Research Topics in Graph Theory and Its Applications

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In memory of my son, Vladik (1987 – 2000)

Contents

Preface

This book includes a number of research topics in graph theory and its applications. The topics are in the form of research projects developed by the author over the last 15 years. We discuss various research ideas devoted to α-discrepancy, strongly perfect graphs, the reconstruction conjectures, graph invariants, hereditary classes of graphs, embedding graphs on topological surfaces, as well as applications of graph theory, such as transport networks and hazard assessments based on unified networks. In addition to the original research ideas presented and methods to address them, there are also examples of impact statements, project resources required, support letters and work plans. The book has a free-form structure that allows the reader freedom, that is, the chapters are independent and can be read in any order.

Another important feature is the inclusion of reviewers' opinions in each chapter, which outline the strengths and weaknesses of various aspects of multiple research projects. The viewpoints of reviewers will be useful for recognising the typical mistakes authors make in research proposals. This book is ideal for developers of grant proposals, as well as for researchers interested in exploring new areas of graph theory and its applications. Advanced students in graph theory may use the topics presented in this book to develop their final-year projects, master's theses or doctoral dissertations.

It is the author's hope that this publication of original research ideas, problems and conjectures will instigate further research, or even a resurgence of interest, in the aforementioned important areas of graph theory.

I am very grateful to my wife and two daughters for their patience and support during the completion of this book.

Chapter 1

α-Discrepancy and Strong Perfectness

1.1 Background and Aims

Discrepancy theory, which originated from number theory, can generally be described as the study of irregularities of distributions in various settings. Classical combinatorial discrepancy theory is devoted to addressing the problem of partitioning the vertex set of a hypergraph into two classes in such a way that all hyperedges are split into approximately equal parts by the classes. That is, it is devoted to measuring the deviation of an optimal partition from a perfect partition when all hyperedges are split into equal parts. It should be noted that many classical results in various areas of mathematics (e.g. geometry and number theory) can be formulated in such terms. Combinatorial discrepancy theory was introduced by Beck in [3] and studied in [2]–[5] and [23]. Füredi and Mubayi [12] indicated that discrepancy theory had "developed into an elaborate field related ... to geometry, probability theory, ergodic theory, computer science, and combinatorics", while Tezuka [26] described its application to finance.

Among practical applications of the theory are image processing and the Monte Carlo methods for high dimensions. An important role in discrepancy theory is played by the fundamental "six-standard-deviation" result [25] and the discrepancy conjecture [4]. An interesting version of discrepancy is α-discrepancy, which occurs when success is measured by minimizing the imbalance of the vertex set while keeping the imbalance of each hyperedge at least α. The basic results in this context are devoted to 1-discrepancy and are concentrated on upper bounds and the Füredi–Mubayi conjecture [12].

The development of graph theory over the last five decades has been strongly influenced by the Strong Perfect Graph Conjecture and perfect graphs introduced by Berge in the early 1960s [6]. Perfect graphs are a fundamental concept in graph theory. This class of graphs has interesting applications, and there are books entirely devoted to perfect graphs (e.g. [7, 13]). The famous Strong Perfect Graph Conjecture, stated by Berge, had been open for about 40 years. Various attempts to prove it gave rise to many powerful methods, important concepts and interesting results in graph theory. Some of those methods affected the development of the theory of modular decomposition and Fulkerson's theory of antiblocking polyhedra. Chudnovsky, Robertson, Seymour and Thomas [9] relatively recently proved the Strong Perfect Graph Conjecture on 179 pages.

In 1978 at a Monday Seminar in Paris, Berge introduced another important class of graphs called strongly perfect graphs. It is a subclass of perfect graphs (see [8]). A graph is *strongly perfect* if every induced subgraph contains an independent set that meets all maximal cliques. The known results on strongly perfect graphs can be found in the survey paper [22]. Unlike perfect graphs, strongly perfect graphs do not have a conjecture similar to the Strong Perfect Graph Conjecture.

