Vibration in Mechanical Systems

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By

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To Kem, Leighton, Jane and Anita

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PREFACE

This book is in the form of lecture notes and was originally created for two semesters of 6 credit hours courses taken by senior undergraduates and first year graduate students majoring in mechanical engineering. The proposed book initially consisted of 15 chapters. They are based on the lecture notes for a required course with similar titles given to junior (and occasionally senior) undergraduate students by the author in the Department of Mechanical Engineering at the University of Calgary from 1981 and at the University of Nebraska, Lincoln since 1996. The emphasis is on fundamental concepts, theory, analysis and design of mechanisms with applications. While it is aimed at senior undergraduates and first year graduate students majoring in mechanical engineering, it is also suitable for senior undergraduates and first year graduates in biological system engineering, ocean engineering, naval architectural engineering, and aerospace engineering.

The original framework of this book was intended to draft for four parts and fifteen chapters in 2018 and its proposed delivery date was October 30, 2020. The title of the proposed book would be: **Vibration, Dynamics, Stability and Bifurcation in Mechanical Systems**. The four parts included: **Part I. Vibration in Mechanical Systems, Part II. Balance, Dynamics in Rotors, Engines, and Machines, Part III. Contact Dynamics in Machinery, and Part IV. Stability and Bifurcation Analysis in Mechanical Systems.**

By February 2020. **Part I. Vibration in Mechanical Systems** had mostly been completed while the author was still active. Now the book is named after Part I. **Vibration in Mechanical Systems**. Owing to the absence of the original author, the author's excellent writing habit of careful and detailed styles might meet challenge in the production of this book.

The author's first book **Nonlinear Random Vibration** was published in 2000. **Vibration in Mechanical Systems** will become his 8th book to be published.

Cho W. S. To

ACKNOWLEDGEMENTS

Vibration in Mechanical Systems was intended to write up the author's knowledge and experience in mechanical engineering based on the courses he taught over thirty-five years, from 1981 in the Department of Mechanical Engineering at the University of Calgary and until his retirement in 2017 from the Department of Mechanical & Materials Engineering at the University of Nebraska, Lincoln. The author and his family sincerely thank many colleagues and friends for their sincere wishes for getting this book published.

The author's family is grateful for the very first encouragement from Professor Neil Popplewell, who is the author's alumus at the Institute of Sound and Vibration Research, University of Souththampton. Professor Popplewell's introduction of his former student was a good start. Many thanks to Professor K. Y. Sze for similar encouragement of getting the book to this stage. Thanks are also due to the author's former students, V. M. Kaladi for allowing the use of figures and tables from his MSc thesis and Jiming Fu, who prepared some drawings for Chapters 2 and 6.

Finally, the author would like to express his gratitude thank the Cambridge Scholars Publishing for their support in the publishing process.

Cho W. S. To

CHAPTER 1

INTRODUCTION

1.1 Background

Vibration or oscillation often refers to the time-varying motion about an equilibrium configuration which is often the stopping configuration of the vibration. Vibrations are important in our daily life. For instance, swings are popularly provided in playgrounds, mechanical clocks and watches keep time by making use of the constant time intervals or periods of the vibrations in their components, vibrations in the musical instruments lead to music, we hear because our eardrums vibrate, etc. Not all vibrations are desirable. Vibrations of land vehicles over bumpy roads and marine vehicles over rough seas may induce motion sickness. Transverse vibrations of high-rise and long-span structures under strong wind can be scary. Excessive vibrations can lead to screw loosening and material fatigue failures which may be catastrophic to the pertinent machines and structures.

1.2 Organization of Presentation

This book is organized into six chapters. The remaining chapters will cover the following content on mechanical vibrations:

- Chapter 2 mentions that the elements capable of holding the kinematic and potential energies are required for mechanical vibrations. The equations of motion for some simple examples of vibration are derived from the energy consideration. Damping which is inevitable in real-world vibrations is also introduced.
- Chapter 3 presents free and forced vibrations of single-degree-of-freedom systems. The equations of motion are derived from not only energy consideration but also Newton's Second Law of Motion. The method for analyzing forced vibrations under non-harmonic and non-periodic excitation forces is introduced.
- Chapter 4 is involved with free and forced vibrations of two-degrees-of-freedom systems. It starts from the forced harmonic vibration and

dynamic absorber. Normal modes and normal mode analysis for forced vibration systems under proportional damping are then introduced. Non-proportional damping is considered by the state vector approach.

