Variational Analysis with Applications in Optimisation and Control

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To my children,

Constantin and Elena

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Preface

The present book focuses on that part of calculus of variations and related applications which combines tools and methods from partial differential equations with geometrical techniques. More precisely, this work is devoted to nonlinear problems coming from different areas, with particular reference to those introducing new techniques capable of solving a wide range of problems. Frequently, we operate in physical problems with a two-time, $t = (t^1, t^2)$, where one component means the intrinsic time and the other one represents the observer time, not having any preference for one of the two components. In this respect, this book provides the latest developments in multidimensional optimization and optimal control. The results presented in this book are based on the author's recent contributions. With various examples and applications to complement and substantiate the mathematical developments, the present book is a valuable guide for researchers, engineers and students in the field of mathematics, operations research, optimal control science, artificial intelligence, management science and economics.

The book is organized in twelve chapters: First, **Chapter** 1 presents necessary and sufficient conditions of efficiency for a class of multiobjective (vector) variational problems involving higher-order derivatives. More precisely, an optimisation problem of minimizing a vector of simple integral functionals subject to higher-order differential equations and/or inequations is investigated. By using the notion of quasiinvexity, sufficient efficiency conditions for a feasible solution are established. In **Chapter** 2, by using the notions of the variational differential system, adjoint differential system and modified Legendrian duality, necessary optimality conditions for a class of signomial constrained optimal control problems are provided. **Chapter** 3 presents a study on sufficient efficiency conditions for a class of multidimensional vector ratio optimisation problems, identified by (MFP), of minimizing a vector of path-independent curvilinear integral functional quotients subject to PDE and/or PDI constraints involving higher-order partial derivatives. Under generalised (ρ, b)-quasiinvexity assumptions, sufficient conditions of efficiency are provided for a feasible solution in (MFP). **Chapter** 4, by using a non-standard Legendrian duality, investigates the Hamiltonian dynamics and formulates a Hamilton-Jacobi type divergence PDE involving higher-order Lagrangians. In **Chapter** 5, necessary and sufficient conditions of efficiency are derived in multiobjective variational control problems which involve multiple integral cost functionals. Under (ρ, b)-quasiinvexity assumptions, sufficient efficiency conditions for a feasible solution are formulated, as well. **Chapter** 6, by using the concept of invexity associated with multiple integral vector functionals, introduces several results of duality for a class of multiobjective fractional variational control problems involving multiple integral cost functionals. Under (ρ, b)-quasiinvexity assumptions, weak, strong and converse duality results are provided. The main goal of **Chapter** 7 is to formulate and prove, under simplified hypothesis, a maximum principle in a mathematical framework governed by geometric tools. More precisely, using some techniques of calculus of variations, the notion of *adjointness* and a geometrical context, necessary optimality conditions are established for two optimal control problems governed by: (i) multiple integral cost functional and (ii) curvilinear integral (mechanical work) cost functional, both subject to fundamental tensor (state variable) evolution as constraint. In both optimisation problems, the control variable is a connection, as well. Finally, as an application of the geometric maximum principle introduced in this chapter, exterior Euler-Lagrange and Hamilton-Pfaff PDEs are obtained. In **Chapter** 8, a KT-pseudoinvex multidimensional control problem is introduced. More exactly, a new condition on the functions which are involved in a multidimensional control problem is formulated and it is proved that a KT-pseudoinvex multidimensional control problem is characterized such that a Kuhn-Tucker point is an optimal solution. The theoretical results are illustrated with an application, as well. In **Chapter** 9, under some assumptions and using a dual gap-type functional, weak sharp solutions are investigated for a multidimensional variational inequality governed by convex multiple integral functional. Moreover, a relation between the minimum principle sufficiency property and weak sharpness of a solution set for the considered variational-type inequality is established. In order to give a better insight into the main results, a numerical application is formulated. **Chapter** 10 introduces a V-KT-pseudoinvex multidimensional vector control problem. A new condition on the functionals which are involved in a multidimensional multiobjective (vector) control problem is introduced and it is shown that a V-KT-pseudoinvex multidimensional vector control problem is described so that all Kuhn-Tucker points are efficient solutions. An illustrative application is also presented. In **Chapter** 11, based on a multidimensional control problem, in short (MCP), a modified multidimensional variational control problem involving first-order partial differential equations (PDEs) and inequality-type constraints is introduced. Optimality conditions for this new variational control problem are formulated and proved, as well. Furthermore, under some generalised convexity assumptions, an equivalence is established between an optimal solution of (MCP) and a saddle-point associated with the Lagrange functional (Lagrangian) corresponding to the modified multidimensional control problem. Also, in order to illustrate the main characterization results and their effectiveness, several applications are presented. **Chapter** 12 investigates optimality conditions for a class of PDE&PDI-constrained variational control problems. Thus, a minimal criterion for a local optimal solution of the considered PDE&PDI-constrained variational control problem to be its global optimal solution is derived. The theoretical development is supported by a suitable nonconvex optimisation problem.

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Bucharest, Romania Savin Treanţă
2019

Chapter 1

Efficiency conditions in vector variational problems involving higher-order derivatives

In this chapter, necessary and sufficient conditions of efficiency are formulated and proved for a class of multiobjective (vector) variational problems involving higher-order derivatives. More precisely, we investigate an optimisation problem of minimizing a vector of simple integral functionals subject to higher-order differential equations and/or inequations. By using the notion of quasiinvexity, sufficient efficiency conditions for a feasible solution are established.

1.1 Introduction and problem description

In this chapter, we extend and further develop some optimisation results regarding the efficiency of a feasible solution for a class of vector non-fractional variational problems. More concretely, we introduce and perform a study on the vector variational problem of minimizing a vector of simple integral functionals, denoted by (MVP), with higher-order differential equation and inequation constraints. The passing from the first-order derivatives to the higher-order derivatives is not a facile task because it requests specific techniques, a new quasiinvexity concept and an appropriate mathematical framework.

Over time, several authors have been interested in the study of vector variational problems by using a generalised convexity/invexity (see [8], [10]-[15], [17], [25], [26], [32], [33], [41], [45], [48]-[50], [56], [58], [60], [66], [68], [80], [82], [84], [87], [88], [94]-[98], [102], [104], [105], [114]-[117], [124], [141], [167]). Also, many of them extended the notion of convexity/invexity and developed a multitime (multidimensional) optimisation theory by using a geometrical language (see [83], [110], [111], [135], [144], [146], [151]-[153]).