The class of absorbantly perfect graphs was introduced by Hammer and Maffray in [15]. A graph is *absorbantly perfect* if every induced subgraph has a minimal dominating set that meets all maximal cliques. Absorbantly perfect graphs are important because they form a superclass of strongly perfect graphs. The well-known classes of bipartite graphs and Meyniel graphs are subclasses of the class of strongly perfect graphs [16, 21]. In fact, we have the following chain of strict inclusions for these subclasses of perfect graphs:

$$\{\text{Bipartite}\} \subset \{\text{Meyniel}\} \subset \{\text{Strongly perfect}\} \subset$$
$$\{\text{Absorbantly perfect}\} \subset \{\text{Perfect}\}.$$

The basic aims of the proposed project are as follows (not in order of priority):

1. Find good bounds for α-discrepancy of hypergraphs.
2. Generalise the Füredi–Mubayi conjecture for α-discrepancy.
3. Formulate a characterisation conjecture on strongly perfect graphs and attempt to prove it.

Research conjectures are important for stimulating research and making progress in the corresponding areas. Any conjecture must ultimately be proved or disproved, but it is important to underline that the disproof of a conjecture does not necessarily mean a 'negative' result. It may yield insights into the problem and result in new conjectures and theorems.

To sum up, the basic idea of the proposed research project is to further develop discrepancy theory, and formulate the characterisation conjecture for strongly perfect graphs and then prove it theoretically. Note that the formulation of the conjecture includes both theoretical research and the development and application of scientific software, which is a new approach to the problem. Moreover, this research proposal meets the funder's mission and strategy because it will attract a talented post-doctoral researcher, stimulate adventure in developing the new approach and build collaboration between the investigator and the post-doctoral researcher.

1.2 α-Discrepancy

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with the vertex set V and the hyperedge set $\mathcal{E} = \{E_1, ..., E_m\}$. One of the main problems in classical combinatorial discrepancy theory is to colour the elements of V by two colours in such a way that all of the hyperedges have almost the same number of elements of each colour. Such a partition of V into two classes can be represented by a function $f : V \to \{+1, -1\}$. For a set $E \subseteq V$, let us define the *imbalance* of E as follows:

$$f(E) = \sum_{v \in E} f(v).$$

First defined by Beck [3], the *discrepancy of* $\mathcal{H}$ *with respect to* f is

$$\mathcal{D}(\mathcal{H}, f) = \max_{E_i \in \mathcal{E}} |f(E_i)|$$

and the *discrepancy* of $\mathcal{H}$ is

$$\mathcal{D}(\mathcal{H}) = \min_{f\colon V \to \{+1,-1\}} \mathcal{D}(\mathcal{H}, f).$$

Thus, the discrepancy of a hypergraph tells us how well all its hyperedges can be partitioned. Spencer [25] proved the fundamental "six-standard-deviation" result that for any hypergraph $\mathcal{H}$ with n vertices and n hyperedges, $\mathcal{D}(\mathcal{H}) \leq 6\sqrt{n}$. As shown in [1], this bound is best possible up to a constant factor. More precisely, if a Hadamard matrix of order $n > 1$ exists, then there is a hypergraph $\mathcal{H}$ with n vertices and n hyperedges such that $\mathcal{D}(\mathcal{H}) \geq 0.5\sqrt{n}$. It is well known that a Hadamard matrix of order between n and $(1 - \epsilon)n$ does exist for any ϵ and sufficiently large n. The following important result, due to Beck and Fiala [4], is valid for a hypergraph with any number of hyperedges: $\mathcal{D}(\mathcal{H}) \leq 2\Delta - 1$, where Δ is the maximum degree of vertices of $\mathcal{H}$. They also posed the following conjecture:

Discrepancy Conjecture [4] *For some constant* K*,*

$$\mathcal{D}(\mathcal{H}) < K\sqrt{\Delta}.$$

Another interesting aspect of discrepancy was discussed by Füredi and Mubayi [12]. A function $g : V \to \{+1, -1\}$ is called an *α-function* of the hypergraph $\mathcal{H}$ if

$$g(E_i) = \sum_{v \in E_i} g(v) \geq \alpha$$

for every hyperedge $E_i \in \mathcal{E}$, that is, each hyperedge has an imbalance at least α. The *α-discrepancy* of $\mathcal{H}$, denoted by $\mathcal{D}_\alpha(\mathcal{H})$, is defined in the following way:

$$\mathcal{D}_\alpha(\mathcal{H}) = \min_g g(V),$$

where the minimum is taken over all α-functions of $\mathcal{H}$. Thus, in this version of discrepancy, the success is measured by minimizing the imbalance of the vertex set V, while keeping the imbalance of every hyperedge $E_i \in \mathcal{E}$ at least α.