- Chapter 5 deals with free and forced vibrations of many degrees of freedom systems. It introduces the Lagrange's equations approach as a generalization of the energy consideration for deriving equations of motion. General treatments of the normal mode analysis and the space vector approach for many degrees of freedom systems are presented.
- Chapter 6 is concerned with continuous systems. Unlike the discrete systems considered in the previous chapters, kinematic and potential energies are distributed in continuous systems which possess an infinite number of degrees of freedom. Free and forced vibration analyses of continuous systems including cables, rods and beams are presented.

CHAPTER 2

OSCILLATORY MOTION

Vibration or oscillation is a continuous exchange of kinematic and potential energy. In Section 2.1 the basic elements capable of storing kinematic and/or potential energy are introduced. The equations of motion for, perhaps, the two simplest vibration systems formed by these elements are derived from energy conservation. Section 2.2 introduces degrees of freedom, discrete systems and continuous systems. Section 2.3 reviews the common forms of damping which dissipate the energy in vibrating systems. In Section 2.4 harmonic and periodic motions are defined. The mathematical procedure for expressing a periodic function into a compound harmonic function is also introduced. Section 2.5 gives a simple example of non-periodic oscillatory motion.

2.1 Elements of Oscillatory Systems

To enable mechanical vibrations, one needs at least a means to store kinetic energy T and a means to store potential energy U. Vibration is a continuous exchange of kinetic energy to and for potential energy. While kinematic energy is held by an object because of its speed, potential energy is held by an object because of its position or deformation.

In mechanical vibrations, the two major forms of potential energy are gravitation energy and elastic energy. From high school physics, we know that when a mass m is raised to a height h above the ground level, the gravitational energy it holds relative to the ground level is mgh where g is the acceleration due to gravity. On the other hand, when a spring with spring constant k is elongated by e relative to its unstretched length, the elastic energy it holds is $ke^2/2$.

In the classical spring-trolley system shown in Figure 2.1(a), a rigid trolley of mass m on light wheels is attached to a rigid wall by a light spring of spring constant k. The upper half of the figure shows the system in which the spring assumes its unstretched length l. The lower half of the figure shows the system in which the trolley is displaced by x towards the right and x would also be the spring extension. Thus, the kinetic and potential energies are

$$T = \frac{1}{2}m\dot{x}^2 \quad \text{and} \quad U = \frac{1}{2}kx^2$$

As usual, a quantity with one and two overdots denote the first and second time derivatives of the quantity under the dot, respectively. In the absence of energy dissipation, the sum of the two energies is a constant, i.e.

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = constant \tag{2.1}$$

The time derivative of the above equation is

$$(kx + m\ddot{x})\dot{x} = 0$$

The trivial solution of $\dot{x} = 0$ does not involve any motion. Our concern is

$$\ddot{x} + \frac{k}{m}x = 0 \tag{2.2}$$

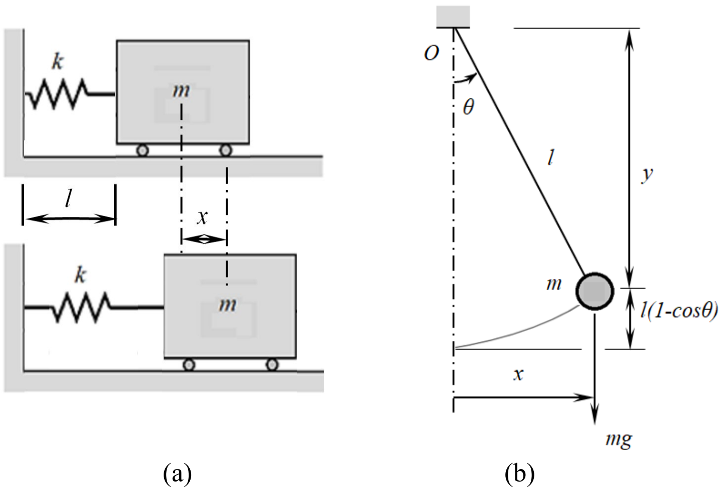


Figure 2.1: (a) A trolley of mass *m* on a smooth horizontal floor, *l* is the unstretched length of the spring. (b) A simple pendulum, *l* is the length of the inextensible string.

The solution to the equation of motion in Equation (2.2) is

$$x = A \sin \omega t + B \cos \omega t \tag{2.3}$$

or, equivalently,

$$x = X \sin(\omega t + \emptyset) \tag{2.4}$$

in which t denotes time, $\omega = \sqrt{k/m}$ is known as the angular velocity of the motion, $X = \sqrt{A^2 + B^2}$ is the amplitude of motion and $\tan \emptyset = B/A$. (A, B) or $(X, \emptyset)$ can be determined by the initial values, i.e. the values at $t = 0$, of x and $\dot{x}$. Equation (2.4) is plotted in Figure 2.2. It can be seen that x oscillates about $x = 0$ and repeats itself at a fixed time interval of $\tau = 2\pi/\omega$ which is known as the period of the vibration. Furthermore, x becomes zero when $\omega t + \emptyset = 0, \pi, 2\pi, \ldots$, etc.

