Quantum Phenomena in Simple Optical Systems

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Saribek Gevorgyan and Menua Gevorgyan

Cambridge Scholars Publishing

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This book first published 2020

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-5275-4433-8
ISBN (13): 978-1-5275-4433-8

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PREFACE

The book is based on original works of the authors of the book published in peer-reviewed scientific journals. Problems of quantum optics in simple optical systems are studied in the book, namely, generation of the second and third harmonics and generation of the second and third subharmonics, as well as two- and three-photon absorption in the case of two- and three-photon perturbations of the absorbing modes, respectively. Quantum optics problems, such as quantum behaviour of unstable optical systems (second and third harmonic generation), as well as problems related to obtaining squeezed, superposition, and entangled states of optical fields in simple optical systems, are considered. Although the authors tried to write fewer formulas and gave a lot of graphic illustrations of the behaviour of the considered optical systems, some knowledge in the field of quantum optics and Monte Carlo algorithms for studying quantum behaviour of systems is required in order to understand the materials. The authors are grateful to Helen Edwards for the idea of writing this book and the moral support shown during the work on the book.

CHAPTER 1

QUANTUM THEORY OF UNSTABLE BEHAVIOR OF THE INTRACAVITY SECOND HARMONIC GENERATION PROCESS

Introduction

All materials in this chapter are taken from [1–7].

This section investigates the dynamics of quantum fluctuations of the number of photons and phases of the fundamental and second harmonic modes in the unstable region of the system for the process of intracavity generation of the second harmonic. The phase and the number of photon distribution functions of these modes are calculated in the positive P-representation [8]. Joint distribution functions of the number of photons and phases of the field modes are also calculated. It is shown that the dynamics of the semiclassical value of the number of photons of the field modes can differ considerably from the dynamics of the solution to the Langevin equations for the number of photons of the same modes. In particular, in the unstable region, depending on the initial state of the system, the dynamics of the semiclassical value of the number of photons can be without oscillations, whereas the solution to the Langevin equations for the same initial state of the system may have strong oscillations. Algorithms for calculating the distribution functions of photon numbers, as well as for calculating the joint distribution function of photon numbers of the interacting modes of the field, are given in the positive P-representation. These algorithms can also be applied to calculate both the phase and the joint distribution function of the number of photons and phases of the interacting modes. It is shown that, at the critical point, the distribution functions of the number of photons of the fundamental mode and of the second harmonic mode have single-peak structures, which are asymmetric with respect to the most probable value of the number of photons of the corresponding modes. In the unstable region, the distribution functions of the number of photons of both modes gradually transform into a two-component structure. Each of the components of the distribution functions represents a state of the mode in which it persists most of the time. In the unstable region, the joint distribution function of the photon numbers of the fundamental and second harmonic mode also has a two-peak structure. The above results were obtained in the case of system evolution from an initial state, which has a normal distribution of stochastic amplitudes of the field modes. It is shown that, in the P-representation, in the unstable region of interaction of the system and in the region of large interaction times, the distribution functions of the phases of the field modes also have a two-component structure, which is in contrast to the stable region. Unlike the corresponding photon number distribution functions, the heights of the peaks of the two components of the state of the phase are equal; this means that the system spends the same amount of time in these components of the state. It is shown that the dynamics of the average value of the phase of the field modes depends strongly on the initial state in the unstable region. In the case of evolution of the system from an initial state, which has a normal distribution of stochastic amplitudes of the field modes, the average values of the phases of the field modes do not change in time in the region of large interaction times, whereas in the case of a coherent initial state of both modes and in the case of a nonzero average value of the phases of the initial state of the system, the average values of the phase oscillate around zero (around the phase of the perturbation field). In the region of instability, in the case of coherent initial states of the field modes, oscillation is observed for both the phase and joint distribution functions of the number of photons and phases, as well as the joint phase distribution functions of the interacting modes of the field. The behaviour of the joint distribution function of the phases of the field modes was studied depending on the system's distance from the critical point. It is shown that, in the unstable region, the joint distribution function of phases of the field modes has a wide dip around the point representing the classical phase matching of the field modes. The latter shows that the classical phase matching has a zero probability of realization in the unstable region. When the system changes into the stable region, a peak is obtained instead of a dip at this point, which corresponds to the classical phase matching of interacting modes.

The dynamics of the quantum entropy, of the Wigner functions, and of the quadrature amplitudes of the field modes are investigated, applying the Monte Carlo wave-function method. It is shown that, depending on the increase in the amplitude of the perturbation field, the quantum entropy of the field modes increases, and the field modes can localize in coherent, squeezed, and unstable states. In the region of a strong perturbation field, the modes of the field first localize in squeezed states, followed by a decay of the squeezed states and a gradual localization of the modes in unstable states.