One of the basic results in α-discrepancy for $\alpha = 1$ is due to Füredi and Mubayi [12]:

Theorem 1.1 [12] *Let $\mathcal{H}$ be an n-vertex hypergraph with hyperedge set $\mathcal{E} = \{E_1, ..., E_m\}$ and suppose that every hyperedge has at least k vertices, where $k \geq 100$. Then*

$$\mathcal{D}_1(\mathcal{H}) \leq 4\sqrt{\frac{\ln k}{k}}n + \frac{1}{k}m.$$

This theorem can be easily re-formulated in terms of graphs by considering the neighbourhood hypergraph of a given graph. Note that $\gamma_s(G)$ is the signed domination number of a graph G, which corresponds to 1-discrepancy of the neighbourhood hypergraph of G.

Theorem 1.2 [12] *If G has n vertices and minimum degree $\delta \geq 99$, then*

$$\gamma_s(G) \leq \left(4\sqrt{\frac{\ln(\delta+1)}{\delta+1}} + \frac{1}{\delta+1}\right)n.$$

Moreover, Füredi and Mubayi [12] found quite good upper bounds for very small values of δ (for $\delta \leq 19$) and, using Hadamard matrices, constructed a δ-regular graph G of order 4δ with

$$\gamma_s(G) \geq 0.5\sqrt{\delta} - O(1).$$

This construction shows that the upper bounds in Theorems 1.1 and 1.2 are off from optimal by at most the factor of $\sqrt{\ln \delta}$. They posed an interesting conjecture that, for some constant C,

$$\gamma_s(G) \leq \frac{C}{\sqrt{\delta}} n,$$

and proved that the above discrepancy conjecture, if true, would imply this upper bound for δ-regular graphs. A strong result of Matoušek [19] shows that the bound is true, but he pointed out that the constant C is rather large: “I believe the constant shouldn't be astronomical — perhaps in the hundreds” (private communication on 15.01.2009). Notice that if $C > 100$, then the bound $\gamma_s(G) \leq \frac{C}{\sqrt{\delta}} n$ is better than the bound of Theorem 1.2 only for huge values of δ. For example, if $\delta = 10^{12}$, then

$$4\sqrt{\ln(\delta + 1)} = 21.03 << C,$$

that is, Theorem 1.2 yields a much better bound than the above conjecture. Hence the constant $C > 100$ in Matoušek's proof makes his result of rather theoretical interest.

Thus, when δ is not huge and at least 99, Theorem 1.2 gives the best upper bound, and for $19 < \delta < 99$, no good upper bound in known. Using the probabilistic method [1], we can prove the following result:

Theorem 1.3 *For any graph G with $\delta > 19$,*

$$\gamma_s(G) \leq \left(1 - \frac{2\widehat{\delta}}{(1+\widehat{\delta})^{1+1/\widehat{\delta}} \, \widetilde{d}_{0.5}^{1/\widehat{\delta}}} \right) n, \qquad (1.1)$$

where $\widehat{\delta} = \lfloor 0.5\delta \rfloor$, $\widetilde{d}_{0.5} = \binom{\delta' + 1}{\lceil 0.5\delta' \rceil}$, $\delta' = \begin{cases} \delta & \text{if } \delta \text{ is odd;} \\ \delta + 1 & \text{if } \delta \text{ is even.} \end{cases}$

This theorem improves the result of Theorem 1.2 for 'relatively small' values of δ. For example, if $\delta(G) = 99$, then by Theorem 1.2, $\gamma_s(G) \leq 0.869n$, while Theorem 1.3 yields $\gamma_s(G) \leq 0.537n$. In fact, we can improve Theorem 1.2 for all values of δ. For instance, we can prove a result which is 1.63 times better than the bound of Theorem 1.2 for very large values of δ, and for $\delta = 10^6$ the improvement is 1.44 times.

