity of a plane wave in the direction of propagation can be expressed by

$$I = \frac{1}{2} \rho_0 c_0 v_0^2 = \frac{1}{2} \frac{p_0^2}{\rho_0 c_0} \tag{2.46}$$

In these equations the intensities are represented by the amplitudes of the particle velocity v_0 and the sound pressure p_0, and therefore the factor is ½. If the root mean squared (rms) of these quantities is used, then the factor ½ disappears. This is common practice, and we use this convention in the following. The intensity $I(r)$ at an arbitrary distance r is expressed as

$$I(r) = I(r_0) \left(\frac{r_0}{r} \right)^2 \tag{2.47}$$

In practice, the decibel scale is often used, which is ten times the logarithm to the base 10 of a power or intensity ratio

$$10 \log I(r) = 10 \log I(r_0) - 20 \log (r/r_0) \tag{2.48}$$

Instead of specifying the strength or power level of an acoustic wave by using the intensity, it is standard practice to characterize the level by the equivalent sound pressure, given that the impedance is $Z = \rho_0 c_0$. The squared sound pressure p^2 at a distance r from a sound source radiating an output of W watts under spherical spreading conditions is

$$p^2 = \frac{\rho_0 c_0 W}{4 \pi r^2} \tag{2.49}$$

Equation (2.49) is expressed in decibels as

$$20 \log p = 10 \log \left(\frac{\rho_0 c_0}{4 \pi} \right) + 10 \log W - 20 \log r \tag{2.50}$$

In underwater acoustics, the sound level is normally not defined by intensity but rather by sound pressure, which is measured in dB relative to a reference pressure. In modern literature, the common reference is 1 μPa (1 micro-Pascal), as used in the second expression of equation (2.51). It is important to note that the standard reference pressures in underwater acoustics and in air acoustics are not the same. The standard pressure reference in underwater acoustics is 1 μPa while the standard refence for air-borne acoustics is 20 μPa. The decibel unit is used to compare differences of like quantities of sound, usually in intensities or power. The decibel is not an absolute unity with a physical dimension; it is a relative unit that expresses sound logarithmically. The term "decibel" is useless unless the standard of comparison is cited. It is important to remember the acoustic impedance of air is about 3600 times lower than the impedance of water. Consequently, the numerical factor becomes approximately 35.5 dB lower in air than in water.

With c_0 = 1500 m/s and ρ_0 = 1000 kg/m^3, the numerical value of the first term in equation (2.50) is approximately 50.8, so that the sound pressure at a distance r from an omnidirectional source that transmits W watts acoustically into the water with the same intensity in all directions becomes

$$\begin{aligned} 20\log p &= 50.8 + 10\log W - 20\log r \quad && \text{dB re 1 Pa} \\ 20\log p &= 170.8 + 10\log W - 20\log r \quad && \text{dB re 1 } \mu\text{Pa} \\ 20\log p &= 70.8 + 10\log W - 20\log r \quad && \text{dB re 1 } \mu\text{bar} \end{aligned} \qquad (2.51)$$

The units are related as follows: 1 μbar = 1 dyne/cm^2 = 0.1 Pa, and 1 μPa = 10^{-6} Pa.

A very high sound pressure in water is one that equals one atmospheric pressure, that is, $p_0 = 10^5$ Pa. Sound pressure of this magnitude is only found in the immediate proximity of a powerful sonar. With this sound pressure, the amplitude of the particle velocity becomes $v_0 = p_0/(\rho_0\, c_0)$ = 0.067 m/s. For a harmonic wave at a frequency of 1 kHz, the particle displacement in this case becomes $v_0/\omega \approx 10^{-5}$m. The intensity is $I = 0.5 p_0 v_0 = 0.33\cdot 10^4$ W/m^2, or I = 0.33 W/cm^2. Note that the particle velocity is considerably smaller than the sound speed and that the particle displacement is very small, even under such high sound pressure.

2.5. The wave equation for modelling

Most problems we encounter in ocean acoustics can be defined in cylindrical coordinates in depth z and radial range r. Equation (2.29) without source term, that is the homogeneous Helmholtz equation, becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\left(\frac{\partial^2 \Phi}{\partial z^2}\right)+\kappa(z)^2\Phi=0 \qquad (2.52)$$

Assuming that the sound speed is a function of depth only and not dependent on range we can assume that the solution is a product of two functions that depend only on the depth and radial distance, respectively. This means assuming a solution of the form

$$\Phi(r,z,\omega)=\Phi_r(r,\omega)\Phi_z(z,\omega) \qquad (2.53)$$

Inserting equation (2.53) into equation (2.52) gives

$$-\frac{1}{\Phi_r(r,\omega)}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\left[\Phi_r(r,\omega)\right]}{dr}\right)\right]=\frac{1}{\Phi_z(z,\omega)}\left[\frac{d^2\left[\Phi_z(z,\omega)\right]}{dz^2}\right]+\kappa(z)^2 \qquad (2.54)$$

The right-hand side of equation (2.54) depends only on z and the left-hand side depends only on r. Both sides must therefore be equal to the same constant, which we set equal to k^2, and which can be referred to as the separation constant. We will soon discover that k is the horizontal wave number. We therefore get two separate differential equations.