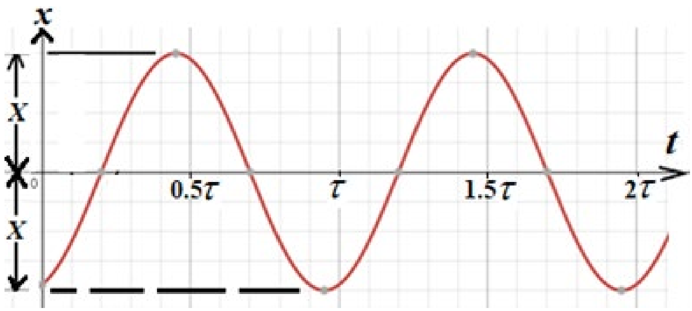


Figure 2.2: A graphical illustration for $X \sin(\omega t + \emptyset)$ where $\tau = 2\pi/\omega$.

Figure 2.1(b) shows the simple pendulum which is another simplest vibration system. A small mass m is tied to a fixed point above it by an inextensible light and taut string of length l. The mass oscillates about the fixed point. When the string makes an angle θ to the vertical, the gravitation energy of the mass relative to its lowest point is

$$U = mgl(1 - cos\theta).$$

On the other hand, the speed of the mass is $l\dot{\theta}$. Thus, the kinematic energy is

$$T = \frac{1}{2} m(l\dot{\theta})^2.$$

Assuming no energy dissipation, the sum of the two energies is a constant, i.e.

$$mgl(1-cos\theta)+\frac{1}{2}m(l\dot{\theta})^2=constant$$

and the time derivative of the energy sum is

$$m(l\dot{\theta})(l\ddot{\theta})+mgl\,sin\theta\,\dot{\theta}=0$$

The trivial static solution given by $\dot{\theta}$ = 0 can be rejected. By adopting the small-angle approximation $sin\theta \approx \theta$, the above equation reduces to

$$\ddot{\theta}+\frac{g}{l}\theta=0. \tag{2.5}$$

Equation (2.5) is of the same form as Equation (2.2) and its solution can again be expressed as

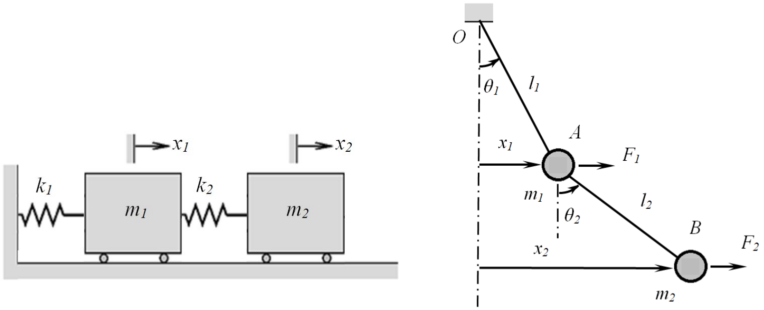
$$\theta=\Theta\sin(\omega t+\emptyset) \tag{2.6}$$

where ω equals $\sqrt{g/l}$ is the angular velocity. Thus, the period of the vibration is $2\pi\sqrt{l/g}$.

2.2 Degrees of Freedom in Mechanical Systems

Degrees of freedom (dofs) are often geometric parameters such as coordinates, displacements and angles of inclination used to describe the time-varying configuration of a vibrating system and, thus, the configurations of all its components. A dof can be dependent or independent. Taking the simple pendulum in Figure 2.1(b) as an example, the horizontal displacement x of the point mass from its lowest point, given by $\theta = 0$, can be taken as a dof. However, one can note that $x = l \sin\theta$. If θ is taken as the independent dof, x will be a dependent dof and vice versa. Unless specified otherwise, dofs often refer to independent dofs. Based on the number of dofs it possesses, a vibration system can be classified as a single dof (sdof), a two dof (tdof) or a many dof (mdof) system. While the vibration systems in Figure 2.1 are sdof systems, those in Figure 2.3 are their generalizations and they are tdof systems.