Let us consider the real interval $I := [t_0, t_1] \subseteq \mathbb{R}$ and

$$f = (f_\alpha) : I \times \mathbb{R}^{n(k+1)} \to \mathbb{R}^p, \quad \alpha = \overline{1, p},$$

$$(f_1(t, x(t), x^{(1)}(t), ..., x^{(k)}(t)), ..., f_p(t, x(t), x^{(1)}(t), ..., x^{(k)}(t))),$$

a C^{k+1}-class function, where $x^{(k)}(t) := \frac{d^k}{dt^k}x(t)$, with $k \geq 1$ a fixed natural number. Also, let be given $g = (g_1, ..., g_m) : I \times \mathbb{R}^{n(k+1)} \to \mathbb{R}^m$, with $m < n$, and $h = (h_1, ..., h_r) : I \times \mathbb{R}^{n(k+1)} \to \mathbb{R}^r$, with $r < n$, two C^{k+1}-class functions. Assume that the C^{k+1}-class Lagrangians

$$f_\alpha(t, x(t), x^{(1)}(t), ..., x^{(k)}(t)), \quad \alpha = \overline{1, p}$$

generate the simple integral functionals

$$F_\alpha\left(x(\cdot)\right) := \int_{t_0}^{t_1} f_\alpha(t, x(t), x^{(1)}(t), ..., x^{(k)}(t))dt, \quad \alpha = \overline{1, p}.$$

Let $C^\infty\left([t_0, t_1], \mathbb{R}^n\right)$ be the space of all functions $x : [t_0, t_1] \to \mathbb{R}^n$ of C^∞-class, with the norm

$$\|x\| := \|x\|_\infty + \sum_{\beta=1}^{k} \|x^{(\beta)}\|_\infty.$$

As usually, for any two vectors $u = (u_1, ..., u_s)$, $v = (v_1, ..., v_s)$ in $\mathbb{R}^s$, we shall consider the following convention

$$u = v \Leftrightarrow u_i = v_i, \quad u \leq v \Leftrightarrow u_i \leq v_i,$$

$$u < v \Leftrightarrow u_i < v_i, \quad u \preceq v \Leftrightarrow u \leq v, \quad u \neq v, \quad i = \overline{1, s}.$$

Also, we underline that the argument of the considered Lagrangians is a graph

$$\chi_x(t) := (t, x(t), x^{(1)}(t), ..., x^{(k)}(t)).$$

Using these ingredients, we formulate the multiobjective variational problem (MVP) (a constrained optimisation problem) as

$$\min_{x(\cdot)} F(x(\cdot)) = \left(\int_{t_0}^{t_1} f_1(\chi_x(t))dt, ..., \int_{t_0}^{t_1} f_p(\chi_x(t))dt\right)$$

$$\text{subject to } x(\cdot) \in \mathtt{F}(I),$$

where the set $\mathtt{F}(I)$ of all feasible solutions is

$$x \in C^\infty\left(I, \mathbb{R}^n\right), \quad x(t_\varepsilon) = x_\varepsilon, \quad x^{(\beta)}(t_\varepsilon) = x_{\beta\varepsilon}, \quad \varepsilon \in \{0, 1\},$$

$$g(\chi_x(t)) \leq 0, \quad h(\chi_x(t)) = 0, \quad t \in I, \quad \beta = \overline{1, k-1}.$$

Th next section introduces some necessary preliminary results which will be used for proving the main results of the present chapter.

1.2 Some preliminary results

In the following, let us start with the case of a single simple integral functional, by considering the following scalar variational problem (SVP),

$$\min_{x(\cdot)} \left\{ I\left(x(\cdot)\right) = \int_{t_0}^{t_1} X\left(\chi_x(t)\right) dt \right\}$$

$$\text{subject to } x\left(\cdot\right) \in \mathcal{F}(I).$$

Further, consider the auxiliary Lagrange function

$$L(\chi_x(t), p(t), q(t), \lambda) := \lambda X\left(\chi_x(t)\right) + p^a(t) g_a(\chi_x(t)) + q^\zeta(t) h_\zeta(\chi_x(t)),$$

with summation over the repeated indices, that allows us to establish necessary conditions of optimality for (SVP).

Theorem 2.1 *Consider that the feasible solution x^0 of the problem (SVP) is an optimal solution and the functions X, g, h are C^{k+1}-class functions. Then there exist a scalar λ and the piecewise smooth functions $p(t)$ and $q(t)$, satisfying*

$$\frac{\partial L}{\partial x}(\chi_{x^0}(t), p(t), q(t), \lambda) - \frac{d}{dt}\frac{\partial L}{\partial x^{(1)}}(\chi_{x^0}(t), p(t), q(t), \lambda)$$

$$+ ... + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial x^{(k)}}(\chi_{x^0}(t), p(t), q(t), \lambda) = 0$$

(higher-order Euler-Lagrange ODEs)

$$p(t) g(\chi_{x^0}(t)) = 0, \quad p(t) \geq 0, \quad (\forall) t \in I.$$

Definition 2.1 The optimal solution $x^0(\cdot)$ of problem (SVP) is called *normal optimal solution* if $\lambda \neq 0$.

Further, without loss of generality, we can assume that $\lambda = 1$.

Definition 2.2 A feasible solution $x^0(\cdot) \in \mathcal{F}(I)$ is called *efficient solution* in (MVP) if there exists no other feasible solution $x(\cdot) \in \mathcal{F}(I)$ such that $F\left(x(\cdot)\right) \preceq F\left(x^0(\cdot)\right)$.

Consider ρ a real number and

$$b : \left[C^\infty\left([t_0, t_1], \mathbb{R}^n\right)\right]^{k+1} \to [0, \infty)$$

a positive functional. Also, consider the notations

$$b\left(x, x^0, x^{0(1)}, ..., x^{0(k-1)}\right) := b_{xx^0},$$

$$\eta\left(t, x, x^{(1)}, ..., x^{(k-1)}, x^{0(k)}\right) := \eta_{txx^0}$$

and $a : I \times \mathbb{R}^{n(k+1)} \to \mathbb{R}$ a real function that determines the simple integral functional

$$A(x(\cdot)) = \int_{t_0}^{t_1} a\left(\chi_x(t)\right) dt.$$

Definition 2.3 The functional $A(\cdot)$ is *[strictly] (ρ, b)-quasiinvex at x^0* if there exist the vector functions $\eta = (\eta_1, ..., \eta_n)$, with the property

$$\frac{d^\zeta \eta_{tx^0x^0}}{dt^\zeta} = 0, \quad \zeta \in \{0, 1, ..., k-1\}, \quad t \in I,$$

and $\theta : \left[C^\infty\left([t_0, t_1], \mathbb{R}^n\right)\right]^{k+1} \to \mathbb{R}^n$ such that, for any x $[x \neq x^0]$, we have

$$\left(A(x) \leq A(x^0)\right) \Longrightarrow (b_{xx^0} \int_{t_0}^{t_1} \eta_{txx^0} \frac{\partial a}{\partial x}(\chi_{x^0}(t))dt$$

$$+b_{xx^0} \int_{t_0}^{t_1} \frac{d\eta_{txx^0}}{dt} \frac{\partial a}{\partial x^{(1)}}(\chi_{x^0}(t))dt + ... + b_{xx^0} \int_{t_0}^{t_1} \frac{d^k\eta_{txx^0}}{dt^k} \frac{\partial a}{\partial x^{(k)}}(\chi_{x^0}(t))dt$$

$$[<] \leq -\rho b_{xx^0} \parallel \theta_{xx^0} \parallel^2).$$

1.3 Necessary and sufficient conditions of efficiency

In the following, in order to formulate and prove the necessary and sufficient conditions for the considered vector optimisation problem, we establish the following auxiliary lemmas.