For the process of intracavity generation of the second harmonic, the dynamics of correlation of quadrature amplitude fluctuations of the fundamental and second harmonic modes is studied, depending on the nonlinear coupling coefficient between the modes. It is shown that, in this system, depending on the nonlinear coupling coefficient, entangled states of the field in relation to variables of quadrature amplitudes may be obtained. It is also shown that, depending on the value of the nonlinear coupling coefficient, the entanglement of the states of field modes, which is related to one quadrature amplitude, may vanish, whereas the states of the field modes remain entangled in relation to the other quadrature amplitude.

1.1 Nonlinear System, Langevin Equations, and Quantum Noise

We consider a model of second harmonic generation inside a two-mode resonator. A nonlinear medium is placed inside a cavity tuned to the frequencies of the fundamental mode ω_1 and the second harmonic mode $\omega_2 = 2\omega_1$. The system is perturbed externally by a coherent field at a frequency of the fundamental mode. The equations for the density matrix of the optical field of this system may be written in the interaction picture as follows:

$$\frac{\partial \rho}{\partial t} = (i\hbar)^{-1}[H_{\text{int}}, \rho] + L_1(\rho) + L_2(\rho). \qquad (1.1.1)$$

Here, the first term represents the external perturbation of the system and the nonlinear interaction of optical fields.

$$H_{\text{int}} = i\hbar E(a_1 - a_1^+) + i\hbar\chi(a_1^2 a_2^+ - a_1^{+2} a_2), \qquad (1.1.2)$$

where a_i^+ and a_i $(i = 1,2)$ are the creation and annihilation operators for the corresponding modes, respectively;χ is the coefficient of coupling between the modes, which is proportional to the nonlinear susceptibility

$\chi^{(2)}$; and E is the amplitude of the classical perturbing field at the frequency ω_1.

The $L_1(\rho)$ and $L_2(\rho)$ superoperators describe the damping of the fundamental and second harmonic modes through the resonator mirrors:

$$L_i(\rho) = -\gamma_i(2a_i\rho a_i^+ - a_i^+a_i\rho - \rho a_i^+a_i), (i = 1,2). \quad (1.1.3)$$

Here, γ_1 and γ_2 are the damping rates of the respective modes.

From Eqs. (1.1.1)–(1.1.3), in the positive P-representation [8], the following Langevin equations may be derived for the stochastic β_i, α_i amplitudes of the field:

$$\frac{\partial\alpha_1}{\partial t} = E - \gamma_1\alpha_1 - 2\chi\beta_1\alpha_2 + (-2\chi\alpha_2)^{1/2}\xi_1(t),$$

$$\frac{\partial\beta_1}{\partial t} = E - \gamma_1\beta_1 - 2\chi\alpha_1\beta_2 - (-2\chi\beta_2)^{1/2}\xi_2(t),$$

$$\frac{\partial\alpha_2}{\partial t} = -\gamma_2\alpha_2 + \chi\alpha_1^2,$$

$$\frac{\partial\beta_2}{\partial t} = -\gamma_2\beta_2 + \chi\beta_1^2, \quad (1.1.4)$$

where $\xi_1(t)$ and $\xi_2(t)$ are independent Langevin sources of noise with the following nonzero correlation functions:

$$< \xi_i(t)\xi_j(t') >= \delta_{ij}\delta(t - t'). \quad (1.1.5)$$

In terms of the stochastic variables of the photon number and phase,

$$n_k = \alpha_k\beta_k, \phi_k = \frac{1}{2i}ln\left(\frac{\alpha_k}{\beta_k}\right) (k = 1,2). \quad (1.1.6)$$

From Eq. (1.1.4) and with the help of the Ito formulas, we get

$$\frac{\partial n_1}{\partial t} = 2En_1^{1/2} - 2\gamma_1 n_1 - 4\chi n_1 n_2^{1/2}\, cos(\phi_2 - 2\phi_1) +$$
$$(-2\chi)^{1/2} n_1^{1/2} n_2^{1/4} \times$$
$$\{\xi_1(t)\, exp[i(\varphi_2 - 2\varphi_1)/2] - \xi_2(t)\, exp[-i(\phi_2 - 2\phi_1)/2]\},$$

$$\frac{\partial n_2}{\partial t} = -2\gamma_2 n_2 + 2\chi n_1 n_2^{1/2}\, cos(\phi_2 - 2\phi_1),$$

$$\frac{\partial \phi_1}{\partial t} = -\frac{E}{n_1^{1/2}} sin(\phi_1) - 2\chi n_2^{1/2}\, sin(\phi_2 - 2\phi_1) +$$
$$\chi n_1^{-1} n_2^{1/2}\, sin(\phi_2 - 2\phi_1) + \frac{1}{2i}(-2\chi)^{1/2} n_1^{-1/2} n_2^{1/4} \times$$
$$\{\xi_1(t)\, exp[i(\phi_2 - 2\phi_1)/2] - \xi_2(t)\, exp[-i(\phi_2 - \phi_1)/2]\},$$

$$\frac{\partial \phi_2}{\partial t} = \chi n_1 n_2^{-1/2}\, sin(2\phi_1 - \phi_2). \qquad (1.1.7)$$