Thus, using our technique, we would like to answer the following natural and important questions: What are good bounds for α-discrepancy of hypergraphs? Also, can we prove the aforementioned conjecture for a small value of C and generalise it for α-discrepancy?

1.3 Strongly Perfect Graphs

A graph G is a *perfect graph* if $\omega(H) = \chi(H)$ for every induced subgraph H of G, where $\omega(H)$ is the clique number of H and $\chi(H)$ is the chromatic number of H. A *hole* in a graph is an induced cycle C_n, $n \geq 5$. A hole is *odd* if it has an odd number of vertices. The complement of a hole is called an *antihole*, and the complement of an odd hole is called an *odd antihole.*

The Strong Perfect Graph Theorem [9] *A graph is perfect if and only if it does not contain any odd holes and odd antiholes as induced subgraphs.*

Let $\beta(G)$ denote the independence number of G and $\Gamma(G)$ denote the upper domination number of G. A graph G is called *upper domination perfect* (Γ-*perfect*) if $\beta(H) = \Gamma(H)$, for every induced subgraph H of G. The *prism* Pr_n ($n \geq 3$) consists of two disjoint cycles

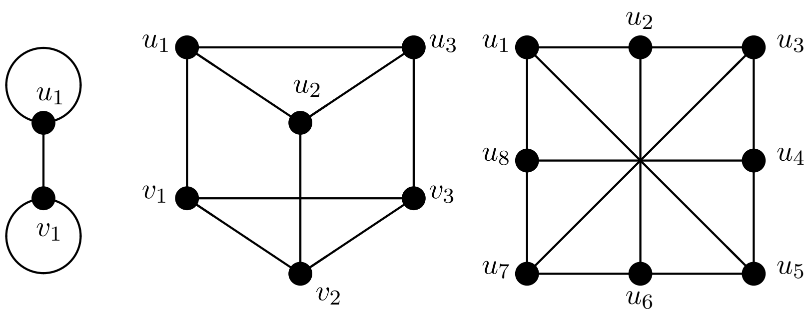
$$C_1 = (u_1, u_2, ..., u_n), \quad C_2 = (v_1, v_2, ..., v_n),$$

and the remaining edges are of the form $u_i v_i$, $1 \leq i \leq n$. The *prism* Pr_1 is two loops connected by the edge $u_1 v_1$, this is the only case where loops are permitted. The *Möbius ladder* Ml_n is constructed from the cycle

$$C = (u_1, u_2, ..., u_{2n})$$

by adding the edges $u_i u_{n+i}$ $(1 \leq i \leq n)$ joining each pair of opposite vertices of C. The odd prisms Pr_1 and Pr_3 and the even Möbius ladder Ml_4 are shown in Figure 1.1.

Two vertex subsets A, B of a graph G *independently match* each other if $A \cap B = \emptyset$, $|A| = |B|$ and all edges between A and B form a perfect matching in $\langle A \cup B \rangle$. A graph G of order $2k$ is called a *W-graph* if there is a partition $V(G) = A \cup B$ such that A and B independently match each other. Clearly, $|A| = |B| = k$. The sets A and B are called *parts*, and the graph G is denoted by $G(A, B)$. A graph H is called a *semi-induced subgraph* of a graph F if $H = H(A', B')$ is a W-graph with parts A' and B', and H is a subgraph of F such that the sets A' and B' independently match each other in the graph F. We say that a graph G is *2-homeomorphic to H* if G can be obtained from H by replacing edges of H by chains of even order $2k$, $k \geq 1$.



Pr_1 Pr_3 Ml_4

Figure 1.1: Odd prisms Pr_1, Pr_3 and even Möbius ladder Ml_4.

Theorem 1.4 [28] *A graph G is Γ-perfect if and only if G does not contain a semi-induced subgraph 2-homeomorphic to the odd prism Pr_{2n+1} $(n \geq 0)$ or the even Möbius ladder Ml_{2m} $(m \geq 1)$.*

Note that not all graphs described in the above theorem are W-graphs, for example the Möbius ladders are not. By the definition, such graphs cannot be semi-induced subgraphs, so there is no contradiction. However, some of graphs 2-homeomorphic to the Möbius ladders are W-graphs and they are 'really' forbidden. Note also that in such graphs the partition into parts is not fixed. The description of the 'pure' list of forbidden W-graphs can be found in [28].