Lemma 3.1 *The feasible solution $x^0(\cdot) \in F(I)$ is an efficient solution of the problem (MVP) if and only if $x^0(\cdot) \in F(I)$ is an optimal solution of each scalar problem $P_l(x^0)$, $l = \overline{1,p}$, defined as*

$$\min_{x(\cdot)} \int_{t_0}^{t_1} f_l\left(\chi_x(t)\right) dt$$

subject to

$$x(\cdot) \in F(I), \quad \int_{t_0}^{t_1} f_j\left(\chi_x(t)\right) dt \leq \int_{t_0}^{t_1} f_j\left(\chi_{x^0}(t)\right) dt, \quad j = \overline{1,p}, \quad j \neq l.$$

Proof. " $\Longrightarrow$ " Consider $x^0(\cdot) \in F(I)$ is an efficient solution of the problem (MVP). Let us suppose there exists $k \in \{1, ..., p\}$ such that $x^0(\cdot) \in F(I)$ is not an optimal solution of the scalar problem $P_k(x^0)$. Consequently, there exists a function $y(\cdot) \in F(I)$ such that

$$\int_{t_0}^{t_1} f_j\left(\chi_y(t)\right) dt \leq \int_{t_0}^{t_1} f_j\left(\chi_{x^0}(t)\right) dt$$

for $j=\overline{1,p},\ j\neq k$, and

$$\int_{t_0}^{t_1} f_k\left(\chi_y(t)\right)dt < \int_{t_0}^{t_1} f_k\left(\chi_{x^0}(t)\right)dt.$$

This contradicts the efficiency of the function $x^0(\cdot)\in \mathtt{F}(I)$ in (MVP). Therefore, we proved the direct implication.

" $\Longleftarrow$ " Let $x^0(\cdot)\in \mathtt{F}(I)$ be an optimal solution of each scalar problem $P_l(x^0),\ l=\overline{1,p}$. Assume that $x^0(\cdot)\in \mathtt{F}(I)$ is not an efficient solution of the problem (MVP). Consequently, there exists a function $y(\cdot)\in \mathtt{F}(I)$ such that

$$\int_{t_0}^{t_1} f_j\left(\chi_y(t)\right)dt \leq \int_{t_0}^{t_1} f_j\left(\chi_{x^0}(t)\right)dt, \quad j=\overline{1,p}$$

and there exists $k\in\{1,...,p\}$ such that

$$\int_{t_0}^{t_1} f_k\left(\chi_y(t)\right)dt < \int_{t_0}^{t_1} f_k\left(\chi_{x^0}(t)\right)dt.$$

But, the function $x^0(\cdot)\in \mathtt{F}(I)$ minimizes the functional $\int_{t_0}^{t_1} f_k\left(\chi_x(t)\right)dt$ on the set of all feasible solutions of problem $P_k(x^0)$. The proof is complete. □

Lemma 3.2 *Consider $l\in\{1,...,p\}$ fixed and $x^0(\cdot)\in \mathtt{F}(I)$ an optimal solution of the scalar problem $P_l(x^0)$. Then there exist the real scalars $\lambda_{jl}\geq 0$ and the piecewise smooth functions $p_l(t)$ and $q_l(t)$ such that*

$$\sum_{j=1}^{p}\lambda_{jl}\frac{\partial f_j}{\partial x}(\chi_{x^0}(t))+p_l(t)\frac{\partial g}{\partial x}(\chi_{x^0}(t))+q_l(t)\frac{\partial h}{\partial x}(\chi_{x^0}(t))$$

$$-\frac{d}{dt}\left\{\sum_{j=1}^{p}\lambda_{jl}\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}(t))+p_l(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t))+q_l(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t))\right\}$$

$$+...+(-1)^k\frac{d^k}{dt^k}\left\{\sum_{j=1}^{p}\lambda_{jl}\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}(t))\right\}$$

$$+(-1)^k\frac{d^k}{dt^k}\left\{p_l(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}(t))+q_l(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}(t))\right\}=0$$

$$(higher\ order\ Euler-Lagrange\ ODEs)$$

$$p_l(t)g(\chi_{x^0}(t))=0, \quad p_l(t)\geq 0, \quad (\forall)t\in I.$$

Proof. Consider $R_l^0 := \int_{t_0}^{t_1} f_l\left(\chi_{x^0}(t)\right) dt = \min_{x(\cdot)} \int_{t_0}^{t_1} f_l\left(\chi_x(t)\right) dt,\ l = \overline{1,p}$. Define the C^{k+1}-class functions, $\phi_j : I \times \mathbb{R}^{n(k+1)} \to \mathbb{R},\ \phi_j(\chi_x(t)) \geq 0,\ j = \overline{1,p},\ j \neq l$, as follows

$$G_j(x(\cdot)) := \int_{t_0}^{t_1} \left[f_j\left(\chi_x(t)\right) - R_j^0 + \phi_j(\chi_x(t))\right] dt = 0.$$

Therefore, the scalar problem $P_l(x^0)$, $l \in \{1, ..., p\}$ fixed, is changed into

$$\max_{x(\cdot)} \int_{t_0}^{t_1} f_l\left(\chi_x\right) dt$$

subject to

$$x \in F(I)\,, \quad G_j(x(\cdot)) = 0, \quad \phi_j(\chi_x) \geq 0, \quad j = \overline{1,p}, \quad j \neq l,$$

or

$$\max_{x(\cdot)} \Big\{ \int_{t_0}^{t_1} f_l\left(\chi_x\right) + \sum_{j=1;\ j\neq l}^{p} \lambda_{jl}\left[f_j\left(\chi_x\right) - R_j^0 + \phi_j(\chi_x)\right] dt \Big\}$$

subject to

$$x \in F(I)\,, \quad \phi_j(\chi_x) \geq 0, \quad j = \overline{1,p}, \quad j \neq l,$$

or, equivalently,

$$\max_{x(\cdot)} \Big\{ \int_{t_0}^{t_1} f_l\left(\chi_x\right) + \sum_{j=1;\ j\neq l}^{p} \lambda_{jl}\left[f_j\left(\chi_x\right) - R_j^0 + \phi_j(\chi_x)\right] dt \Big\} \qquad (*)$$

subject to

$$x\left(\cdot\right) \in F(I)\,, \quad -\phi_j(\chi_x) \leq 0, \quad j = \overline{1,p}, \quad j \neq l.$$