The systems portrayed in Figures 2.1 and 2.3 are also known as discrete vibration systems in the sense that the configuration and, thus, the energies of the system can be fully specified by a finite number of dofs. In these systems, the components carrying the kinematic and elastic energies are typically taken to be rigid and light, respectively. Contrary to the discrete system is the continuous system in which the configuration and, thus, the energies of the system cannot be fully specified by a finite number of dofs.



(a) (b)

Figure 2.3: (a) A two-trolley system. (b) A double pendulum.

Taking the beam with both ends simply-supported in Figure 2.4 as an example, its kinematic and elastic energies (Rao, 2004, 864) are

$$T = \frac{1}{2}\int \mu \left(\frac{\partial w}{\partial t}\right)^2 dx \quad , \; U = \frac{1}{2}\int EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

in which *w*, μ and *EI* are the vertical deflection, the mass per unit length and the flexural rigidity of the beam, respectively. Both energies are distributed in every infinitesimal segment of the beam. In principle, the beam can take up any admissible configurations that satisfy the zero-deflection $w = 0$ and zero-moment $d^2w/dx^2 = 0$ conditions at $x = 0$ and $x = L$ as well as the combinations of these configurations. A set of admissible configurations specified in terms of the *w* is

$$\{w = \sin(nx/L)\,, n = 1,2,3, \ldots\},$$

see Figure 2.5 for $n = 1$ to 5, and their combination deals to

$$w = \sum_{n=1}^{\infty} c_n \sin(nx/L)$$

in which c_ns are the dofs. Though the summation index can go up to infinity, only the first few terms are of practical importance as the subsequent terms will die out quickly due to energy dissipation or damping to be introduced in Section 2.3. The dof used in a continuous system often refers to the amplitude of a displacement mode. For this reason, these dofs are sometimes termed as modal dofs or coefficients.

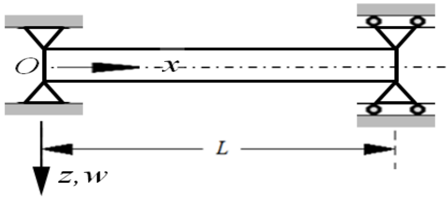


Figure 2.4: A beam with both ends simply supported.

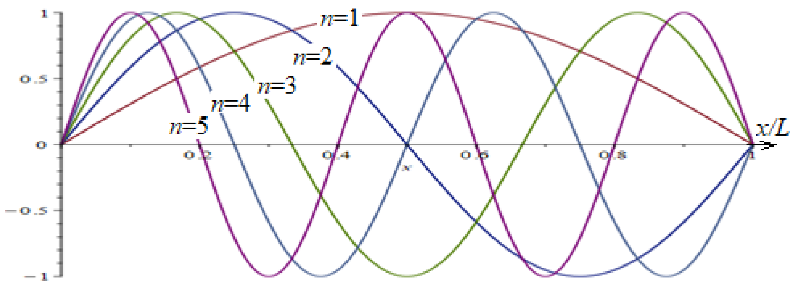


Figure 2.5: Some admissible configurations for the beam with both ends simply supported.

2.3 Viscous, Structural and Coulomb Damping

Damping which opposes the motion and dissipates the energy of a vibration system has not been considered in the discussed examples. In this section, viscous, structural and Coulomb dampings which are the most abundant forms of damping are introduced (Rao, 2004, 139,157) (Thomson and Dahleh, 1993, 27,35,72).

Viscous force or drag is experienced by a body moving in a fluid medium. Consider the viscous fluid between two plates at separation h shown in Figure 2.6. The lower plate is fixed and the upper plate is moving at velocity V. The velocities of the fluid in contact with the two plates will move at the same velocities as the plates. Between the plates, the fluid velocity u can be approximated as linearly varying, i.e.

$$u = \frac{Vy}{h}$$

where *y* is the distance from the lower fixed plate.

Applying the Newtonian law of viscosity, the horizontal shear stress would be proportional to the velocity gradient *du*/*dy* of the fluid, i.e.

$$shear\ stress = \mu\frac{du}{dy} = \frac{\mu V}{h}$$

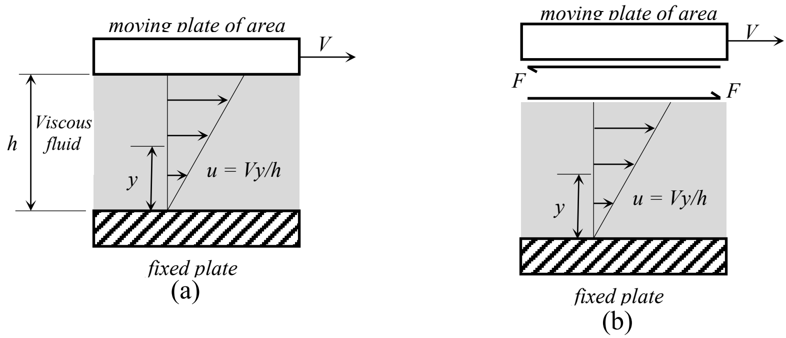


Figure 2.6: (a) Two parallel plates with a viscous fluid in between. (b) The viscous force *F* experienced by the moving plate.