Theorem 1.5 [14] *Any absorbantly perfect graph is Γ-perfect.*

Theorem 1.5 implies that any strongly perfect graph is Γ-perfect. A graph is called *strongly Γ-perfect* if it is both perfect and Γ-perfect. Since strongly perfect graphs are perfect, we obtain

$$\{\text{Strongly perfect}\} \subset \{\text{Strongly } \Gamma\text{-perfect}\} \subset \{\text{Perfect}\}.$$

The Strong Perfect Graph Theorem and Theorem 1.4 imply the following characterisation:

Theorem 1.6 *A graph G is strongly Γ-perfect if and only if*

1. *G does not contain odd holes and odd antiholes as induced subgraphs and*
2. *G does not contain a semi-induced subgraph 2-homeomorphic to the odd prism Pr_{2n+1} $(n \geq 0)$ or the even Möbius ladder Ml_{2m} $(m \geq 1)$.*

The class of strongly Γ-perfect graphs is 'much more narrow' than the class of perfect graphs because of forbidden semi-induced subgraphs 2-homeomorphic to the odd prism Pr_{2n+1} $(n \geq 0)$ or the even Möbius ladder Ml_{2m} $(m \geq 1)$. Theoretically, these forbidden semi-induced subgraphs can be replaced by an equivalent family $\mathcal{F}$ of forbidden induced subgraphs. This family would have an exponential growth of graphs. Indeed, $\mathcal{F}$ has 1 graph of order 6; 14 graphs of order 8 and 228 graphs of order 10 [14, 28].

Thus, we believe that the class of strongly Γ-perfect graphs is much 'closer' to strongly perfect graphs than to perfect graphs. Therefore, the missing link in the following conjecture, which is based on Theorem 1.6, is 'reasonably small' and can be determined by our approach.

Conjecture on Strongly Perfect Graphs *A graph G is strongly perfect if and only if*

1. *G does not contain odd holes and odd antiholes as induced subgraphs and*
2. *G does not contain a semi-induced subgraph 2-homeomorphic to the odd prism* Pr_{2n+1} $(n \geq 0)$ *or the even Möbius ladder* Ml_{2m} $(m \geq 1)$ *and*
3. *'Missing Link'.*

1.4 Computational Aspects

Besides graph-theoretic research, the proposed research project suggests developing and applying software to the hereditary class of strongly perfect graphs in order to obtain a characterisation conjecture for the class. Theoretically, each hereditary class can be characterised in terms of forbidden induced subgraphs. However, if the number of forbidden induced subgraphs is large, then it can be practically impossible to provide an explicit list of such subgraphs and a characterisation in different terms must be looked for. Another problem could be that the number of forbidden induced subgraphs is reasonable, but the method for obtaining the corresponding characterisation is difficult and cannot be implemented without specialised software.

Given a hereditary class $\mathcal{P}$, the first step in tackling the problem is to obtain the list of all minimal forbidden induced subgraphs of order at most 10. Let us denote this list by $\mathcal{F}$. This step is very important and crucial for choosing subsequent steps. In fact, the list $\mathcal{F}$ gives rise to various ideas and conjectures about the class. Moreover, it provides a characterisation of the class $\mathcal{P}$ for graphs of order at most 10. To fulfil the difficult task of finding the list $\mathcal{F}$ for the class $\mathcal{P}$, we will use our database of all non-isomorphic graphs of order at most 10. There are 12,293,434 graphs in the database. It is necessary to develop a program implementing the following procedure:

We put $\mathcal{F} = \emptyset$, where $\mathcal{F}$ is the list of minimal forbidden induced subgraphs for $\mathcal{P}$ of order at most 10.

Step 1 Take the next graph G from the database. If there are no more graphs, then exit.

Step 2 Verify whether G belongs to $\mathcal{P}$. If so, go to Step 1.

Step 3 Verify whether G contains one of the graphs from $\mathcal{F}$ as an induced subgraph. If it does contain any of the forbidden graphs, then go to Step 1.