Let us consider the following Lagrangian

$$V_l(\chi_x, p_l, q_l, \gamma_l, a_j) := \gamma_l f_l\left(\chi_x\right) + \gamma_l \left\{ \sum_{j=1;\ j\neq l}^{p} \lambda_{jl}\left[f_j\left(\chi_x\right) - R_j^0 + \phi_j(\chi_x)\right] \right\}$$

$$+ p_l(t) g(\chi_x) + q_l(t) h(\chi_x) - \sum_{j=1;\ j\neq l}^{p} a_j(t)\phi_j(\chi_x),$$

where $\gamma_l \in \mathbb{R},\ \gamma_l \geq 0$, and $p_l : I \to \mathbb{R}^m,\ q_l : I \to \mathbb{R}^r,\ a_j : I \to \mathbb{R},\ a_j \geq 0,\ j = \overline{1,p},\ j \neq l$, are piecewise smooth functions. The function x^0 being an optimal solution for $(*)$, the following necessary conditions (see Valentine [162]) are fulfilled

$$\frac{\partial V_l}{\partial x}(\chi_{x^0}, p_l, q_l, \gamma_l, a_j) - \frac{d}{dt}\frac{\partial V_l}{\partial x^{(1)}}(\chi_{x^0}, p_l, q_l, \gamma_l, a_j) + ... + (-1)^k \frac{d^k}{dt^k}\frac{\partial V_l}{\partial x^{(k)}}(\chi_{x^0}, p_l, q_l, \gamma_l, a_j) = 0$$

$$p_l(t)g(\chi_{x^0}(t)) = 0, \quad p_l(t) \geq 0, \quad t \in I$$
$$a_j(t)\phi_j(\chi_{x^0}(t)) = 0, \quad a_j(t) \geq 0, \quad j = \overline{1,p}, \quad j \neq l$$
$$\gamma_l \geq 0, \quad \lambda_{jl} \geq 0, \quad j = \overline{1,p}, \quad j \neq l.$$

Concretely, it results

$$\gamma_l \frac{\partial f_l}{\partial x}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \left[\frac{\partial f_j}{\partial x}(\chi_{x^0}) + \frac{\partial \phi_j}{\partial x}(\chi_{x^0})\right] + p_l(t)\frac{\partial g}{\partial x}(\chi_{x^0}) + q_l(t)\frac{\partial h}{\partial x}(\chi_{x^0})$$
$$- \sum_{j=1;\, j\neq l}^{p} a_j(t)\frac{\partial \phi_j}{\partial x}(\chi_{x^0}) - \frac{d}{dt}\left\{\gamma_l \frac{\partial f_l}{\partial x^{(1)}}(\chi_{x^0})\right\}$$
$$-\frac{d}{dt}\left\{\sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \left[\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) + \frac{\partial \phi_j}{\partial x^{(1)}}(\chi_{x^0})\right] + p_l(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + q_l(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0})\right\}$$
$$+\frac{d}{dt}\left\{\sum_{j=1;\, j\neq l}^{p} a_j(t)\frac{\partial \phi_j}{\partial x^{(1)}}(\chi_{x^0})\right\} + ... + (-1)^k \frac{d^k}{dt^k}\left\{\gamma_l \frac{\partial f_l}{\partial x^{(k)}}(\chi_{x^0})\right\}$$
$$+(-1)^k \frac{d^k}{dt^k}\left\{\sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \frac{\partial \phi_j}{\partial x^{(k)}}(\chi_{x^0})\right\}$$
$$+(-1)^k \frac{d^k}{dt^k}\left\{p_l(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + q_l(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0})\right\}$$
$$+(-1)^{k+1} \frac{d^k}{dt^k}\left\{\sum_{j=1;\, j\neq l}^{p} a_j(t)\frac{\partial \phi_j}{\partial x^{(k)}}(\chi_{x^0})\right\} = 0,$$

or, equivalently,

$$\gamma_l \frac{\partial f_l}{\partial x}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \frac{\partial f_j}{\partial x}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} \left[\gamma_l \lambda_{jl} - a_j(t)\right] \frac{\partial \phi_j}{\partial x}(\chi_{x^0}) \tag{$\star$}$$
$$+p_l(t)\frac{\partial g}{\partial x}(\chi_{x^0}) + q_l(t)\frac{\partial h}{\partial x}(\chi_{x^0}) - \frac{d}{dt}\left\{\gamma_l \frac{\partial f_l}{\partial x^{(1)}}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0})\right\}$$
$$-\frac{d}{dt}\left\{\sum_{j=1;\, j\neq l}^{p} \left[\gamma_l \lambda_{jl} - a_j(t)\right] \frac{\partial \phi_j}{\partial x^{(1)}}(\chi_{x^0}) + p_l(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + q_l(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0})\right\}$$
$$+... + (-1)^k \frac{d^k}{dt^k}\left\{\gamma_l \frac{\partial f_l}{\partial x^{(k)}}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0})\right\}$$
$$+(-1)^k \frac{d^k}{dt^k}\left\{\sum_{j=1;\, j\neq l}^{p} \left[\gamma_l \lambda_{jl} - a_j(t)\right] \frac{\partial \phi_j}{\partial x^{(k)}}(\chi_{x^0})\right\}$$

$$+(-1)^k \frac{d^k}{dt^k}\left\{p_l(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + q_l(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0})\right\} = 0.$$

We impose the following conditions: $\gamma_l\lambda_{jl} - a_j(t) = 0,\ j = \overline{1,p},\ j \neq l$, for any $t \in I$, $\gamma_l = \lambda_{ll} \geq 0,\ \lambda_{jl} = \gamma_l\lambda_{jl} \geq 0,\ j = \overline{1,p},\ j \neq l$. Rewriting $(\star)$, it follows

$$\lambda_{ll}\frac{\partial f_l}{\partial x}(\chi_{x^0}) + \sum_{j=1;\ j\neq l}^{p} \lambda_{jl}\frac{\partial f_j}{\partial x}(\chi_{x^0}) + p_l(t)\frac{\partial g}{\partial x}(\chi_{x^0})$$

$$+q_l(t)\frac{\partial h}{\partial x}(\chi_{x^0}) - \frac{d}{dt}\left\{\lambda_{ll}\frac{\partial f_l}{\partial x^{(1)}}(\chi_{x^0})\right\}$$

$$-\frac{d}{dt}\left\{\sum_{j=1;\ j\neq l}^{p} \lambda_{jl}\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) + p_l(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + q_l(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0})\right\}$$

$$+... + (-1)^k \frac{d^k}{dt^k}\left\{\lambda_{ll}\frac{\partial f_l}{\partial x^{(k)}}(\chi_{x^0}) + \sum_{j=1;\ j\neq l}^{p} \lambda_{jl}\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0})\right\}$$

$$+(-1)^k \frac{d^k}{dt^k}\left\{p_l(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + q_l(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0})\right\} = 0$$

and the proof is complete. □

Definition 3.1 The feasible solution $x^0(\cdot) \in \mathtt{F}(I)$ is called *normal efficient solution* of (MVP) if it is a normal optimal solution for at least one of the scalar problems $P_l(x^0),\ l = \overline{1,p}$.

In the next result, the normal necessary efficiency conditions associated to (MVP) are established.