In the relation, μ is the fluid property known as viscosity and is a measure of how "sticky" the fluid is. Thus, the force resisting the motion of the upper plate of area *A* is

$$F = A \times shear\ stress = \frac{\mu A}{h}V = cV$$

in which $c = \mu A/h$ is the damping coefficient. In vibration control, viscous damping is mostly realized in a dashpot in which fluid flows through the narrow gap between the piston and the cylinder and/or orifice(s) connecting the two sides of the piston. The velocity of the fluid flow and, thus, the damping force is proportional to the time derivative of the dashpot's extension. A commonly adopted symbol for the dashpot is shown in Figure 2.7 in which the single-dof spring-trolley system is damped by a dashpot of damping coefficient *c*.

Assuming the displacement takes up the form of Equation (2.4), the viscous damping force would be

$$F = c\dot{x} = cX\omega\,cos(\omega t + \emptyset). \tag{2.7}$$

The energy dissipated by dashpot per vibration cycle is

$$W_d = \int_{one\ cycle} F dx = \int_{t_o}^{t_o+\frac{2\pi}{\omega}} c\dot{x}^2 dt = c(X\omega)^2 \int_{t_o}^{t_o+\frac{2\pi}{\omega}} cos^2(\omega t + \emptyset)\, dt$$
$$= \pi c\omega X^2. \tag{2.8}$$

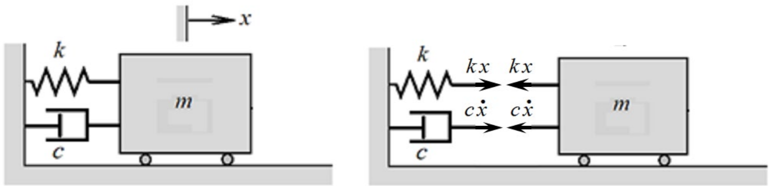


Figure 2.7: A spring-trolley system damped by a dashpot with damping coefficient *c*.

In structural or material damping, the energy dissipated per vibration cycle takes the following form

$$W_d = \alpha X^2 \tag{2.9}$$

in which α is a constant independent of the angular frequency of the vibration. Figure 2.8 shows a typical hysteresis loop in the stress-strain plot of solid materials. The area inside the loop is the energy dissipated per unit volume of the material in a loading-unloading or vibration cycle. It is trivial that the area inside the loop is proportional to the square of the vibration amplitude if the loop contracts or expands equally in all directions for different vibration amplitudes,

Comparing Equation (2.8) and Equation (2.9) leads to the following equivalent viscous damping coefficient for the structural damping,

$$c_{eq} = \frac{\alpha}{\pi\omega}$$

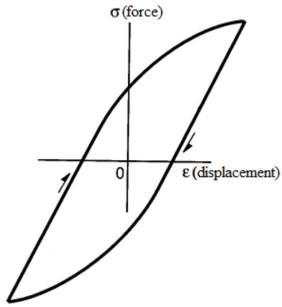


Figure 2.8: Hysteresis effect in the stress-strain and force-displacement plots.

Coulomb damping occurs when two dry solid surfaces rub against each other. The magnitude of the Coulomb damping force or, simply, dry friction is proportional to the normal reaction force between the two contacting surfaces. Assuming the normal reaction force is constant, Figure 2.9 plots the viscous damping force(in solid line) and dry friction force(in chained line) against the velocity. As the forces always oppose the velocity, they are negative when the velocity is positive and vice versa. For the Coulomb damping, the energy dissipation per vibration cycle of amplitude X is

$$W_d = 4X \cdot dry\ friction\ force$$

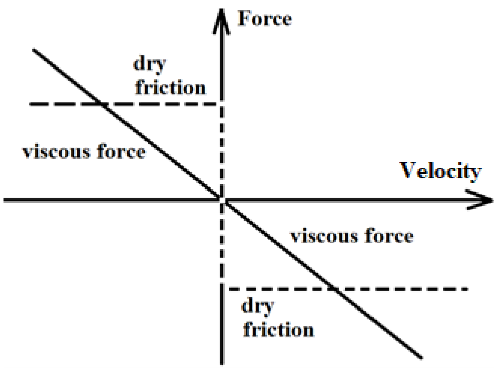


Figure 2.9: Viscous damping force (in solid line) and dry friction force (in chained line) versus the velocity.