In a semiclassical approach, i.e., neglecting the noise sources in (1.1.7), for average photon numbers in the steady-state limit ($\gamma_{1,2} t >> 1$), we obtain

$$E(n_1^0)^{1/2} - \gamma_1 n_1^0 - 2\chi n_1^0 (n_2^0)^{1/2} = 0, \qquad -\gamma_2 n_2^0 + \chi n_1^0 (n_2^0)^{1/2} = 0,$$
$$\phi_1^0 = \phi_2^0 = 0. \qquad (1.1.8)$$

After linearization with respect to small fluctuations ($\delta n_i = n_i - n_i^0$, $\delta\phi_i = \phi_i - \phi_i^0$) near the steady-state solutions n_i^0, ϕ_i^0 ($i = 1,2$), equations (1.1.7) are converted to

$$\frac{\partial}{\partial t}\begin{bmatrix}\delta n_1\\ \delta n_2\end{bmatrix} = A_n \begin{bmatrix}\delta n_1\\ \delta n_2\end{bmatrix}, \quad \frac{\partial}{\partial t}\begin{bmatrix}\delta \phi_1\\ \delta \phi_2\end{bmatrix} = A_\phi \begin{bmatrix}\delta \phi_1\\ \delta \phi_2\end{bmatrix}, \qquad (1.1.9)$$

where A_n and A_ϕ are as follows:

$$A_n = \begin{bmatrix} -\gamma_1 - 2\chi(n_2^0)^{1/2} & -2\gamma_2 \\ 2\chi(n_2^0)^{1/2} & \gamma_2 \end{bmatrix},$$

$$A_\phi = \begin{bmatrix} -\gamma_1 + 2\chi(n_2^0)^{1/2}(1-(n_1^0)^{-1}) & -\chi(n_2^0)^{1/2}(2-(n_1^0)^{-1}) \\ 2\gamma_2 & -\gamma_2 \end{bmatrix}. \quad (1.1.10)$$

We assume that the classically characterized driving field phase is equal to zero. The eigenvalues λ_1 and λ_2 of the $A_n$matrix and $\lambda_3$and λ_4 of the A_φ matrix for $n_1 >> 1$ are

$$\lambda_{1,2} = -\frac{1}{2}(\gamma_1 + \gamma_2 + 2\chi(n_2^0)^{1/2}) \pm \frac{1}{2}\left[(\gamma_1 - \gamma_2 + 2\chi(n_2^0)^{1/2})^2 - 16\chi\gamma_2(n_2^0)^{1/2}\right]^{1/2},$$

$$\lambda_{3,4} = -\frac{1}{2}(\gamma_1 + \gamma_2 - 2\chi(n_2^0)^{1/2}) \pm \frac{1}{2}\left[(\gamma_1 - \gamma_2 - 2\chi(n_2^0)^{1/2})^2 - 16\chi\gamma_2(n_2^0)^{1/2}\right]^{1/2}. \quad (1.1.11)$$

The real parts of $\lambda_1$and λ_2 are always negative, but the λ_3 and λ_4 coefficients at the critical values

$$n_{2cr} = \left(\frac{\gamma_1+\gamma_2}{2\chi}\right)^2, n_{1cr} = \frac{\gamma_2(\gamma_1+\gamma_2)}{2\chi^2}, \ E_{cr} = (2\gamma_1 + \gamma_2)\left[\frac{\gamma_2(\gamma_1+\gamma_2)}{2\chi^2}\right]^{1/2} \quad (1.1.12)$$

become imaginary. It means that, at this point (the Hopf bifurcation point), the small fluctuations of the phase do not damp. This is a physical reason for the unstable behaviour of the entire optical system (to right of the bifurcation point).

The Langevin system of equations (1.1.4) can be written in the following differential form:

$$d\alpha_1 = (\varepsilon - \alpha_1 - 2k\beta_1\alpha_2)d\tau + (-2k\alpha_2)^{1/2}w_1(\tau)(d\tau)^{1/2},$$

$$d\beta_1 = (\varepsilon * -\beta_1 - 2k\alpha_1\beta_2)d\tau + (-2k\beta_2)^{1/2}w_2(\tau)(d\tau)^{1/2},$$

$$d\alpha_2 = (-r\alpha_2 + k\alpha_1^2)d\tau,$$

$$d\beta_2 = (-r\beta_2 + k\beta_1^2)d\tau. \quad (1.1.13)$$

Here, the quantity $\tau = \gamma_1 t$ is the scaled time, $r = \gamma_2/\gamma_1$ is the ratio of the damping rates of the modes inside the cavity, $k = \chi/\gamma_1$ is the scaled constant of coupling between the modes, and $\varepsilon = E/\gamma_1$ is the scaled perturbation at the frequency of the fundamental mode. The independent noise sources $w_1(\tau)$ and $w_2(\tau)$ have zero mean values: $< w_1(\tau) >=< w_2(\tau) >= 0$. Nonzero values only have the means of squares of these quantities:

$$< [w_1(\tau)]^2 >=< [w_2(\tau)]^2 >= 1. \quad (1.1.14)$$

Without noise sources and at long times of interaction, the system of equations (1.1.13) has a stable stationary solution only for small values of the perturbing field ($\varepsilon < \varepsilon_{cr}$). The quantity $\varepsilon_{cr}$is the Hopf bifurcation point and is determined by the formula

$$\varepsilon_{cr} = (2 + r)[r(1 + r)/(2k^2)]^{1/2}. \quad (1.1.15)$$

When $\varepsilon > \varepsilon_{cr}$, small time fluctuations of phases of the fundamental and second harmonic modes cease damping. The system loses its stability around the stationary solutions to Eqs. (1.1.13) without noise terms. In this region, classical solutions for numbers of photons get into a selfoscillation regime. Our calculations are based on a simulation of noise sources. We model these sources by the following formulas [9]:

$$w_1(\tau) = [-2\,ln(z_1)]^{1/2}\,cos(2\pi z_2),$$

$$w_2(\tau) = [-2\,ln(z_1)]^{1/2}\,sin(2\pi z_2), \qquad (1.1.16)$$

where z_1 and z_2 are independent random numbers with a uniform distribution in the interval (0;1). For the random quantities in expression (1.1.16), we have

$$< w_i(\tau) >= 0,\ < w_i(\tau)w_j(\tau) >= \delta_{ij}\ (i,j = 1,2). \qquad (1.1.17)$$

For the solution of the system of equations (1.1.13), the Euler numerical technique for differential equations is used, which is not quick; however, unlike Runge-Kutta methods, it is more correct for solving equations with Langevin noise sources [10]. This approach was initially developed in [11] to calculate the dynamics of the photon numbers in SHG. The lack of numerical solutions to the Langevin equations, in particular, the appearance of nonphysical spikes at large times and at a sufficient distance to the right of the bifurcation point, is discussed in [10]. It was shown in [12] that the deterministic part of the quantum stochastic equations for a wavefunction can be solved with the Runge-Kutta methods, but the stochastic part should be simulated with the more reliable Euler's method.

1.2 Semiclassical and Quantum Solutions of Langevin Equations

Let us consider the semiclassical solutions of Eqs. (1.1.13) for photon numbers that were for the first time discovered in [13]. Figure 1.1 (curve 1) shows the photon number dynamics of the second harmonic with the initial conditions $\alpha_i(0) = \beta_i(0) = 1$, $(i = 1,2)$, using the semiclassical approach (the noise sources in (1.1.13) are neglected). As one can see, under such conditions, the self-pulsing regime is absent. Curve 2 represents the real part of the photon number

$$Re(\beta_2 \alpha_2) = Re\, n_2 \qquad (1.2.1)$$

for a single solution of Langevin equations (with noise sources) under the same conditions. The imaginary part $Im(\beta_2 \alpha_2)$ is not taken into account, as it becomes zero when averaged over an ensemble with a large number of realizations.

The absence of a self-pulsing regime in the semiclassical approach can be connected with the fact that the initial values β_i, α_i are absolutely real, so the initial phase is equal to the phase in the steady-state solution ($\delta\Phi = 0$) [13]. Meanwhile, the presence of quantum noise in realizations of Langevin equations leads to small phase fluctuations around the steady-state solution. This is the reason for the change of the system into the self-pulsing region in a separate solution. That is why it is interesting to consider the quantum dynamics of the phase in SHG in order to describe and further investigate the instability in this process.

In the region of instability $E > E_{cr}$, the temporal behaviour of the number of photons of the fundamental mode is analogous to the temporal behaviour of the number of photons of the second harmonic mode.

1.3 Quantum Dynamics of Photon Numbers of the Modes

First, let us study the temporal behaviour of the photon numbers of the fundamental mode and of the second harmonic in the instability region of the system. The average value of the number of photons of the optical system is calculated as the mathematical expectation of the stochastic quantities: $n_i(\tau) = Re\{\alpha_i(\tau)\beta_i(\tau)\}$, where

$$\langle n_i(\tau)\rangle = \lim_{N\to\infty}\left(\frac{1}{N}\sum_{(j)=1}^{N} n_i^{(j)}(\tau)\right), \quad (i = 1,2). \qquad (1.3.1)$$

Here, (j) is the realization number and N is the total number of realizations of the set of equations (1.1.14).

In this section, we calculate the quantum dynamics of the number of photons of the system in the case of a normal distribution of the initial values of stochastic amplitudes of the fundamental mode and of the second harmonic.

$$\alpha_1(0) = [-2\,ln(z_1)]^{1/2}\,cos(2\pi z_2) + i[-2\,ln(z_1)]^{1/2}\,sin(2\pi z_2),$$

$$\beta_1(0) = \alpha_1^*(0),$$

$$\alpha_2(0) = [-2\,ln(z_3)]^{1/2}\,cos(2\pi z_4) + i[-2\,ln(z_3)]^{1/2}\,sin(2\pi z_4),$$

$$\beta_2(0) = \alpha_2^*(0) \quad , \qquad (1.3.2)$$

where z_1, z_2, z_3, z_4 are independent random numbers with a uniform distribution on the interval (0…1). In particular, for these initial values of the amplitudes of the optical field modes, we have

$$< \alpha_i(0) >=< \beta_i(0) >= 0,$$

$$< \beta_i(0)\alpha_j(0) >= 2\delta_{ij}\ (i, j = 1,2). \qquad (1.3.3)$$

We also investigate the quantum dynamics of photon numbers of the interacting modes of the optical system in the case of evolution of the system from initial coherent states of the fundamental and second harmonic modes.