Theorem 3.1 ([Normal] necessary efficiency conditions for (MVP)) *If $x^0(\cdot) \in \mathtt{F}(I)$ is a [normal] efficient solution of the problem (MVP), then there exist $\lambda \in \mathbb{R}^p$, $p : I \to \mathbb{R}^m$ and $q : I \to \mathbb{R}^r$ satisfying the following conditions*

$$\sum_{j=1}^{p} \lambda_j\frac{\partial f_j}{\partial x}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x}(\chi_{x^0}(t)) + q(t)\frac{\partial h}{\partial x}(\chi_{x^0}(t))$$

$$-\frac{d}{dt}\left\{\sum_{j=1}^{p} \lambda_j\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t)) + q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t))\right\}$$

$$+... + (-1)^k \frac{d^k}{dt^k}\left\{\sum_{j=1}^{p} \lambda_j\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}(t))\right\}$$

$$+(-1)^k \frac{d^k}{dt^k}\left\{q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}(t))\right\} = 0,$$

$(higher\ order\ Euler - Lagrange\ ODEs)$

$$p(t)g(\chi_{x^0}(t)) = 0, \quad p(t) \geq 0, \quad (\forall)t \in I,$$
$$\lambda \geq 0, \quad e^t\lambda = 1, \quad e^t = (1, 1, ..., 1) \in \mathbb{R}^p.$$

Proof. Taking into account Lemma 3.1, we get $x^0(\cdot) \in F(I)$ is an optimal solution of each problem $P_l(x^0)$, $l = \overline{1,p}$. Therefore, if $x^0(\cdot) \in F(I)$ is [normal] optimal solution in $P_l(x^0)$, $l \in \{1, ..., p\}$ fixed, then the relations which appear in Lemma 3.2 are fulfilled $[\lambda_{ll} = 1]$. Making summation over $l = \overline{1,p}$ of all relations in Lemma 3.2 and setting

$$\sum_{l=1}^{p} \lambda_{jl} = \tilde{\lambda}_j, \quad \sum_{l=1}^{p} p_l(t) = \tilde{p}(t), \quad \sum_{l=1}^{p} q_l(t) = \tilde{q}(t),$$

we get the following relations

$$\sum_{j=1}^{p} \tilde{\lambda}_j \frac{\partial f_j}{\partial x}(\chi_{x^0}) + \tilde{p}(t)\frac{\partial g}{\partial x}(\chi_{x^0}) + \tilde{q}(t)\frac{\partial h}{\partial x}(\chi_{x^0})$$
$$-\frac{d}{dt}\left\{\sum_{j=1}^{p} \tilde{\lambda}_j \frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) + \tilde{p}(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + \tilde{q}(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0})\right\}$$
$$+... + (-1)^k\frac{d^k}{dt^k}\left\{\sum_{j=1}^{p} \tilde{\lambda}_j \frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}) + \tilde{p}(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + \tilde{q}(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0})\right\} = 0,$$
$$\tilde{p}(t)g(\chi_{x^0}(t)) = 0, \quad \tilde{p}(t) \geq 0, \quad (\forall)t \in I,$$
$$\tilde{\lambda}_j \geq 0, \quad [\tilde{\lambda}_j \geq 1].$$

By dividing with $S = \sum_{j=1}^{p} \tilde{\lambda}_j \geq 1$ and denoting $\lambda_j = \tilde{\lambda}_j/S$, $p(t) = \tilde{p}(t)/S$, $q(t) = \tilde{q}(t)/S$, we obtain

$$\sum_{j=1}^{p} \lambda_j \frac{\partial f_j}{\partial x}(\chi_{x^0}) + p(t)\frac{\partial g}{\partial x}(\chi_{x^0}) + q(t)\frac{\partial h}{\partial x}(\chi_{x^0})$$
$$-\frac{d}{dt}\left\{\sum_{j=1}^{p} \lambda_j \frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) + p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0})\right\}$$
$$+... + (-1)^k\frac{d^k}{dt^k}\left\{\sum_{j=1}^{p} \lambda_j \frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}) + p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0})\right\} = 0,$$
$$p(t)g(\chi_{x^0}(t)) = 0, \quad p(t) \geq 0, \quad (\forall)t \in I,$$
$$\lambda \geq 0, \quad e^t\lambda = 1, \quad e^t = (1, 1, ..., 1) \in \mathbb{R}^p$$

and the proof is complete. □

Let us establish the second main result of this section: sufficient efficiency conditions for the variational problem (MVP).

Theorem 3.2 (Sufficient efficiency conditions for (MVP)) *Assume that Theorem* 3.1 *is fulfilled and there exist the vector functions* η *and* θ *satisfying Definition* 2.3. *Also, consider that the following statements are true:*

a) the functionals $\int_{t_0}^{t_1} f_l(\chi_x(t))dt$, *for* $l \in \{1,...,p\}$, *are* (ρ_l^1, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

b) $\int_{t_0}^{t_1} p(t)g(\chi_x(t))dt$ *is* (ρ^2, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

c) $\int_{t_0}^{t_1} q(t)h(\chi_x(t))dt$ *is* (ρ^3, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

d) one of the integral functionals $\int_{t_0}^{t_1} f_l(\chi_x(t))dt$, $l \in \{1,...,p\}$, $\int_{t_0}^{t_1} p(t)g(\chi_x(t))dt$ *and* $\int_{t_0}^{t_1} q(t)h(\chi_x(t))dt$ *is strictly* (ρ, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;* $(\rho = \rho_l^1, \rho^2$ or $\rho^3)$

e) $\sum_{l=1}^{p} \lambda_l \rho_l^1 + \rho^2 + \rho^3 \geq 0 \quad (\rho_l^1, \rho^2, \rho^3 \in \mathbb{R})$.

Then the point $x^0(\cdot)$ *is an efficient solution for* (MVP).

Proof. Assume that $x^0(\cdot)$ is not an efficient solution for (MVP). Then, for $l = \overline{1,p}$, there exists $x(\cdot) \in F(I)$, $x(\cdot) \neq x^0(\cdot)$, such that

$$\int_{t_0}^{t_1} f_l(\chi_x(t))dt \leq \int_{t_0}^{t_1} f_l(\chi_{x^0}(t))dt$$

and there exists at least $k \in \{1, 2, ..., p\}$ with

$$\int_{t_0}^{t_1} f_k(\chi_x(t))dt < \int_{t_0}^{t_1} f_k(\chi_{x^0}(t))dt.$$

Using the hypothesis a), it follows

$$b_{xx^0} \int_{t_0}^{t_1} \left[\eta_{txx^0} \frac{\partial f_l}{\partial x}(\chi_{x^0}(t)) + \frac{d\eta_{txx^0}}{dt} \frac{\partial f_l}{\partial x^{(1)}}(\chi_{x^0}(t))\right] dt$$

$$+ ... + b_{xx^0} \int_{t_0}^{t_1} \left[\frac{d^k \eta_{txx^0}}{dt^k} \frac{\partial f_l}{\partial x^{(k)}}(\chi_{x^0}(t))\right] dt \leq -\rho_l^1 b_{xx^0} \parallel \theta_{xx^0} \parallel^2 .$$

Multiplying by $\lambda_l \geq 0$ and making summation over $l = \overline{1,p}$, we obtain

$$b_{xx^0} \int_{t_0}^{t_1} \left[\eta_{txx^0} \lambda \frac{\partial f}{\partial x}(\chi_{x^0}(t)) + \frac{d\eta_{txx^0}}{dt} \lambda \frac{\partial f}{\partial x^{(1)}}(\chi_{x^0}(t))\right] dt \tag{3.1}$$

$$+ ... + b_{xx^0} \int_{t_0}^{t_1} \left[\frac{d^k \eta_{txx^0}}{dt^k} \lambda \frac{\partial f}{\partial x^{(k)}}(\chi_{x^0}(t)) \right] dt \leq - \left(\sum_{l=1}^{p} \lambda_l \rho_l^1 \right) b_{xx^0} \parallel \theta_{xx^0} \parallel^2 .$$

