Linear and Integer Programming

Linear and Integer Programming

By

Sanaullah Khan, Abdul Bari
and Mohammad Faisal Khan

Cambridge
Scholars
Publishing

Linear and Integer Programming

By Sanaullah Khan, Abdul Bari and Mohammad Faisal Khan

This book first published 2019

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-5275-3913-X
ISBN (13): 978-1-5275-3913-6

CONTENTS

PREFACE

Linear Programming (LP) is a process in which a real-life problem is transformed into a mathematical model consisting of a linear objective function of several decision variables to be maximized (or minimized) through an algorithmic procedure in the presence of a finite number of linear constraints. If the decision variables are required to be integers, the LP model is called an integer linear programming (ILP) model.

A wide variety of real-life problems arising in economic optimization, management science, strategic planning and other areas of knowledge can be easily transformed into the format of linear programming or integer linear programming. Besides their importance in a large number of practical applications, the algorithms of LP and ILP can be visualized through simple geometrical concepts, thus making the algebraic derivations easy to understand.

The first chapter of the book consists of the definitions of the terms used in the remainder. The basic concepts are illustrated through simple prototype examples. The reader is required to know intermediate level mathematics. The understanding of the basic concepts present in the development of algorithms for LP and ILP is essential as many real-life problems demand specific adaptations of standard techniques.

In Chapter 2 we start by developing the basic concept of linear programming. Dantzig's simplex method is then developed, and the simple iterative steps of the algorithm are illustrated by solving a numerical example. The concepts of duality and sensitivity analysis are also developed in this chapter. We explain how perturbations in the cost and constraint coefficients affect the shadow prices.

In Chapter 3 we discuss integer linear programming. In order to observe the breadth of its applicability, some important situations are first formulated as ILP problems. Then in this chapter we develop the enumerative procedures for solving ILP problems. The well-known branch and bound technique is discussed and illustrated with numerical examples.

A search enumeration technique suitable for (0-1) IP problems developed by Balas is also discussed in this chapter.

In Chapter 4 we develop the cutting plane techniques for solving ILP problems. Gomory's mixed integer cut is also developed. The primal and dual all integer methods are discussed in detail. The column tableau presentation of an LPP is explained (parallel to the row representation) as it simplifies the addition of new cutting planes when solving an ILP problem by cutting plane techniques.

All suggestions for further improvement of the book will be received thankfully by us, to serve the cause of imparting good, correct and useful information to the students.

The book is basically written as a textbook for graduate students in operations research, management science, mathematics and various engineering fields, MBA and MCA. It can be used as a semester course in linear and integer programming.

Sanaullah Khan
Abdul Bari
Mohammad Faisal Khan

CHAPTER 1

INTRODUCTION

Linear and integer programming techniques have brought tremendous advancements in the field of optimization. Optimization is the science of selecting the best of many possible decisions in a complex real-life situation. Thus it is required in almost all branches of knowledge today.

The development of optimization techniques, and specially the linear and integer linear programming methods, requires some mathematical background at undergraduate level. In order to provide a self-contained treatment in the book, we first explain the important terms used in subsequent chapters.

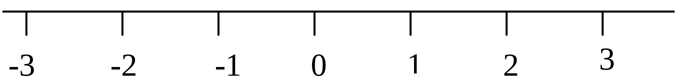
1.1 NUMBERS

Natural Numbers

The natural numbers are the numbers 1,2,3,…. with which we count.

Integers

The integers are the numbers 0, ± 1, ± 2 etc. These may be considered as particular points along a straight line (the real line) on either side of 0 thought of as distances, measured from an origin at zero (0). It is conventional to place positive numbers to the right of 0 and negative numbers to the left. The successive integers are one unit apart.



Whole Numbers

The whole numbers are the natural numbers including 0, namely 0,1,2,3 etc.

Prime Numbers

Included in the natural numbers are the prime numbers that are divisible by 1 and themselves only, e.g. 1,3,5,7 etc.

Rational Numbers

If p is an integer and q is a natural number and p and q have no common factor then $\frac{p}{q}$ is a rational number. Integers are rational numbers with q=1. For example, -5/4 is a rational number since -5 is an integer and 4 is a natural number, and there is no common factor. A rational number can be represented either as a terminating decimal or a non-terminating but repeating (recurring) decimal, e.g. 2/7= .285714285714….