Figure 1.2 demonstrates the dynamics of the average value of the number of photons of the fundamental mode in the region of unstable behaviour of the system (curve 1). According to (1.1.15), the critical perturbation of the system is $\varepsilon_{cr} = 30$. The function was calculated for the value of perturbation $\varepsilon = 50$. To calculate the quantum dynamics of the number of photons of the fundamental mode, we used 1000 realizations for the set of equations (1.1.13) with initial values given in (1.3.2). Curve 2 represents the behaviour of the number of photons of one of the realizations of the set of equations (1.1.13) with the initial values of the stochastic amplitudes of the field modes $\alpha_j(0) = i$ and $\beta_j(0) = -i$ $(j = 1,2)$. Curve 2 was calculated for the same parameter values as for curve 1. At large interaction times, oscillations of curve 2 show that, in this region, the system is in a self-oscillating regime. In the region of instability, the dynamics of the number of photons of the fundamental mode, averaged over an ensemble of realizations of the system (described by the set of equations (1.1.13)), has no oscillations (curve 1), if the interacting modes develop from an initial state that has a normal distribution of stochastic amplitudes (1.3.2) for both modes.

Figure 1.3 shows the time behaviour of the average number of photons of the second harmonic mode (curve 1) in the case of evolution of both modes of the system from an initial state with a normal distribution of the stochastic amplitudes. To calculate this function, we used 1000 independent realizations for the set of equations (1.1.13). The function was calculated for the same parameters and perturbation of the system as the curves in Fig. 1.2. The time behaviour of the second harmonic mode in the region of instability is similar to the behaviour of the fundamental mode. Curve 2 in Fig. 1.3 represents one realization of the second harmonic mode for the case where the system evolves from the same initial values of the stochastic amplitudes for which curve 2 in Fig. 1.2 was calculated.

Figure 1.4 demonstrates the quantum dynamics of the number of photons of the fundamental mode (curve 1) and of the second harmonic (curve 2) for the case of evolution of the system from the following initial values of the stochastic amplitudes of the field modes: $\alpha_j(0) = 1 + i$ and $\beta_j(0) = 1 - i$ $(j = 1,2)$. To obtain each curve, we used 20 000 independent realizations for the set of equations (1.1.13). Here, unlike in the previous cases, the average numbers of photons of the modes oscillate in the region of large evolution times. The amplitude of oscillations of the fundamental mode is larger than the amplitude of oscillations of the second harmonic. These two cases of evolutions of the system show that, in the region of instability and in the region of long interaction times, the quantum behaviour of the number of photons of the modes depends strongly on the initial state of these modes.

1.4 Distribution Functions for Photon Numbers of the Modes

In this section, we study distribution functions for the number of photons of the fundamental mode and the second harmonic in the vicinity of the bifurcation point of the system. First, let us study the distribution functions of the fundamental mode in the region of large interaction times. Let us describe the algorithm for calculating this function. To calculate the distribution function, first, a segment in the phase space of the number of photons of the fundamental mode should be chosen. This is the region outside of which the distribution function vanishes. This region can be easily found by studying the dynamics of several realizations of the number of photons of the fundamental mode. We divide the length of this segment into N_p equal parts and calculate the quantity

$$\Delta n_1 = (n_{1\max} - n_{1\min})/N_p. \quad (1.4.1)$$

Then, we choose an array of numbers $A(i)$ with the dimension $N_p + 1$ $(i = 0, N_p)$ and equate all of its elements to zero.

Next, a cycle of calculations begins, and one of its steps is presented below. We calculate the number of photons of the fundamental mode for an instant of time by using the solution to the set of equations (1.1.13). After this, we calculate

$$i = Int[(n_1(\tau) - n_{1\min})/\Delta n_1], \quad (1.4.2)$$

where Int means the calculation of the integer part of the expression in square brackets. The value of the array element should be increased by unity as follows:

$$A(i) \rightarrow A(i) + 1. \quad (1.4.3)$$

Then, calculations are repeated until the necessary number of realizations N is obtained. The array elements $A(i)$ represent approximate values of the unnormalized distribution function for the number of photons of the fundamental mode for the instant of time τ at the points $n_{1min} + i\Delta n_1$, $(i = 0, N_p)$. We construct a curve that passes through these points and normalizes the obtained function to unity. The normalized curve represents an approximate plot of the distribution for the number of photons of the fundamental mode in the positive P representation.