Step 4 Update the list of minimal forbidden subgraphs $\mathcal{F}$ because a new minimal forbidden graph was found. Go to Step 1.

It is not difficult to see that this procedure correctly determines the list $\mathcal{F}$. Indeed, in Step 3 we only consider graphs not belonging to $\mathcal{P}$. If such a graph G is not minimal, then it must contain a minimal forbidden graph of smaller order from $\mathcal{F}$. Because our database contains all graphs of order up to 10, such a minimal graph of smaller order must be already in $\mathcal{F}$ and it will be found in G by the procedure. Thus, the procedure is correct. This procedure basically requires to develop and implement a recognition algorithm (in Step 2) for the class of strongly perfect graph and also a subgraph isomorphism algorithm (in Step 3). The latter algorithm was developed, implemented and successfully tested before. It is important to emphasise that the above procedure is feasible if we apply it to all graphs of order at most 10. For this database of graphs we ran various computer programs, which contained exponential algorithms, and running time was always very reasonable.

The conjecture will be thoroughly tested using random graphs of order between 11 and 20. Let $\mathcal{F}$ denote all graphs forbidden in the Conjecture on Strongly Perfect Graph (including the missing link). A graph G is called *$\mathcal{F}$-free* if it does not contain any member of $\mathcal{F}$. If the conjecture is not true, then there is an $\mathcal{F}$-free graph G which is not strongly perfect. We will generate a huge number of random $\mathcal{F}$-free graphs and verify whether they are strongly perfect:

Step 1 Take a random graph G of order 10 from the database of all graphs of order 10. If this graph is not $\mathcal{F}$-free, then do this step again.

Step 2 Generate a random number r between 1 and 10, and repeat the next step r times.

Step 3 Add a vertex v to G and randomly generate edges between v and the vertices of G. If the resulting graph is not $\mathcal{F}$-free, then remove the edges incident to v and generate them randomly again — do this until the resulting graph H is $\mathcal{F}$-free.

Step 4 Using the definition, verify whether H is strongly perfect. If so, go to Step 1. Otherwise, H is a counterexample to the conjecture.

Note that any $\mathcal{F}$-free graph of order between 11 and 20 can potentially be reached using the method of the above procedure. There are 3,063,185 strongly perfect graphs of order 10 and 12,005,168 graphs of order 10. Therefore, a strongly perfect graph of order 10 is randomly generated with the probability 0.255. Since the conjecture is true for all graphs of order at most 10, we conclude that an $\mathcal{F}$-free graph of order 10 in Step 1 is randomly generated with the same probability 0.255. It is very easy to see that Step 3 is feasible too. Thus, the above procedure will correctly verify the Conjecture on Strongly Perfect Graphs. If a counterexample H is found, then the conjecture must be rectified and the procedure must be run again.

There are no general methods for proving characterisation results. Therefore, it will be necessary to develop specific methods to prove the conjecture for some interesting cases and then try to prove the entire conjecture by further developing these methods. For example, it seems feasible to prove the conjecture for $K_{1,3}$-free graphs because in this case the families of forbidden graphs described in the second part of the Conjecture on Strongly Perfect Graphs can be replaced by three simple families of forbidden induced graphs.

Another approach would be to use the Strong Perfect Graph Theorem, whose proof provides a description of the structure of perfect graphs. Roughly speaking, a graph is perfect if and only if it either belongs to one of the four simple classes of graphs (bipartite graphs and their complements, line graphs and their complements) or can be obtained from these graphs by applying some operations. Using this structure and the families of forbidden graphs in the conjecture other than holes and antiholes, it might be possible to prove the entire conjecture. We are convinced that an essential progress in proving the conjecture will be achieved.

1.5 Beneficiaries and Dissemination

Füredi and Mubayi [12] indicated that discrepancy theory had "developed into an elaborate field related ... to geometry, probability theory, ergodic theory, computer science, and combinatorics", while Tezuka [26] describes its application to finance. Among other practical applications, image processing and the Monte Carlo methods in high dimensions can be mentioned. Ramirez-Alfonsi and Reed in their book [20] pointed out many important applications of perfect graphs and their links to other areas of mathematics. "These applications include frequency assignment for telecommunication systems, integer programming and optimisation" [10]. Among other applications, complexity and coding theories may be mentioned. For example, Diestel ([11], page 111) underlines that "the class of perfect graphs has duality properties with deep connections to optimisation and complexity theory, which are far from understood".