Fraction

The term fraction is sometimes used to express numbers of the form p/q where p is an integer and q is a natural number. The numbers may have common factors. For example, -4/6 is a fraction since -4 is an integer and 6 is a natural number. There is a common factor, 2, so that -4/6=-2/3 which is a rational number.

Irrational Numbers

Numbers such as π = 3.14159…, √2=1.41421…, √3, 2+√3 etc., which are non-repeating or non-terminating decimals, are called irrational numbers. These can only be written down approximately as numbers. (It is not possible to find any integers, p & q such that p/q represents such numbers. e.g. 22/7 is not an exact representation of π, it is only an approximation.)

Real Numbers

Rational and irrational numbers taken together are the real numbers. e.g. 1, 2.13, 4.2,π, √2 etc.

Complex Numbers

The numbers such as 2+3i, 4-2i, where i^2=-1, are known as complex numbers. (By introducing complex numbers, it is possible to write down the solution of equations such as x^2= -1).

For example $-\sqrt{-9} = i\sqrt{9} = 3i$: $3i^2 = -9$

Equality and Inequality

Real numbers may be thought of as being distributed along a line. In order to perform operations with them, we need certain symbols that enable us to compare two numbers. If a and b are real numbers then:

$a=b$ means a is equal to b

$a>b$ means a is greater than b (a lies to the right of b on the real line)

$a<b$ means a is less than b (a lies to the left of b on the real line)

$a\geq b$ means a is greater than or equal to b

$a\leq b$ means a is less than or equal to b

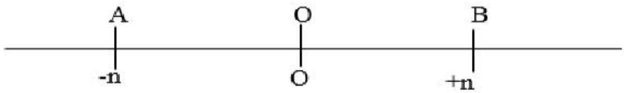
$a\neq b$ means a is not equal to b (either $a<b$ or $a>b$)

$a\cong b$ or $a\approx b$ means a is approximately equal to b (thus $\sqrt{2}\cong 1.4142$).

If the same operation is performed on each side of an inequality, the direction of the inequality remains unchanged except when multiplied by a negative number in which case the inequality changes its direction. For example, the inequality $4>3$ when multiplied on both sides by 2 gives $8>6$; If we add -8 on both sides we get $-4>-5$; When divided by 4 we get the unchanged inequality $1>3/4$. However, if we multiply both sides by -1, the inequality gets reversed as $-4<-3$.

Modulus of a Real Number

The modulus or absolute value of a real number gives its magnitude, regardless of sign. In terms of the numbers on the real line, the modulus is the distance of the number from 0 and is denoted by placing a vertical line on either side of the number. Let us consider the points A and B at a distance of n units on either side of O on the real line (see figure below).



If $n>0$ then OA represents $-n$ and OB represents $+n$

The modulus of $+n = |+n| = n =$ length of $OB>0$.

The modulus of $-n = |-n| = n$=length of OA =length of OB>0

Alternatively, the modulus of n may be thought of as the positive square root of n^2 so that $|n| = \sqrt{n^2}$.

Interval

A section of the real line is called an interval. The interval $a \leq x \leq b$ is said to be closed since the end points are included in it. This closed interval is denoted by [a,b]. The interval $a < x < b$ is said to be open as it does not include the end points and is denoted by (a,b).

Factorial

The product of all the integers from 1 to n, where n is a whole number, is called n factorial and is denoted by n!

Thus

$$4! = 4.3.2.1 = 24$$

$$n! = n\,(n-1)\,(n-2)\,\ldots\,2.1 = n\,(n-1)!$$

$$0! = 1$$

Reciprocal

The reciprocal of a real number a is $b = \frac{1}{a}$ provided $a \neq 0$. An alternative notation of the reciprocal is a^{-1}. The reciprocal of $\pm\infty$is 0 and the reciprocal of 0 is $\pm\infty$.

Proportional and Inversely Proportional

A quantity a is proportional to another quantity b (written as $a \propto b$) if a=kb where k is a constant and $\overline{a}$ is inversely proportional to $\overline{b}$ written as ($\overline{a} \propto 1/\overline{b}$).

Indices

If a quantity is written in the form A^b then b is the index of A.