Figure 1.5 shows the distribution function for the number of photons of the fundamental mode ($n_1 = Re(\beta_1\alpha_1)$) at the critical point of the system ($\varepsilon = \varepsilon_{cr} = 30$). To calculate this function, we used 100 000 independent numerical solutions of the set of Langevin equations (1.1.13) with the initial values of stochastic amplitudes of the field modes (1.3.2). The function was calculated in the range of large interaction times ($\tau = 10$) and for the parameter values k = 0.1 and r = 1. This function is asymmetric relative to the most probable value. To the right of the point of the most probable value of the number of photons, the probability of realization is larger than that observed to the left.

Figure 1.6 demonstrates the distribution function for the number of photons of the fundamental mode in the instability region ($\varepsilon = 50$) and for large interaction times ($\tau = 10$). To calculate this function, we used 100 000 independent numerical realizations of the set of Langevin equations (1.1.13). The function was calculated for the same values of the parameters k = 0.1 and r = 1. In this case, the distribution function for the number of photons has two most probable values, each of which corresponds to a state in which the system mostly resides. After the passage of the critical point, as the system penetrates deep into the unstable region, the distribution function of the number of photons gradually changes from the single-component asymmetric structure (Fig.1.5) to the two-component structure (Fig.1.6).

In the region of long interaction times and in the region of instability, the behaviour of the distribution function of the number of photons of the second harmonic is analogous to the behaviour of the distribution function for the number of photons of the fundamental mode. In Fig.1.7, the distribution function for the number of photons of the second mode is presented in the region of large interaction times (τ = 10) and at the critical point of the system ($\varepsilon = \varepsilon_{cr} = 30$). The function was calculated for the same parameters of the system that were used to calculate the distribution function shown in Fig. 1.5 (k = 0.1 and r = 1). To obtain this function, we used 100 000 independent realizations of the set of Langevin equations with the initial values (1.3.2). Similarly to the distribution function for the number of photons of the fundamental mode at the critical point (Fig.1.5), this function is also asymmetric relative to the most probable value. However, in this case, the probability of realizing a number of photons that is smaller than the most probable number is greater than the probability of realizing a number of photons that is greater than the most probable number.

Figure 1.8 presents a distribution function for the number of photons of the second harmonic in the region of large interaction times and in the instability region. The function was obtained using 100 000 independent realizations of the set of Langevin equations (1.1.13) with the initial independent values (1.3.2). The function was calculated for the parameters k = 0.1 and r = 1. The distribution function of the number of photons of the second harmonic has a two-peak structure, as well as the distribution function of the number of photons of the fundamental mode. Each of the most probable values of the distribution function corresponds to a state in which the system resides most of the time.

1.5 Joint Distribution Functions for Photon Numbers of the Modes

In this section, we study the behaviour of the joint distribution function of the number of photons of the fundamental mode and of the second harmonic.

To calculate this function, we use the following algorithm. In the phase space of the number of photons of the fundamental mode and of the second harmonic, we choose a rectangle with the vertices $(n_{1\min}, n_{2\min})$, $(n_{1\min}, n_{2\max})$, $(n_{1\max}, n_{2\max})$, and $(n_{1\max}, n_{2\min})$. This is the region of the phase space outside of which the joint distribution function of the number of photons of the fundamental and second harmonic modes vanishes. We divide the edges of this rectangle into N_p equal parts and calculate the quantities $\Delta n_1 = (n_{1\max} - n_{1\min})/N_p$ and $\Delta n_2 = (n_{2\max} - n_{2\min})/N_p$. Then, we determine the two-dimensional array of numbers $A(i,j)$ with the dimension $(N_p + 1) \times (N_p + 1)$, where $i, j = 0, N_p$, and equate all its elements to zero. Next, a cycle of calculations begins (one of its steps is presented below). Using the solution to the set of equations (1.1.13), we calculate the number of photons of the fundamental mode and of the second harmonic at the instants of time τ, $n_1(\tau)$ and $n_2(\tau)$. Then, we use the following formulas:

$$i = Int[(n_1(\tau) - n_{1\min})/\Delta n_1],$$

$$j = Int[(n_2(\tau) - n_{2\min})/\Delta n_2]. \quad (1.5.1)$$

Here, as in formula (1.4.2), Int means the calculation of the integer part of the expression in square brackets. We increase the value of the array element $A(i,j)$ with the number (i,j) by 1 as follows:

$$A(i,j) \rightarrow A(i,j) + 1. \quad (1.5.2)$$

After this, calculations are repeated until the required number of realizations N is obtained.

The array elements $A(i,j)$ represent approximate values of the unnormalized joint distribution function of the number of photons of the fundamental mode and of the second harmonic at the points $(n_{1\min} + i\Delta n_1,\ n_{2\min} + j\Delta n_2)$.

Further, we construct a surface that passes through the points $(n_{1\min} + i\Delta n_1,\ n_{2\min} + j\Delta n_2,\ A(i,j))$ of the three-dimensional space. This surface represents the unnormalized joint distribution function of the number of photons of the fundamental mode and of the second harmonic at the instant of time τ. Then, we normalize this function.