Also, the following inequality holds

$$\int_{t_0}^{t_1} p(t) g(\chi_x(t)) dt \leq \int_{t_0}^{t_1} p(t) g(\chi_{x^0}(t)) dt$$

and (see b)) it follows

$$b_{xx^0} \int_{t_0}^{t_1} \left[\eta_{txx^0} p(t) \frac{\partial g}{\partial x}(\chi_{x^0}(t)) + \frac{d\eta_{txx^0}}{dt} p(t) \frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t)) \right] dt \tag{3.2}$$

$$+ ... + b_{xx^0} \int_{t_0}^{t_1} \left[\frac{d^k \eta_{txx^0}}{dt^k} p(t) \frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}(t)) \right] dt \leq -\rho^2 b_{xx^0} \parallel \theta_{xx^0} \parallel^2 .$$

The equality (see c))

$$\int_{t_0}^{t_1} q(t) h(\chi_x(t)) dt = \int_{t_0}^{t_1} q(t) h(\chi_{x^0}(t)) dt$$

is fulfilled and implies

$$b_{xx^0} \int_{t_0}^{t_1} \left[\eta_{txx^0} q(t) \frac{\partial h}{\partial x}(\chi_{x^0}(t)) + \frac{d\eta_{txx^0}}{dt} q(t) \frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t)) \right] dt \tag{3.3}$$

$$+ ... + b_{xx^0} \int_{t_0}^{t_1} \left[\frac{d^k \eta_{txx^0}}{dt^k} q(t) \frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}(t)) \right] dt \leq -\rho^3 b_{xx^0} \parallel \theta_{xx^0} \parallel^2 .$$

Making the sum (3.1) + (3.2) + (3.3), side by side, and taking into account d), we have

$$b_{xx^0} \int_{t_0}^{t_1} \eta_{txx^0} \left[\lambda \frac{\partial f}{\partial x}(\chi_{x^0}(t)) + p(t) \frac{\partial g}{\partial x}(\chi_{x^0}(t)) + q(t) \frac{\partial h}{\partial x}(\chi_{x^0}(t)) \right] dt$$

$$+ b_{xx^0} \int_{t_0}^{t_1} \frac{d\eta_{txx^0}}{dt} \left[\lambda \frac{\partial f}{\partial x^{(1)}}(\chi_{x^0}(t)) + p(t) \frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t)) + q(t) \frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t)) \right] dt$$

$$+ ... + b_{xx^0} \int_{t_0}^{t_1} \frac{d^k \eta_{txx^0}}{dt^k} \left[\lambda \frac{\partial f}{\partial x^{(k)}}(\chi_{x^0}(t)) + p(t) \frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}(t)) \right] dt$$

$$+ b_{xx^0} \int_{t_0}^{t_1} \frac{d^k \eta_{txx^0}}{dt^k} \left[q(t) \frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}(t)) \right] dt < - \left(\sum_{l=1}^{p} \lambda_l \rho_l^1 + \rho^2 + \rho^3 \right) b_{xx^0} \parallel \theta_{xx^0} \parallel^2 .$$

Using e), it follows that $b_{xx^0} > 0$ and the foregoing inequality can be rewritten as

$$\int_{t_0}^{t_1} \eta_{txx^0} \left[\lambda \frac{\partial f}{\partial x}(\chi_{x^0}(t)) + p(t) \frac{\partial g}{\partial x}(\chi_{x^0}(t)) + q(t) \frac{\partial h}{\partial x}(\chi_{x^0}(t)) \right] dt$$

$$+\int_{t_0}^{t_1}\frac{d\eta_{txx^0}}{dt}\left[\lambda\frac{\partial f}{\partial x^{(1)}}(\chi_{x^0}(t))+p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t))+q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t))\right]dt$$

$$+...+\int_{t_0}^{t_1}\frac{d^k\eta_{txx^0}}{dt^k}\left[\lambda\frac{\partial f}{\partial x^{(k)}}(\chi_{x^0}(t))+p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}(t))\right]dt$$

$$+\int_{t_0}^{t_1}\frac{d^k\eta_{txx^0}}{dt^k}\left[q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}(t))\right]dt<-\left(\sum_{l=1}^{p}\lambda_l\rho_l^1+\rho^2+\rho^3\right)\parallel\theta_{xx^0}\parallel^2.$$

or, using the formula of integration by parts, we get

$$\int_{t_0}^{t_1}\eta_{txx^0}\left[\lambda\frac{\partial f}{\partial x}(\chi_{x^0}(t))+p(t)\frac{\partial g}{\partial x}(\chi_{x^0}(t))+q(t)\frac{\partial h}{\partial x}(\chi_{x^0}(t))\right]dt$$

$$+\eta_{txx^0}\left[\lambda\frac{\partial f}{\partial x^{(1)}}(\chi_{x^0}(t))+p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t))+q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t))\right]\Big|_{t_0}^{t_1}$$

$$-\int_{t_0}^{t_1}\eta_{txx^0}\frac{d}{dt}\left[\lambda\frac{\partial f}{\partial x^{(1)}}(\chi_{x^0}(t))+p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t))+q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t))\right]dt$$

$$+...+(-1)^k\int_{t_0}^{t_1}\eta_{txx^0}\frac{d^k}{dt^k}\left[\lambda\frac{\partial f}{\partial x^{(k)}}(\chi_{x^0}(t))+p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}(t))\right]dt$$

$$+(-1)^k\int_{t_0}^{t_1}\eta_{txx^0}\frac{d^k}{dt^k}\left[q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}(t))\right]dt<-\left(\sum_{l=1}^{p}\lambda_l\rho_l^1+\rho^2+\rho^3\right)\parallel\theta_{xx^0}\parallel^2.$$

Considering the boundary conditions $x(t_\varepsilon)=x_\varepsilon,\ x^{(\beta)}(t_\varepsilon)=x_{\beta\varepsilon},\ \varepsilon=0,1,\ \beta=\overline{1,k-1}$ (see $x(t_\varepsilon)=x_\varepsilon=x^0(t_\varepsilon),\ x^{(\beta)}(t_\varepsilon)=x_{\beta\varepsilon}=x^{0(\beta)}(t_\varepsilon)$), and knowing that $\dfrac{d^\zeta\eta_{tx^0x^0}}{dt^\zeta}=0,\ \zeta\in\{0,1,2,...,k-1\},\ (\forall)\ t\in I$ (see Definition 2.3), the previous inequality becomes

$$\int_{t_0}^{t_1}\eta_{txx^0}\left[\lambda\frac{\partial f}{\partial x}(\chi_{x^0}(t))+p(t)\frac{\partial g}{\partial x}(\chi_{x^0}(t))+q(t)\frac{\partial h}{\partial x}(\chi_{x^0}(t))\right]dt$$

$$-\int_{t_0}^{t_1}\eta_{txx^0}\frac{d}{dt}\left[\lambda\frac{\partial f}{\partial x^{(1)}}(\chi_{x^0}(t))+p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t))+q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t))\right]dt$$

$$+...+(-1)^k\int_{t_0}^{t_1}\eta_{txx^0}\frac{d^k}{dt^k}\left[\lambda\frac{\partial f}{\partial x^{(k)}}(\chi_{x^0}(t))+p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}(t))\right]dt$$

$$+(-1)^k\int_{t_0}^{t_1}\eta_{txx^0}\frac{d^k}{dt^k}\left[q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}(t))\right]dt<-\left(\sum_{l=1}^{p}\lambda_l\rho_l^1+\rho^2+\rho^3\right)\parallel\theta_{xx^0}\parallel^2.$$