Rules for operating with indices are:

i. $A^b \times A^c = A^{b+c}$

ii. $(A^b)^c = A^{b\times c} = A^{bc}$

iii. $A^0 = 1, A \neq 0$

iv. $A^{-b} = 1/A^b$

Field

A system of numbers F is called a field if knowing that a,b∈ F; a+b, a-b, a.b, a/b, ka etc. also belong to F.

1.2 SETS

A set is defined as a collection or aggregate of objects. There are certain requirements for a collection or aggregate of objects to constitute a set. These requirements are:

(a) The collection or aggregate of objects must be well defined; i.e. we must be able to determine equivocally whether or not any object belongs to this set.

(b) The objects of a set must be distinct; i.e. we must be able to distinguish between the objects and no object may appear twice.

(c) The order of the objects within the set is immaterial; i.e. the set (a,b,c) is the same as the set (b,c,a).

Example- The collection of digits 0,1,2,…,9 written as {0,1,2,3,4,5,6,7,8,9} is a set.

Example- The letters in the word MISSISSIPPI satisfy all the requirements for a set as the 4 letters in the word are M, I, S and P well defined, distinct and the order of the letters is immaterial.

We use curly brackets or braces “{ }” to designate a set. It is customary to name a set using capital letters such as A, C, S, X, etc.

Elements or Members of a Set

The objects which belong to a set are called its elements or members. The elements or the members of the set are designated by one of two methods: (1) the roster method (tabular form), or (2) the descriptive method (set builder form).

The roster method involves listing within braces all members of the set. The descriptive method consists of describing the membership of the set in a manner such that one can determine if an object belongs in the set. E.g. in the roster method, the set of digits would appear as D={0,1,2,3,4,5,6,7,8,9} while in the descriptive method it would appear as

$$D=\{x|x = 0,1,2,3,4,5,6,7,8,9\}.$$

The Greek letter ∈ (epsilon) is customarily used to indicate that an object belongs to the set. If D represents the set of digits, then 2 ∈ D means that 2 is an element of D. The symbol ∉ (epsilon with slashed line) represents non-membership. i.e. "is not an element of", or "does not belong to".

Finite and Infinite Sets

A set is termed finite or infinite depending upon the number of elements in the set. The set D above is finite, since it has only ten digits. The set N of positive integers or natural numbers is infinite, since the process of counting continues infinitely.

Equal Sets

Two sets A and B are said to be equal, written A=B, if every element of A is in B and every element of B is in A. For example, the set A={0,1,2,3,4} and B={1,0,2,4,3} are equal.

Subset

A set *A* is a subset of another set *B* if every element in *A* is in *B*. For example, If *B*={0,1,2,3} and *A*={0,1,2}, then every element in *A* is in *B*, and *A* is a subset of B.

The symbol used for subset is ⊆; *A* is a subset of *B* is written as *A*⊆*B*. The set of rational numbers generally denoted by *Q* has the following important subsets:

N = {1,2,3,…}; the set of all counting numbers or natural numbers.

W = {0,1,2,3,…}; the set of whole numbers.

I = {…-5,-4,-3,-2,-1,0,1,2,…}; the set of integers.

F = {a/b | a,b….∈N}; the set of all fractions.

A is a “proper subset” of another set B if all the elements in A are in B, but not all the elements in B are in A. It is designated by the symbol $\subset$. For A to be a proper subset of B, B must have all elements that are in A plus at least one element that is not in A; we write $A \subset B$.

The Set N of natural numbers is a proper subset of the set I of integers.

Universal Set

The term “universal set” is used for the set that contains all the elements the analyst will wish to consider. For example, if the analyst were interested in certain combinations of the English letters then the universal set would be all the letters of the English alphabet.

Null Set

The set which contains no element is called the null set or empty set. It is designated by the Greek letter φ (phi). Thus an empty set $\varphi = \{\}$ is a null set as there is no element in it. It should be noted that $\{0\}$ is not a null set. It is a set containing 0.

The universal set and the null set are subsets of the universal set.

Venn Diagram

The Venn diagram, named after the English logician John Venn (1834–83), consists of a rectangle that conceptually represents the universal set. Subsets of the universal set are represented by circles drawn within the rectangle, see Fig. 1.1.