Figure 1.9 demonstrates the joint distribution function of the number of photons of the fundamental mode and of the second harmonic at the critical point ($\varepsilon = \varepsilon_{cr} = 30$) and in the region of large interaction times ($\tau = 10$). The function was calculated using 50 000 independent realizations of the set of equations (1.1.13) with the initial values (1.3.2) and for the values of the parameters of the system k = 0.1 and r = 1. The function has a single peak structure but is asymmetric relative to the most probable value of the distribution function. A combination of larger values of the photon number of the fundamental mode with smaller values of photon number of the second harmonic relative to the most probable value of the distribution function has a larger probability of realization than a combination of smaller values of the photon number of the fundamental mode with larger values of the photon number of the second harmonic.

Figure 1.10 demonstrates a joint distribution function of the number of photons of the fundamental mode and of the second harmonic in the region of instability ($\varepsilon = 50$) and in the range of large interaction times (τ = 10). To calculate this function, we used 50 000 independent numerical solutions to the set of Langevin equations (1.1.13) with the initial values (1.3.2). The function was calculated for the parameters of the system k = 0.1 and r = 1. It has a two-peak structure. Each of the most probable values of the distribution function corresponds to a state in which the system spends the majority of its time.

1.6 Dynamics of Phase Fluctuations

In this section, we study the behaviour of the distribution function of phases (1.1.6) $Re\,\phi_i$ $(i = 1{,}2)$ of the fundamental and second harmonic modes in the case of different initial states of the interacting modes. The algorithm for calculating the phase distribution function of the field modes is similar to the algorithm for calculating the photon number distribution function; hence, we do not provide it here.

Figure 1.11 shows the dynamics of the fundamental mode phase distribution function in the case of the initial coherent states of interacting modes $\alpha_j(0) = \sqrt{2}$ and $\beta_j(0) = \alpha_j(0)^*$, where $j = 1{,}2$. The function is calculated based on 100 000 independent trajectories of Eqs. (1.1.13) for the following parameter values: $\varepsilon = 50$, $k = 0.1$, and $r = 1$. At these values, we have $\varepsilon_{cr} = 30$. At the moment $\tau = 0$, the phase distribution function of this mode is infinitely narrow. The distribution function broadens gradually with time and, after some time ($\tau = 5$), passes from a single-peak to a two-peak shape. After acquiring a two-peak structure, it does not change its shape later. The two peaks are symmetric with respect to the zero phase and determine the two components of the phase state of this mode. The phase distribution function of the second harmonic displays a similar time behaviour.

In Fig. 1.12 the dynamics of a certain realization of the fundamental mode phase is shown. The dashed lines correspond to the two most probable values of the phase. The time of transition of the system between these states is about the stay time in them.

Figure 1.13 shows the dynamics of the phase distribution function of the second harmonic mode in the case of Gaussian initial conditions (1.3.2) of the interacting modes and for the following values of parameters: $\varepsilon = 50$, $k = 0.1$, and $r = 1$. The function was calculated using 50 000 independent realizations for the set of equations (1.1.13). At the beginning (t = 0), the function has a sharp single peak form. It is seen in Fig. 3 that the distribution function gradually widens as time progresses and eventually acquires a two-peak form. These two peaks are the two most probable values of the stochastic phase; in fact, with the further evolution of the system, a new macroscopic state is formed.

Figure 1.14 shows the dynamics of the average value of the real part of the second harmonic mode phase in the case of evolution of the system from an initial coherent state with the values of stochastic amplitudes $\alpha_1(0) = 1 + i$, $\beta_1(0) = 1 - i$, $\alpha_2(0) = 1 - i$, and $\beta_2(0) = 1 + i$ (curve 1) and in the case of evolution from an initial state, which has a normal distribution of stochastic amplitudes of the field modes (1.3.2) (curve 2). Both curves represent the temporal behaviour of the average value of the phase of the second-harmonic mode in the unstable region and are calculated for the following values of system parameters: $\varepsilon = 50$, $k = 0.1$, and $r = 1$. Each curve is calculated using 5000 independent solutions of the system of equations (1.1.13). In the case of evolution from an initial state with a normal distribution of the stochastic amplitudes of the field modes, the average value of the phase of the second harmonic mode does not change with time and is zero (the value of the phase of the perturbation field). In the case of evolution of the system from initial coherent states of the field modes (curve 1), the average value of the phase of the mode does not have a stationary value and oscillates in time around the zero value of the phase. These curves show that, in the unstable region, the dynamics of the average value of the phases of the interacting modes depend strongly on the initial state of these modes. In the case of system evolution from the initial states mentioned above, the average value of the phase of the fundamental mode has a similar temporal behaviour: in the case of normal distributions of the initial values of stochastic amplitudes of the interacting modes, the average value of the phase of the fundamental mode does not change in time and is zero, whereas in the case of coherent initial states of both modes it oscillates around the zero value of the phase.