2.4 Harmonic and Periodic Motions

An oscillatory motion is termed a harmonic or a simple harmonic motion if the dof, say x, describing the motion can be expressed as

$$x = X\sin(\omega t + \emptyset)$$

Hence, the spring-trolley and the simple pendulum sdof systems discussed in Section 2.1 are undergoing simple harmonic motions. An oscillatory motion x is periodic if it repeats itself at a fixed interval of time, i.e.

$$x(t+\tau) = x(t) \tag{2.10}$$

where τ is the fixed time interval and is known as the period of the motion. It is trivial that harmonic motion is a special type of periodic motion with period $2\pi/\omega$.

A periodic function of period τ can always be expressed as a Fourier series which is a summation series of sine and cosine functions of periods τ/n where $n = 0, 1, 2, 3, \ldots$, i.e.

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(\frac{2n\pi}{\tau}t) + b_n\sin(\frac{2n\pi}{\tau}t)\right) \tag{2.11}$$

and the first term arises from $n = 0$ [1,2]. In other words, a periodic motion can be regarded as a compound harmonic motion. Furthermore,

$$a_n\cos(\frac{2n\pi}{\tau}t) \quad \text{and} \quad b_n\sin(\frac{2n\pi}{\tau}t)$$

are termed as the n-th harmonic components of $x(t)$. To find a_ns and b_ns for a given $x(t)$, both sides of (2.11) can be multiplied with 1, $\cos(\frac{2m\pi}{\tau}t)$ and $\sin(\frac{2m\pi}{\tau}t)$, and integrated from t_o to $t_o+\tau$, namely

$$\begin{aligned}
\int_{t_o}^{t_o+\tau} x(t)dt &= \int_{t_o}^{t_o+\tau} a_0 dt \\
&\quad + \sum_{n=1}^{\infty}\int_{t_o}^{t_o+\tau}\left(a_n\cos(\frac{2n\pi}{\tau}t) + b_n\sin(\frac{2n\pi}{\tau}t)\right)dt, \\
\int_{t_o}^{t_o+\tau} x(t)\cos(\frac{2m\pi}{\tau}t)\,dt &= \int_{t_o}^{t_o+\tau} a_0\cos(\frac{2m\pi}{\tau}t)\,dt \\
&\quad + \sum_{n=1}^{\infty}\int_{t_o}^{t_o+\tau}\cos(\frac{2m\pi}{\tau}t)\left(a_n\cos(\frac{2n\pi}{\tau}t) + b_n\sin(\frac{2n\pi}{\tau}t)\right)dt, \\
\int_{t_o}^{t_o+\tau} x(t)\sin(\frac{2m\pi}{\tau}t)\,dt &= \int_{t_o}^{t_o+\tau} a_0\sin(\frac{2m\pi}{\tau}t)\,dt \\
&\quad + \sum_{n=1}^{\infty}\int_{t_o}^{t_o+\tau}\sin(\frac{2m\pi}{\tau}t)\left(a_n\cos(\frac{2n\pi}{\tau}t) + b_n\sin(\frac{2n\pi}{\tau}t)\right)dt
\end{aligned} \tag{2.12}$$

in which m is a non-zero positive integer and t_o is an arbitrary instant of time. By invoking

$$\int_{t_o}^{t_o+\tau} \cos(\frac{2m\pi}{\tau} t)\, dt = \int_{t_o}^{t_o+\tau} \sin(\frac{2n\pi}{\tau} t)\, dt = \int_{t_o}^{t_o+\tau} \cos(\frac{2m\pi}{\tau} t) \sin(\frac{2n\pi}{\tau} t)\, dt = 0\,,$$

$$\int_{t_o}^{t_o+\tau} \cos(\frac{2m\pi}{\tau} t) \cos(\frac{2n\pi}{\tau} t)\, dt = \int_{t_o}^{t_o+\tau} \sin(\frac{2m\pi}{\tau} t) \sin(\frac{2n\pi}{\tau} t)\, dt = \frac{\delta_{mn}}{2}$$

where δ_{mn} is equal to unity when $m = n$ and zero otherwise, the three relations in Equation (2.12) yield

$$a_0 = \frac{2}{\tau} \int_{t_o}^{t_o+\tau} x(t) dt \;\;,\;\; a_n = \frac{2}{\tau} \int_{t_o}^{t_o+\tau} x(t) \cdot \cos(\frac{2n\pi}{\tau} t)\, dt,$$

$$b_n = \frac{2}{\tau} \int_{t_o}^{t_o+\tau} x(t) \cdot \sin(\frac{2n\pi}{\tau} t)\, dt,$$

respectively.