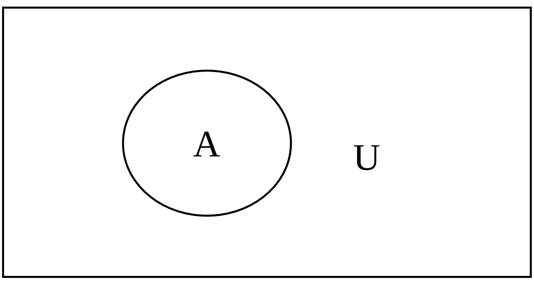


Fig 1.1 A Venn diagram

Operations on Sets

There are certain operations on sets which combine the given sets to yield another set. Such operations are easily illustrated through the use of the Venn diagram, as will be seen below.

Complementation

Let A be any subset of a universal set U. The complement of A is the subset of elements of U that are not members of A and is denoted by A' (read ‘A complement’). Fig. 1.2 shows the Venn diagram for the complement A' of the set A. A' is represented by the shaded region.

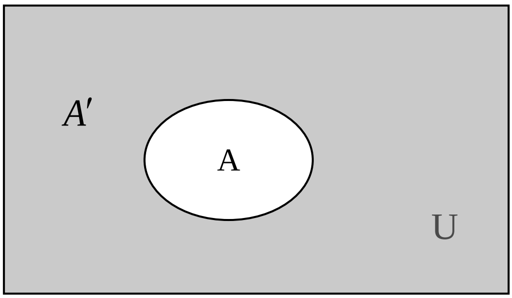


Fig 1.2 Complement of a set

Example: For the universal set D = {0,1,2,3,4,5,6,7,8,9}, the complement of the subset A = {0,1,3,5,7,9}, is $A' = \{ 2,4,6,8 \}$.

Intersection

If A and B are any two sets, then the intersection of A and B is the set of all elements which are in A and also in B and is denoted by A∩B (read ‘A intersection B’). See Fig. 1.3.

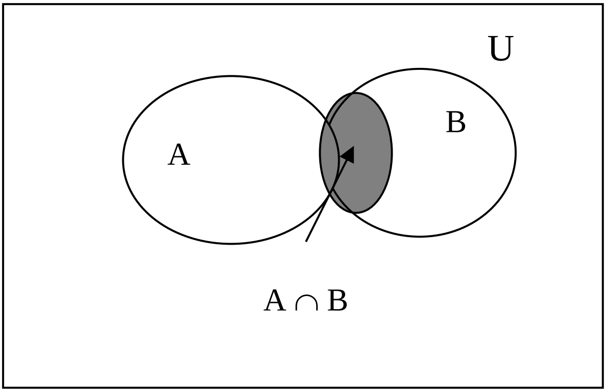


Fig 1.3 Intersection of two sets.

Example: For the universal set D = {0,1,2,3,4,5,6,7,8,9} with subsets A = {0,1,2,5,7,9} and B = {0,3,4,5,9}, the intersection of A and B is given by A∩B = {0,5,9}. From the definition it follows that A∩U =A.

Union

The union of the sets A and B is the set of all elements of A together with all the elements of B and is denoted by A∪B (read as A union B) See Fig. 1.4.

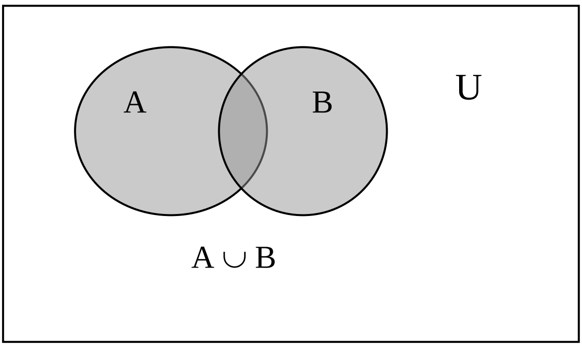


Fig 1.4 Union of two sets

Example: For the universal set D = {0,1,2,3,4,5,6,7,8,9} with subset A = {0,1,2,5,7,9} and B = {0,3,4,5,9}, A∪B = {0,1,2,3,4,5,7,9}.

From the definitions, it follows that

$$A \cup U = U$$

$$A \cup A' = U.$$

Disjoint Sets

Two sets *A* and *B* are said to be disjoint (or mutually exclusive) if there is no element in common, i.e. if $A \cap B = \varphi$.

For example, the sets A = {1,2,3} and B = {4,5,6} are disjoint.