The depth equation

$$\frac{d^2\Phi_z(z,\omega)}{dz^2}=-\left(\kappa(z)^2-k^2\right)\Phi_z(z,\omega) \qquad (2.55)$$

The range equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi_r(r,\omega)}{dr}\right)=-k^2\Phi_r\left(r,\omega\right) \qquad (2.56)$$

The depth equation can be rewritten as

$$\frac{d^2\Phi_z(z,\omega)}{dz^2} = -\gamma^2(z)\Phi_z(z,\omega) \tag{2.57}$$

since $c(z)$ is a function of depth only. The vertical wave number $\gamma(z)$ in the z direction is defined by equation (2.55) and given as

$$\gamma^2(z) = \kappa^2(z) - k^2 = \left(\frac{\omega}{c_0(z)}\right)^2 - k^2 \tag{2.58}$$

This relationship between the horizontal and vertical wave numbers can be depicted geometrically in Figure 2.2.

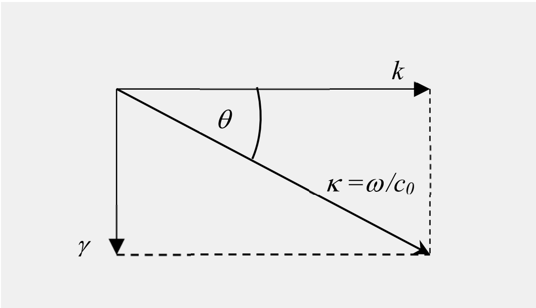

Figure 2.2. Wave number components. The component in the horizontal r direction is k, while γ is the component in the z direction, and $\kappa = \omega/c_0$.

The range equation (2.56) is the Bessel equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi_r}{\partial r}\right) + k^2\Phi_r = 0 \tag{2.59}$$

The solution to equation (2.59) is most conveniently expressed by the Hankel function

$$\Phi_r = AH_0^{(1)}(kr) + BH_0^{(2)}(kr) \tag{2.60}$$

In this form the first term represents an outgoing wave and the second term an ingoing wave, as is clear from the asymptotic form of the Hankel function for kr going to infinity

$$H_0^{(1)}(kr) \approx \sqrt{\frac{2}{\pi kr}} \exp\left(ikr - i\frac{\pi}{4}\right)$$
$$H_0^{(2)}(kr) \approx \sqrt{\frac{2}{\pi kr}} \exp\left(-ikr + i\frac{\pi}{4}\right) \tag{2.61}$$

We see that the two equations express the waves decaying proportionally with $r^{-1/2}$, which signifies cylindrical spreading. In the case of an omni-directional point source, the field only depends on the range from the source with the solution

$$\Phi = A\frac{\exp(i\kappa r)}{r} + B\frac{\exp(-i\kappa r)}{r} \tag{2.62}$$

This represents two outgoing and ingoing spherical waves with amplitude decaying proportionally with r, which is called spherical spreading. Both spherical and cylindrical spreading is often referred to as geometrical spreading

For the special case in which the sound speed is constant with depth, the solution of equation (2.57) has the form

$$\Phi_z(k,z,\omega) = A_+ \exp(i\gamma z) + A_- \exp(-i\gamma z) \tag{2.63}$$

The solution in equation (2.63) represents a downward (A_+) and an upward (A_-) propagating plane wave. The amplitudes A_+ and A_- are given by the boundary conditions, and the vertical wave number γ is given by

$$\gamma = \sqrt{\kappa^2 - k^2} \qquad k \le \kappa$$
$$\gamma = i\sqrt{k^2 - \kappa^2} \qquad k > \kappa \tag{2.64}$$

Note that when the wave number is imaginary, the sign is selected to make $\Phi_z(k,z,\omega)$ decay with increasing depth.

Consider a point source at $(r, z) = (0, z_s)$ in cylindrical coordinates in equation (2.29) and equation (2.57) becomes

$$\frac{d^2\Phi_z(z,\omega)}{dz^2} + \gamma^2(z)\Phi_z(z,\omega) = \frac{S(\omega)\delta(z - z_s)}{2\pi} \tag{2.65}$$