2.5 Non-Periodic Motion

Contrary to the periodic motion, the non-periodic motion is a type of motion that does not repeat itself as described in Equation (2.10).

A typical example of non-periodic motion is a ball dropping vertically downward from a height. The ball will be rebounded by the floor. Let the coefficient of restitution be $e < 1$, the velocity (upward as positive) of the ball before the first rebound be $-u$ and the air resistance be negligible, the following list shows the velocity, the time of flight and the maximum height reached by the ball after each rebound.

	velocity after the rebound	time of flight after the rebound	maximum height after the rebound
Rebound 1	eu	$2eu/g$	$(eu)^2/(2g)$
Rebound 2	e^2u	$2e^2u/g$	$(e^2u)^2/(2g)$
:	:	:	:
Rebound n	e^nu	$2e^nu/g$	$(e^nu)^2/(2g)$

The motion of the ball is non-periodic because both the time of flight and the maximum height of the ball after each rebound are diminishing. Of course, the motion becomes periodic if $e = 1$ which implies no energy is lost in the rebound.

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Rao, S.S. 2004. Mechanical Vibrations, Fourth Edition. New Jersey: Pearson.

Thomson, W. T., and Dahleh, M.D. 1993. Theory of Vibration with Applications, Fifth Edition. Englewood Cliffs, New Jersey: Prentice-Hall.

Exercise Questions

Q1. A light rigid rod OA of length $3l$ is hanged to a smooth pivot O. A mass m is attached to A whilst two horizontal springs of spring constant k are attached to the rod at l and $2l$ from O as shown in Figure 2Q1. When the rod is vertical, the spring force is zero. Assuming that the swinging angle is small, derive the kinematic, gravitational and elastic energies of the system. Hence, find the frequency of the oscillation.

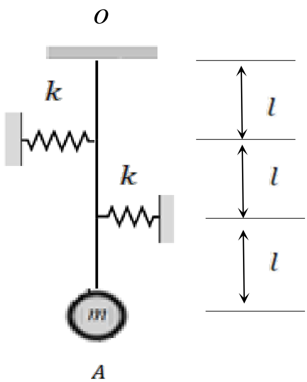

Figure 2Q1: A light rod OA pivoted at O with attached mass and springs.

Q2. A pendulum is formed by hanging a uniform rigid rod of mass m and length l to a smooth pivot. Assuming that the swinging angle is small, derive the kinematic and potential energies of the rod. Hence, find the frequency of the oscillation.

Q3. The initial displacement and initial velocity of a simple harmonic motion of frequency 5 Hz are 0.01 m and 0.1 m/s, respectively. Determine the displacement solution.

Q4. The following figure shows a rectangular waveform $x(t)$ of period $2\pi/\omega$ and magnitude A. Determine the Fourier series of $x(t)$.

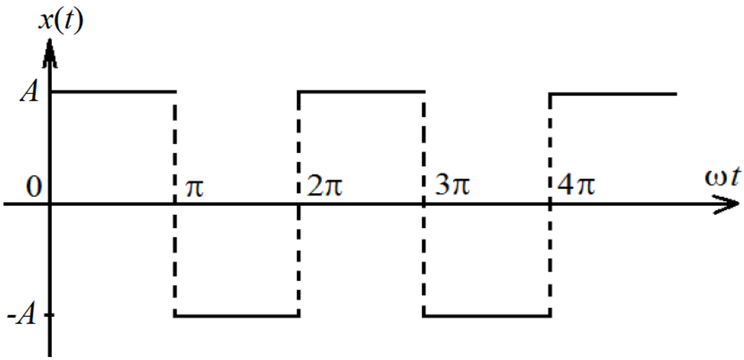

Figure 2Q4: A rectangular waveform $x(t)$.

CHAPTER 3

FREE AND FORCED VIBRATION OF SINGLE DEGREE OF FREEDOM SYSTEMS

In translating the oscillatory or oscillatory mechanical systems in the physical world into mathematical models in the conceptual world there is always a question of how accurate and realistic of representation of the physical system. It is generally logical to approach the problem by choosing a simplest model and if one desires, more realistic model can be formulated at a later stage. The simplest model is generally governed by the economy of solution. This simplest model can often provide information and insight to the problem at hand.