Cartesian Product

If *A* and *B* are two sets, then the cartesian product of the sets, designated by *A*x*B*, is the set containing all possible ordered pairs (a,b) such that $a \in A$ and $b \in B$. If the set A contains the elements a_1, a_2 and a_3 and the set B contains the elements b_1 and b_2, then the cartesian product AxB is the set

$$AxB = \{(a_1,b_1), (a_1,b_2),(a_2,b_1),(a_2,b_2),(a_3,b_1),(a_3,b_2)\}$$

or

$$AxB = \left\{ (a,b) | a \in A, b \in B \right\}$$

1.3 LOGARITHMS

Definition: If for a positive real number a (a>0), $a^m = b$, we say that m is the logarithm of b to the base a. In symbols we write $m = \log_a b$

e.g. $\log_3 9 = 2$ (because $3^2 = 9$).

The logarithm to base 10 of A ($=10^m$) is defined to be the index m so that $\log_{10} A = m$, The basic concept used in working with logarithms to base 10 is that of expressing a real positive number A in the form $A = 10^m$. By writing another number B as 10^n, we have

$$AB = (10^m)(10^n) = 10^{m+n}.$$

Thus multiplication of A and B is transformed by this process into the addition of their indices m and n.

$$\log_{10}(AB) = m+n$$

These concepts may be generalized to the logarithm to any base a, where $a > 0$.

Let $A = a^M$ and $B = a^N$,

Then $log_a A = M$and $log_a B = N$

From the laws of indices, we note that

(i) $log_a 1 = 0$ as $a^0 = 1$

(ii) $log_a a = 1$ as $a^1 = a$

(iii) For $a > 1$, $log_a 0 = -\infty$ as $a^{-\infty} = 0$

(iv) For $a < 1$, $\log_a 0 = \infty$ as $a^{\infty} = 0$

(v) $log_{10} 1\,00 = 2$ as $10^2 = 100$

(vi) $log_{10} .001 = -3$ as $10^{-3} = .001$

(vii) $log_a \ (AB) = log_a A + log_a B$

(viii) $log_a \left(\frac{A}{B}\right) = log_a A - log_a B$

(ix) $log\ A^B = B\ log\ A$

(x) $log_a A = log_b A \,.\, log_a b$

(xi) $log_{10} A = log_e A \times 2.3026.$

The most commonly occurring bases are 10 and the irrational number e = 2.71828…… The number e gives rise to natural or Napierian logarithms with $\log_e$ x denoted by ln x.

1.4 SEQUENCES AND SERIES

Sequences

A sequence $\{x_n\}$ is an ordered set of symbols of the form

$$x_1, x_2, \ldots\ldots, x_n, \ldots,$$

where n is a number, and is such that each value of x is associated in turn with the natural numbers. Thus x_1 is associated with the number 1, x_2 with the number 2, and in general the term x_n is associated with number n. Some simple examples of sequences are:

$\{2n\}$, for which $x_1 = 2$, $x_2 = 4, \ldots, x_n = 2n, \ldots$ and

$\{x^n\}$, which stands for the sequence $x^1, x^2, x^3, \ldots, x^n, \ldots$

Series

Given a sequence $\{x_n\}$, it is possible to form a second sequence denoted by $\{s_n\}$ with terms s_n that are the sums of the first n terms of the sequence $\{x_n\}$. That is,

$$S_n = x_1 + x_2 + \ldots + x_{n-1} + x_n = S_{n-1} + x_n$$

The terms s_n are sometimes referred to as finite series. For any given sequence $\{x_n\}$, it is possible to write down an infinite series S

Where $S = x_1 + x_2 + x_3 + \ldots + x_n + \ldots$

The Arithmetic Series

The sequence $\{x_n\}$ with $x_n = a + (n-1)d$, where a and d are some constants is called an arithmetic progression (AP).

A sequence $\{S_n\}$ may be constructed by summing the terms of the sequence $\{x_n\}$, as

$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)d]$.

The quantity $\{S_n\}$ is called a arithmetic series of order n.

The Geometric Series

The sequence $\{x_n\}$ defined by $x_n = ar^{n-1}$, where a is a number and $r > 0$ is the common ratio, is called a geometric progression (GP). The sequence $\{S_n\}$where $S_n = a(1 + r + \cdots + r^{n-1}) = a\frac{r^n - 1}{r - 1}$ for $r \neq 1$. is called a geometric series.