In the unstable region, not only the dynamics of the average values of the phases but also the dynamics of the phase distribution function of the interacting modes of the field, depend strongly on the initial state of the optical system. In the region of large times, if the system evolves from initial coherent states with zero values of phases of both modes of the system (see Fig. 1.11), as well as if the system evolves from an initial state in which the stochastic amplitudes of the interacting modes of the field have normal distributions (see Fig. 1.13), the distribution functions of the stochastic phases of the modes do not change in time and have a two-component structure.

However, the dynamics of the phase distribution of the field modes changes dramatically in the case of evolution of the optical system from initial coherent states of both modes with a nonzero phase. Figure 6.15 shows the temporal behaviour of the distribution function of the second harmonic mode phase in the case of evolution of the optical system from an initial coherent state with the following values of the stochastic amplitudes of the field modes: $\alpha_1(0) = 1 + i$, $\beta_1(0) = 1 - i$, $\alpha_2(0) = 1 - i$, and $\beta_2(0) = 1 + i$. Here, Figs. 6.15a to 6.15d represent the distribution function of the real part of the phase of the second harmonic mode $Re(\phi_2)$ at the times of evolution of the system $\gamma t = 5.3$, $\gamma t = 6$, $\gamma t = 6.6$, and $\gamma t = 7.3$, respectively. These time points correspond to points a, b, c, and d in Fig. 1.14. The dynamics of the distribution function is calculated for the parameter values $k = 0.1$, $r = 1$, and $\varepsilon = 50$. For calculating the dynamics of this function, 50 000 independent realizations of the Langevin system of equations (1.1.13) were used. At the time point $\gamma t = 5.3$, the average value of the phase of the second harmonic mode is zero (see Fig. 1.14). At this point, the second harmonic mode is localized in a two-component state with the same probability of detection of this mode of the field in each component of the state (Fig. 1.15a). Then, from a two-component state, the system gradually localizes into one component of the state in which the average value of the phase of the second harmonic mode has a positive value (Fig. 1.15b). From here, the system changes back into a two-component state with the same probability of detection of the second harmonic mode in each component of the state (Fig. 1.15c). The average value of the phase of the second harmonic mode in this state is reset to zero. After that, the system localizes in the other component of the state, where the average value of the phase of the second harmonic mode is negative. This behaviour of the phase distribution function of the second harmonic mode repeats itself with the further evolution of the optical system.

In the unstable region, in the case of evolution of the optical system from initial coherent states with nonzero phases of stochastic amplitudes of both modes, the temporal behaviour of the distribution function of the phase of the fundamental mode $Re\,\phi_1$ is similar to the temporal behaviour of the distribution function of the second harmonic mode phase.

In Fig. 1.16 we demonstrate the dynamics of the phase distribution function for the fundamental mode in the case of the initial coherent states of both modes $\alpha_j(0) = i\sqrt{2}$ and $\beta_j(0) = \alpha_j(0)^*$ ($j = 1,2$), and for the following parameter values: $\varepsilon = 50$, $k = 0.1$, and $r = 1$. The function was calculated using 100 000 independent realizations for the set of equations (1.1.13). In this case, there is no stationary solution for the distribution function, and it passes to an oscillation regime after broadening. The fundamental mode localizes gradually in one component of the state from the two-component state, which has the same probability of detecting the mode in each component of the state. Next, the system returns to the two-component state, and after that localizes in the other component of the state.

In Fig. 1.17 we plot the phase distribution function of the fundamental mode at the moment of time $\tau = 9$ in the case of initial coherent states of both modes $\alpha_j(0) = \sqrt{2}$ and $\beta_j(0) = \alpha_j(0)^*$, where $j = 1,2$ ($k = 0.1$ and $r = 1$), versus the amplitude of the driving field ε. The function was calculated using 100 000 independent realizations for the set of equations (1.1.13). At $\varepsilon = 30$ (point of bifurcation), the system is in a one-component state with zero phase values. When the system passes the bifurcation point, the system branches into a two-component state with opposite values of the phases of the components. The fundamental mode can be detected with the same probability in both of these two components of the state of the mode.

In the region of large interaction times, the dependence of the distribution function of the phase of the second harmonic mode on the amplitude of the perturbation field is similar to the dependence of the distribution function of the phase of the fundamental mode shown in Fig. 1.17.

1.7 Dynamics of Joint Phase Fluctuations of the Modes and Self-Phase Matching

The above-described technique can also be applied for the calculation of the joint distribution function of the phases of the fundamental mode and second harmonic. This function characterizes the phase matching between the interacting modes. Below the critical point, where the system has stable classical solutions for the photon numbers and phases, the phase fluctuations are considerably smaller than unity. In this case, the most probable values of phase pairs coincide with their classical values, and the equality $Re\,\phi_1 = Re\,\phi_2 = 0$ determines the classical phase matching between the modes. Figure 1.18 shows the joint distribution function of phases of the fundamental mode and second harmonic in the case of initial vacuum states of both modes ($\alpha_j(0) = \beta_j(0) = 0$, where $j = 1{,}2$) and for the following parameter values: $r = 1$, $k = 0.1$, and $\varepsilon = 10$. The function was calculated using 100 000 independent realizations for the set of equations (1.1.13). The function shows the classic matching of the system mode phases.