The results of this project will be of benefit to the academic and technological organisations interested in graph theory, combinatorial methods and algorithms. The new results may be useful to solve various problems associated with graph and network structures. The algorithms developed will be used in a complex Windows application for graph theory, called GRAPHOGRAPH, which can be used as a research and learning tool for graph theory. This software will be freely available via Internet in future.

In order to reach the community of interested researchers, we will publish our results in an internationally recognised journal such as the *Journal of Graph Theory* or *Discrete Mathematics.* Moreover, we will present our ideas/results at the *International Workshop on Graph-Theoretic Concepts in Computer Science* (Europe) and the *British Combinatorial Conference.*

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1.7 Resources and Work Plan

Table 1.1 provides a summary of resources required for the project in terms of full economic costs.

Directly Allocated Costs

The duration of the project is 24 months, and the PI is anticipating actively researching approximately one day per week because of many additional commitments (e.g. research, teaching, grant applications, scientific software, editorship, supervision of a PhD student). The time frame of 24 months is appropriate for a project of this nature due to the volume of work to be undertaken.

Table 1.1: Summary of costs.

Type of Costs	**Sub-type**	**Full Economic Costs** (£)
Directly Allocated Costs	Investigators	20,694
	Estates Costs	15,473
Directly Incurred Costs	Staff	73,242
	Travel & Subsistence	1,700
	Other Costs (PC)	2,315
Indirect Costs		71,630
	Total	**185,054**

Directly Incurred Costs

Staff

The research proposed here requires considerable effort at a high technical level relying on expertise in graph theory and computer programming. The proposal for a post-doctoral researcher full time for 24 months is, in our view, a cost-effective approach to meeting these requirements. The starting salary of £29,704 (the start of the RF Grade) is necessary to attract a researcher at the desired level of expertise to ensure project success and high-quality research and implementation.

Travel and Subsistence

Travel expenses are needed to exchange ideas with British and foreign specialists in the field and to disseminate the work. The post-doctoral researcher will take part at the *International*

Workshop on Graph-Theoretic Concepts in Computer Science (Europe) and the *British Combinatorial Conference* (unknown location). The results of the research will be published in the internationally recognised *Journal of Graph Theory* or *Discrete Mathematics*.

Other Incurred Costs

A high-performance PC for a post-doctoral researcher will be essentially required to carry out the project. The PC must have specifications sufficient to carry out the project, taking into consideration that there will be many programming and testing activities that are resource intensive.

To carry out this research, the PI has all the resources and infrastructure at the University of the West of England, Bristol, and therefore does not require additional equipment for this project. The work is genuine and will be done using available infrastructure.

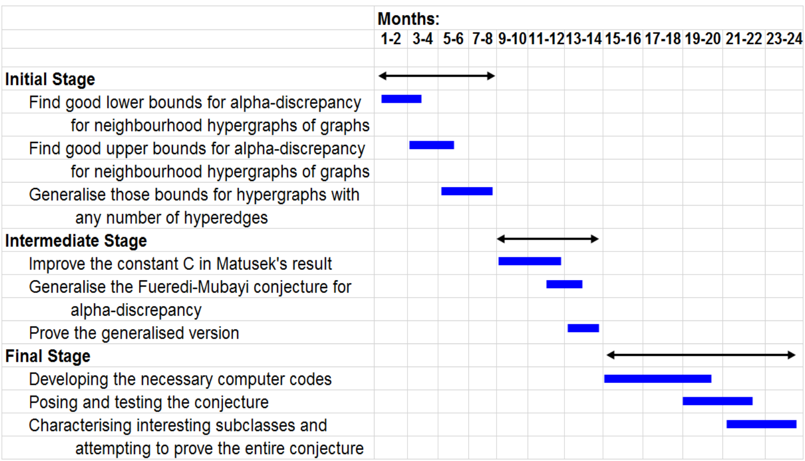

Figure 1.2: Work plan.