1.5 MATRICES

Matrix

The matrix is a rectangular array of numbers enclosed by brackets [ ] or ( ). The number of rows and columns determine the dimension or order of the matrix. A matrix with $m$rows and n columns is called an $m \times n$ matrix.

If a_{ij} is the element in the i^{th} row and j^{th} column of a given matrix A then we write $\underset{m\times n}{A} = [a_{ij}],\ 1 \leq i \leq m, 1 \leq j \leq n.$

A matrix in which the number of rows equals to the number of columns (m=n) is called a square matrix. A square matrix in which at least one diagonal element is non-zero and the rest of the elements are zero is called a diagonal matrix.

A diagonal matrix having all its diagonal elements equal to one is called an identity (or a unit) matrix and is denoted by I or I_n.

For example, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Two matrices $A = \left(a_{ij}\right)$ and $B = (b_{ij})$ of the same order, say $m \times n$, are equal if $a_{ij} = b_{ij}$ for all $i = 1, \dots, n$ and $j = 1, \dots, n$. A matrix is called a null matrix if all its elements are equal to zero.

A square matrix is called an upper triangular matrix if all the entries below the diagonal are zero. Similarly, a lower triangular matrix is one in which all the elements above the diagonal are zero.

Addition and Subtraction

The sum of matrices $A \quad = [a_{ij}]$ and $B \quad = [b_{ij}]$ is defined only when A and B have the same order.

The sum is then

A+B= $[a_{ij} + b_{ij}] = [c_{ij}]$

A-B is defined as A+(-B). It is called the difference of A and B.

Multiplication by a Scalar

A matrix may be multiplied by a scalar quantity. Thus, ifλ is a scalar and A is an $m \times n$matrix, then$A$ is a matrix, each element of which is λ times the corresponding element of A.

Matrix Multiplication

Two matrices may be multiplied together only if the number of columns in the first matrix equals the number of rows in the second matrix. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then the product is defined as

AB=C=$(c_{ij})_{mxp}$, where $c_{ij} = \sum_{r=1}^{n} a_{ir}\, b_{rj}$ for i=1,2,…m; j=1,2,….p

Laws of Algebra for Multiplication

(i) A (BC)=(AB) C : Associative law holds.

(ii) A (B+C)=AB + AC : Distributive law holds.

Transpose of a Matrix

The transpose of a matrix A of order $m \times n$is the matrix of order $n \times m$, denoted by A^Tor A', such that the (i,j)th element of A is the (j,i)th element of A'.

The following properties can be shown to hold:

(i) $(A + B)' = A' + B'(A + B\quad)' = A' + B'$

(ii) $(AB)' = B'A'$

(iii) $(A')' = A$

(iv) $(kA)' = kA'$

Symmetric Matrix

A square matrix A is said to be symmetric if it is equal to its transposed matrix, i.e. if A =A'. A symmetric matrix can be constructed by multiplying the given matrix by its transpose. Thus $A'A$ and AA' are both symmetric.

Skew Symmetric Matrix

A matrix$A$ is said to be skew symmetric if $A = -A'$. It is always a square matrix whose diagonal elements are zero. It can be shown that any matrix A can be written in the form

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

Orthogonal Matrix

A square matrix$A$ is said to be orthogonal if $AA' = I$

Trace of a Matrix

The trace of a square matrix$A$, denoted by tr$A$, is defined as the sum of its diagonal elements, i.e. tr A =$\sum_i a_{ii}$.

(i) If A and B are square matrices of the same order then

tr(A+B)=trA + trB

(ii) If C and D are such that CD and DC are both defined, then CD

and DC are both square, and tr CD=tr DC.

Non-Singular Matrices

A matrix A is called non-singular if there exists a matrix B, such that AB=BA=I. It follows that both A and B must be square matrices of the same order.

Inverse of a Matrix

Given a matrix A, if there exists a square matrix B such that AB=BA =I, then B is called the inverse of A. It is denoted byA−1. B is the left inverse of A if BA =I. C is the right inverse of A if AC =I.

The following properties can be shown to hold:

(i). $(AB)^{-1} = B^{-1}A^{-1}$

(ii). $(A^{-1})' = (A')^{-1}$

(iii). An inverse matrix is unique

(iv). $(A^{-1})^{-1} = A$

(v) A zero matrix has no inverse.