The present chapter therefore begins with the presentation of translational and rotational elements of oscillatory systems, formulation and analysis of simplest dynamic systems. The single degree-of-freedom (dof) systems without applied forces are dealt with in Section 3.1. Section 3.2 deals with single dof linear systems under harmonic forces. Section 3.3 introduces single dof linear systems under periodic forces. Section 3.4 is concerned with non-harmonically and non-periodic forced single dof linear systems. Questions and their solutions are included in Section 3.5.

3.1 Free Vibration of Single Degree of Freedom Systems

The presentation in this section is as follows. First, the systems in the physical world as shown in Figure 3.1 are given. Second, their corresponding models in the conceptual world are presented in Figure 3.2. Third, the formulation and analysis of one of the models in the conceptual world is selected to illustrate the solution steps.

Stage 1: ***Systems in Physical World***

There are many systems in the engineering physical world. It suffices to give two common examples for the purpose of illustrating the process of modelling. These two systems in the physical world are the automobile and television tower as sketched in Figure 3.1.

Stage 2: ***Systems in Conceptual World***

The systems in Figure 3.1 may be represented by the simplest mathematical models, known as *lumped-parameter models* or *discrete systems*, in the conceptual world. They are illustrated in Figure 3.2.

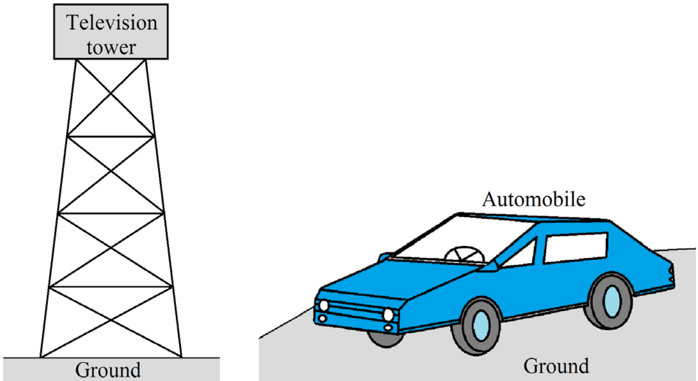

Figure 3.1: Systems in physical world.

Stage 3: ***Formulation and Solution***

The equation of motion governed each of the conceptual world models sketched in Figure 3.2 may now be derived. In general, the equation of free vibration for a single dof system may be obtained by one of the following approaches:

- Definition of simple harmonic motion,
- Knowledge of oscillatory motion,
- Law of conservation of energy,
- Newton's second law of motion, and
- Method of virtual work or virtual power.

Consider the discrete system on the left-hand side (lhs) in Figure 3.2. The latter and its free-body diagram (FBD) are included in Figure 3.3.

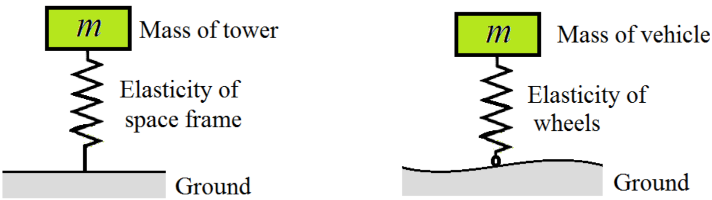

Figure 3.2: Single dof models of systems in Figure 3.1.

By Newton's second law of motion for the lumped mass,

$$m\ddot{x} = \sum F = mg - k(\Delta + x) - c\dot{x},$$

in which m is the mass, k spring constant, and c damping coefficient of the system, respectively. The symbol $\sum F$ denotes summing of the forces. The spring force $k\Delta$ is equal to the gravitational force mg acting on the mass m. That is, $mg = k\Delta$.

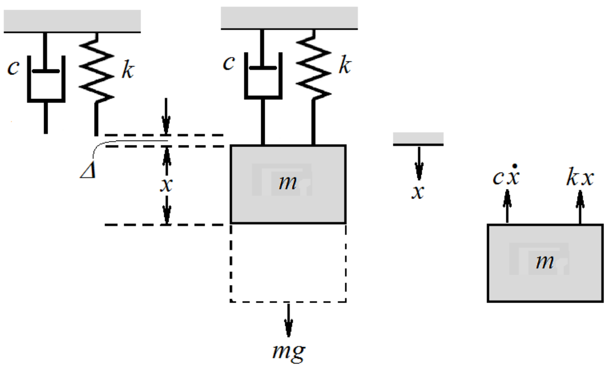

Figure 3.3: Single dof system and free-body diagram.

Therefore, the equation of motion for the above system is

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{3.1}$$

The first, second and third terms on the lhs of Equation (3.1) are the inertia, damping, and restoring forces, respectively.