By using Theorem 3.1, it results

$$0<-\left(\sum_{l=1}^{p}\lambda_l\rho_l^1+\rho^2+\rho^3\right)\parallel\theta_{xx^0}\parallel^2.$$

Applying the hypothesis e) and $\| \theta_{xx^0} \|^2 \geq 0$, we find a contradiction. Therefore, the point x^0 is an efficient solution to (MVP). The proof is complete. □

Replacing the integrals from hypotheses b), c), of Theorem 3.2, by the integral

$$\int_{t_0}^{t_1} [p(t)g(\chi_x(t)) + q(t)h(\chi_x(t))]\, dt,$$

the following statement is obtained.

Corollary 3.1 (Sufficient efficiency conditions for (MVP)) *Assume that Theorem* 3.1 *and Definition* 2.3 *are fulfilled and the following assertions are true:*

a) the functionals $\int_{t_0}^{t_1} f_l(\chi_x(t))dt$, *for* $l \in \{1, ..., p\}$, *are* (ρ_l^1, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* $\eta,\ \theta$;

b′) $\int_{t_0}^{t_1} [p(t)g(\chi_x(t)) + q(t)h(\chi_x(t))]\, dt$ *is* (ρ^2, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* $\eta,\ \theta$;

d′) one of the simple integral functionals

$$\int_{t_0}^{t_1} f_l(\chi_x(t))dt, \quad l \in \{1, ..., p\},$$

$$\int_{t_0}^{t_1} [p(t)g(\chi_x(t)) + q(t)h(\chi_x(t))]\, dt,$$

is strictly (ρ, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* $\eta,\ \theta$ $\quad (\rho = \rho_l^1$ *or* $\rho^2)$;

e′) $\sum_{l=1}^{p} \lambda_l \rho_l^1 + \rho^2 \geq 0 \quad (\rho_l^1, \rho^2 \in \mathbb{R})$.

Then the point $x^0(\cdot)$ *is an efficient solution for* (MVP).

Chapter 2

Controlled signomial dynamical systems as constraints in variational control problems

In this chapter, by using the notions of the variational differential system, adjoint differential system and modified Legendrian duality, we provide necessary optimality conditions for a class of signomial constrained optimal control problems.

2.1 Controlled signomial dynamical systems

Optimal control theory (see [3], [34], [40], [70], [113]), due to important applications in various branches of pure and applied science, has attracted many researchers over the years. For instance, Wagner [164] provided a Pontryagin-type maximum principle associated with some Dieudonné-Rashevsky type problems involving Lipschitz functions. Also, Udrişte [161], by using the *multi-time* concept, under simplified hypothesis, formulated and proved a maximum principle based on multiple/curvilinear integral cost functional and PDE constraints of m-flow type. Later, Treanţă and Vârsan [136] proved that solutions associated with an extended affine control system can be obtained as a limit process using solutions for a parameterized affine control system and weak small controls.

In this chapter, inspired by the previously mentioned research works and in accordance with [137], [144] and [145], we formulate and prove necessary conditions of optimality for a new class of optimal control problems involving signomial type constraints.

Further, for $x = (x^1, ..., x^n) \in R^n$ we shall write $x > 0$ if $x^i > 0,\ i = \overline{1,n}$, and $x \geq 0$ if $x^i \geq 0,\ i = \overline{1,n}$. The set $R^n_+ = \{x \in R^n : x \geq 0\}$ is said to be the *positive orthant* and, most of the time, we shall use the *open positive orthant* $\mathbf{P}^n = \{x \in R^n : x > 0\}$. On the set $\mathbf{P}^n$, we consider the distinct monomials of the form $v^k = v^k(x) = (x^1)^{\alpha_{1k}} \cdots (x^n)^{\alpha_{nk}},\ k = \overline{1,m}$, where α_{ik} are real numbers. If a^i_k are real numbers, then the functions $a^i_k v^k$, with summation upon k, are called *signomials*. The *controlled signomial dynamical systems* are defined as follows:

$$\dot{x}^i(t) = a^i_k v^k\left(x(t), u(t)\right), \quad i = \overline{1,n},$$

where $v^k\left(x,u\right) := (x^1)^{\alpha_{1k}} \cdots (x^n)^{\alpha_{nk}} (u^1)^{\gamma_{1k}} \cdots (u^r)^{\gamma_{rk}},\ \alpha_{ik}, \gamma_{\beta k} \in R,\ k = \overline{1,m},\ i = \overline{1,n},\ \beta = \overline{1,r},\ t \in I \subseteq R$, and $\mathbf{P}^r \ni u = (u^\beta),\ \beta = \overline{1,r}$, is a *control.*

In the following, let us consider an optimal control problem based on a simple integral cost functional, constrained by a controlled signomial dynamical system:

$$\max_{u(\cdot), x_{t_0}} I\left(u(\cdot)\right) = \int_0^{t_0} X\left(x(t), u(t)\right) dt \tag{1.1}$$

subject to

$$\dot{x}^i(t) = a^i_k v^k\left(x(t), u(t)\right), \quad i = \overline{1,n},\ k = \overline{1,m} \tag{1.2}$$

$$u(t) \in U,\ \forall t \in [0, t_0]; \quad x(0) = x_0,\ x(t_0) = x_{t_0}. \tag{1.3}$$

In the aforementioned optimal control problem we used the following terminology and notations: $t \in [0, t_0]$ is *parameter of evolution*, or the *time*; $[0, t_0] \subset R_+$ is the *time interval*; $x : [0, t_0] \to \mathbf{P}^n,\ x(t) = (x^i(t)),\ i = \overline{1,n}$, is a C^2-class function, called *state vector*; $u : [0, t_0] \to \mathbf{P}^r,\ u(t) = (u^\beta(t)),\ \beta = \overline{1,r}$, is a continuous *control vector*; U is the set of all admissible controls; the *running cost* $X\left(x(t), u(t)\right)$ is a C^1-class function, called *autonomous Lagrangian.*

Through this chapter, the summation over the repeated indices is assumed. Further, we introduce the *Lagrange multiplier* $p(t) = (p_i(t))$, also called *co-state variable (vector)*, and a new Lagrange function

$$L\left(x(t), u(t), p(t)\right) = X\left(x(t), u(t)\right) + p_i(t)\left[a^i_k v^k\left(x(t), u(t)\right) - \dot{x}^i(t)\right].$$