Remark: If a matrix does not have an inverse it is said to be singular.

Determinant of a Matrix

Associated with any square matrix $A = [a_{ij}]$ having n^2elements is a number called the determinant of A and denoted by $|A|$ or det.A given by $|A| = \sum(\pm)a_{1i}a_{2j} \dots a_{nr}$, the sum being taken over all permutations of the second subscripts. The assignment of the + or – sign is explained below.

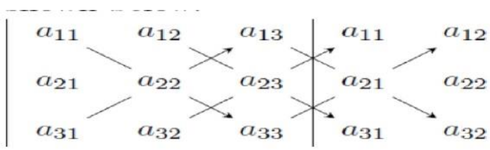
The determinant of a 2x2 matrix A_2 is given by

$$|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$$

The determinant of a 3x3 matrix A_3 is given by

$$|A_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

The determination of a plus or minus sign is easily done by employing the following scheme. First repeat the first two columns of the matrix; next draw diagonals through the components as shown below:



The products of the components on the diagonals from upper left to lower right are summed, and the products of the components on the diagonals from lower left to upper right are subtracted from this sum.

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Note that, if the square matrix is of order 1, $|A|$ is not the absolute value of the number A. E.g. for the matrix of order 1 given by A= (-3), $|A| = -3$.

Example: find the determinant of $A = \begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & 6 \\ 3 & 0 & -2 \end{pmatrix}$

$$|A| = \left| \begin{array}{ccc|cc} 1 & 3 & -4 & 1 & 3 \\ 2 & -1 & 6 & 2 & -1 \\ 3 & 0 & -2 & 3 & 0 \end{array} \right.$$

$$|A| = 2 + 54 + 0 - 12 - 0 + 12 = 56$$

Determinants have the following properties:

(i). $|A'| = |A|$

(ii). $|A| = 0$ if and only if A is singular.

(iii). $|AB| = |A||B|$

(iv). If two rows (or two columns) are interchanged then the value of the determinant changes only in sign.

(v). If each element in a row (or column) is multiplied by a scalar then the value of the determinant is also multiplied by the scalar. It follows that for an $n \times n$ matrix A, $|kA| = k^n|A|$., where k is a scalar.

(vi). If a multiple of one row is added to another, the value of the determinant remains the same (a similar result also holds for the columns).

(vii). The value of the determinant of an upper triangular matrix is given by the product of the elements on the leading diagonal.

Because of this last property, an efficient way of calculating the value of a determinant is by reducing it to triangular form. The reduction to a triangular form is done by using the properties (iv), (v) and (vi) , known as elementary transformations.

Minors

The determinant of the sub-matrix formed by deleting one row and one column from a given square matrix is called the minor associated with the element lying in the deleted row and deleted column.

Consider a 3×3 matrix A given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then the minor associated with a_{11} will be

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, which is obtained by deleting the first row and first column of A.

Cofactors

A cofactor is a minor with its proper sign, where the sign is determined by the subscripts of the element with which the cofactor is associated. If the sum of the subscripts is even then it will have a plus sign, and if the sum is odd then it will have a minus sign.

For example, in the matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

the cofactor of a_{11} is $+\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ while the cofactor of a_{12} is $-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

The Cofactor Matrix

Associated with a matrix A is another matrix whose elements are the cofactors of the corresponding elements of A. Such a matrix is called the cofactor matrix.

The Adjoint Matrix

The transpose of the cofactor matrix is called the adjoint matrix.

$$Adj.A = [cofactor.A]'$$

1.6 FINDING THE INVERSE OF A MATRIX

Let A be a non-singular matrix, i.e. $|A| \neq 0$.

It can be shown that the inverse of A is given by

$$A^{-1} = \frac{AdjA}{|A|}$$

where $AdjA$ is the adjoint matrix of A.