In this way, we change the initial optimal control problem into a free optimisation problem

$$\max_{u(\cdot), x_{t_0}} \int_0^{t_0} L\left(x(t), u(t), p(t)\right) dt$$

subject to

$$u(t) \in U,\ p(t) \in P,\ \forall t \in [0, t_0]$$

$$x(0) = x_0,\ x(t_0) = x_{t_0},$$

where P is the set of co-state variables, which will be defined later. The *control Hamiltonian*

$$H\left(x(t), u(t), p(t)\right) = X\left(x(t), u(t)\right) + a^i_k p_i(t) v^k\left(x(t), u(t)\right),$$

or, equivalently,

$$H = L + p_i \dot{x}^i \quad \text{(modified Legendrian duality)}$$

permits us to rewrite the previous optimal control problem as follows

$$\max_{u(\cdot), x_{t_0}} \int_0^{t_0} \left[H\left(x(t), u(t), p(t)\right) - p_i(t)\dot{x}^i(t)\right] dt$$

subject to

$$u(t) \in U,\ p(t) \in P,\ \forall t \in [0, t_0]$$

$$x(0) = x_0,\ x(t_0) = x_{t_0}.$$

2.1.1 Variational and adjoint signomial differential systems

Let us suppose that (1.2) is satisfied. Fix the control $u(t)$ and a corresponding solution $x(t)$ of (1.2). Let $x(t, \varepsilon)$ be a differentiable variation of the state variable $x(t)$, fulfilling

$$\dot{x}^i(t, \varepsilon) = a^i_k v^k \left(x(t, \varepsilon), u(t)\right)$$

$$x(t, 0) = x(t), \quad i = \overline{1, n}.$$

Denote by $y^i(t) := x^i_\varepsilon(t, 0)$. Taking the partial derivative with respect to ε, evaluating at $\varepsilon = 0$, we obtain the following system

$$\dot{y}^i(t) = a^i_k v^k_{x^j} \left(x(t), u(t)\right) \cdot y^j(t),$$

called *variational signomial differential system.* The differential system

$$\dot{p}_j(t) = -a^i_k p_i(t) v^k_{x^j} \left(x(t), u(t)\right)$$

is called the *adjoint signomial differential system* of the previous variational differential system since the scalar product $p_i(t) \cdot y^i(t)$ is a first integral of the two systems. Indeed, we have

$$\frac{d}{dt}\left[p_i(t) \cdot y^i(t)\right] = 0.$$

2.2 Necessary conditions of optimality

Let $\hat{u}(t) = \left(\hat{u}^\beta(t)\right),\ \beta = \overline{1, r}$, be a continuous control vector defined on the closed interval $[0, t_0]$, with $\hat{u}(t) \in Int\,U$, which is an optimal point for the aforementioned control problem. Consider $u(t, \varepsilon) = \hat{u}(t) + \varepsilon h(t)$ a variation of the optimal control vector $\hat{u}(t)$, where h is an arbitrary continuous vector function. We have $\hat{u}(t) \in Int\,U$ and, since a continuous function on a compact interval $[0, t_0]$ is bounded, there exists $\varepsilon_h > 0$ such that $u(t, \varepsilon) = \hat{u}(t) + \varepsilon h(t) \in Int\,U,\ \forall\ |\varepsilon| < \varepsilon_h$. This ε is a "small" parameter used in our variational arguments.

Define $x(t, \varepsilon)$ as the state variable corresponding to the control variable $u(t, \varepsilon)$, i.e.,

$$\dot{x}^i(t, \varepsilon) = a^i_k v^k \left(x(t, \varepsilon), u(t, \varepsilon)\right), \quad i = \overline{1, n},\ \forall t \in [0, t_0]$$

and $x(0, \varepsilon) = x_0$. As well, consider (for $|\varepsilon| < \varepsilon_h$) the function (integral with parameter)

$$I(\varepsilon) := \int_0^{t_0} X\left(x(t, \varepsilon), u(t, \varepsilon)\right) dt.$$

Since $\hat{u}(t)$ is an optimal control variable we get $I(0) \geq I(\varepsilon), \ \forall \ |\varepsilon| < \varepsilon_h$. Also, for any continuous vector function $p(t) = (p_i)(t) : [0, t_0] \to R^n$, we have

$$\int_0^{t_0} p_i(t) \left[a^i_k v^k \left(x(t,\varepsilon), u(t,\varepsilon)\right) - \dot{x}^i(t,\varepsilon)\right] dt = 0.$$

The variations involve

$$L\left(x(t,\varepsilon), u(t,\varepsilon), p(t)\right) = X\left(x(t,\varepsilon), u(t,\varepsilon)\right)$$

$$+p_i(t) \left[a^i_k v^k \left(x(t,\varepsilon), u(t,\varepsilon)\right) - \dot{x}^i(t,\varepsilon)\right]$$

and the associated function (integral with parameter)

$$I(\varepsilon) = \int_0^{t_0} L\left(x(t,\varepsilon), u(t,\varepsilon), p(t)\right) dt.$$

Now, assume that the co-state variable $p(t) = (p_i(t))$ is of C^1-class. The control Hamiltonian with variations

$$H\left(x(t,\varepsilon), u(t,\varepsilon), p(t)\right) = X\left(x(t,\varepsilon), u(t,\varepsilon)\right) + a^i_k p_i(t) v^k \left(x(t,\varepsilon), u(t,\varepsilon)\right)$$

changes the above integral with parameter as follows

$$I(\varepsilon) = \int_0^{t_0} \left[H\left(x(t,\varepsilon), u(t,\varepsilon), p(t)\right) - p_i(t)\dot{x}^i(t,\varepsilon)\right] dt.$$

Differentiating with respect to ε, evaluating at $\varepsilon = 0$, and using the formula of integration by parts, it follows

$$I'(0) = \int_0^{t_0} \left[H_{x^j}\left(x(t), \hat{u}(t), p(t)\right) + \dot{p}_j(t)\right] \cdot x^j_\varepsilon(t,0) dt$$

$$+ \int_0^{t_0} H_{u^\beta}\left(x(t), \hat{u}(t), p(t)\right) \cdot h^\beta(t) dt - \left(p_i(t) \cdot x^i_\varepsilon(t,0)\right)\big|_0^{t_0},$$

where $x(t)$ is the state variable corresponding to the optimal control variable $\hat{u}(t)$. We must have $I'(0) = 0$, for any continuous vector function $h(t) = \left(h^\beta(t)\right), \ \beta = \overline{1,r}$. Also, the functions $x^i_\varepsilon(t,0)$ solve the following Cauchy problem

$$\nabla_t x^i_\varepsilon(t,0) = a^i_k v^k_x\left(x(t,0), u(t)\right) \cdot x_\varepsilon(t,0) + a^i_k v^k_u\left(x(t,0), u(t)\right) \cdot h(t)$$

$$t \in [0, t_0], \quad x_\varepsilon(0,0) = 0.$$

Consequently, we obtain

$$\frac{\partial H}{\partial u^\beta}\left(x(t), \hat{u}(t), p(t)\right) = 0, \quad \forall t \in [0, t_0]. \tag{2.1}$$