Example: Let $A = \begin{pmatrix} 7 & -3 \\ 3 & 4 \end{pmatrix}$. Then $|A| = 37, Cofactor\ A = \begin{pmatrix} 4 & -3 \\ 3 & 7 \end{pmatrix}$, $AdjA = \begin{pmatrix} 4 & 3 \\ -3 & 7 \end{pmatrix}$. Thus $A^{-1} = \begin{pmatrix} \frac{4}{37} & \frac{3}{37} \\ \frac{-3}{37} & \frac{7}{37} \end{pmatrix}$

Computing the inverse directly by using this formula or by using the definition of the inverse is quite lengthy as can be seen in the following simple case:

consider the 2×2 matrix

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

We know that the inverse of A is another matrix A^{-1}such that $AA^{-1} = I$. Let

$$B = A^{-1} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Then by definition,

$$\mathrm{AA}^{-1} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From the definition of the equality of matrices, we get

$$\begin{aligned} 4b_{11} + 2b_{21} &= 1 \\ 4b_{12} + 2b_{22} &= 0 \\ 1b_{11} + 3b_{21} &= 0 \\ 1b_{12} + 3b_{22} &= 1 \end{aligned}$$

These are four equations in four unknowns, which on solution give

$$b_{11} = \frac{3}{10}, b_{12} = -\frac{1}{5}, b_{21} = -1, b_{22} = \frac{2}{5}$$

Hence,

$$A^{-1} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & -\frac{1}{5} \\ -\frac{1}{10} & -\frac{2}{5} \end{pmatrix}$$

We can check our computations as

$$AA^{-1} = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{3}{10} & \frac{-1}{5} \\ \frac{-1}{10} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It is obvious that finding the inverse of a matrix of order higher than 2x2 would be a rather cumbersome task using the direct method. The Gaussian Elimination procedure described below is a systematic procedure for higher order matrices.

Finding the Inverse by the Gaussian Elimination Method

Consider an $n \times n$ matrix$A$. The procedure starts by augmenting the matrix A with the identity matrix I_n. The $n \times 2n$ augmented matrix $A|I_n$then undergoes the following row operations until the matrix A is transformed to I_n and the augmented identity matrix I_n is transformed to a matrix B. This new matrix B will be the inverse of the matrix$A$.

The row operations are:

(i) adding a multiple of one row to another;

(ii) multiplying a row by a nonzero constant;

(iii) interchanging the rows.

The row operations on $(A|I)$are carried out such that A is first made upper triangular; then the elements above the diagonal are made zero; and finally the diagonal elements are made to equal one. If A is singular, this method results in a zero row being produced in the left hand part of the augmented matrix.

Let us find the inverse of A = $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 6 & 9 \end{pmatrix}$, if it exists.

The Augmented matrix is

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 6 & 9 & 0 & 0 & 1 \end{array}\right)$$

Step 1. Subtract the first row from the second row. This operation is denoted as $R_2 = R_2 - R_1$. Also subtract twice the first row from the third. This is denoted as $R_3 = R_3 - 2R_{1.}$ The two operations make the off-diagonal elements in the first column equal to zero. The new matrix is

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right)$$

The other steps are given below.

Step 2.

$$R_3 = R_3 - 4R_2,$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{array}\right)$$

Step 3.

$$R_1 = R_1 - 2R_3, R_2 = R_2 - R_3,$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -3 & 8 & -2 \\ 0 & 1 & 0 & -3 & 5 & -1 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right)$$

Step 4.

$$R_1 = R_1 - R_2$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & -3 & 5 & -1 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{array}\right)$$

The A matrix has now been transformed to I_3.

Therefore, $A^{-1} = \begin{pmatrix} 0 & 3 & -1 \\ -3 & 5 & -1 \\ 2 & 4 & 1 \end{pmatrix}$, which is the inverse of A $= \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 6 & 9 \end{pmatrix}$.

Check:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 6 & 9 \end{pmatrix}\begin{pmatrix} 0 & 3 & -1 \\ -3 & 5 & -1 \\ 2 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.7 VECTORS

A vector is a row (or column) of numbers in a given order. The position of each number within the row (or column) is meaningful. Naturally a vector is a special case of a matrix in that it has only a single row (or a single column). It is immaterial whether a vector is written as a row or a column. Vectors are generally denoted by lower case bold letters. The components bear suffixes for specifying the order. A vector is usually thought of as a column of numbers. The row representation will be denoted by its transpose:

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \underline{x}^T = (x_1, \dots, x_n)$$

Thus in general, a row vector is a $1 \times n$ matrix, where n=2,3,….. and a column vector is an $m \times 1$ matrix, where m=2,3….. The numbers in a vector are referred to as the elements or components of the vector. A vector consists of at least two components.