Kindergarten of Fractional Calculus

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By

Shantanu Das

Cambridge
Scholars
Publishing

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This book first published 2020

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-5275-4498-2
ISBN (13): 978-1-5275-4498-7

Dedicated to my mother, the late Smt.Purabi Das (1934-2018), and my blind father, the late Sri.Soumendra Kumar Das (1926-2009), as well as all of my school and college teachers.

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FOREWORD

যাদবপুর বিশ্ববিদ্যালয়
কলকাতা-৭০০০৩২, ভারত

*JADAVPUR UNIVERSITY
KOLKATA-700 032, INDIA

Boundaries between different branches of science seem to be melting away, as knowledge progresses. It is as if we are moving towards a unified interdisciplinary 'natural science', held together by a framework of mathematics. Novel materials, as well as familiar, but less understood complex systems, lie on the overlapping areas between the compartments we use for classifying 'subjects', such as physics, biology, geoscience, chemical engineering and so on. We need new techniques for understanding and modelling these systems. As we know, Hooke's law works well for many solid materials within the elastic limit, as does Newton's law for viscous forces, in the case of many simple fluids. However, what about the other familiar materials we always use – things like sticky pastes, ductile metals, starch gels, and so on? The simple linear laws which work quite well under limiting conditions are not of much use here. An ingenious way of treating such systems, suggested by a group of scientists, was to implement *generalized calculus* in the governing equations, allowing the order of differentiation to take on non-integer values. This opens up a huge field to play with concerning such peculiar and non-conformist systems. However, of course, a vast amount of intimidating mathematical techniques have to be worked out in order to actually *solve* problems using this crazy idea.

In this kindergarten, or play-school, Shantanu Das develops the ideas behind this new field of mathematics, starting from a most elementary level up to actual applications in different areas of science. Shantanu Das has already published a very successful book on this subject, and has been working in this field for many years having applied generalized calculus in control systems, as well as in the study of visco-elastic materials, ion-conducting polymers and other fields. He has also delivered lectures and courses on the subject at different universities and institutes. I believe the reader will enjoy the experience at this kindergarten and graduate to higher levels in order to make fruitful use of the ideas introduced here.

Sujata Tarafdar
Professor of Physics Jadavpur University
Kolkata 700032, INDIA
sujata_tarafdar@hotmail.com

PREFACE

This book has been compiled from various ‘hand-written teaching notes’ for classroom lectures and ‘presentations’ that I delivered on the topic of fractional calculus at various universities and institutes, from 2005 onwards. The classroom lectures were termed as a ‘Kindergarten of Fractional Calculus’. I must say that book writing is very tough compared to delivering lectures interactively on a blackboard. The name ‘Kindergarten of Fractional Calculus’ (for the classroom talks) suggests that the treatment is very simple; unconventional, yet serious. It suggests that, without going into ‘formal’ theories ,this topic could be developed following a ‘just do it’ attitude, building on our knowledge of classical calculus and mathematics gained at high school and college. I employ this same, informal style in this book. Therefore, I request purists excuse me for this unconventional and non- formal approach, as it is particularly practical.

Why did I write this book in such an unconventional way? It is because my students liked the way I presented this complex subject in class, as it made it easy to grasp and interesting for them, as I first discussed the paradoxes and conditions of the subject, before going on, at a later stage, to discuss its formalism. In this book, I have not dealt with physical interpretations; rather, I took time to write each small step that goes towards every derivation, which is essential in order to develop the subject. This being said, the first chapter and appendix contain a number of relevant definitions and descriptions for terms such as analyticity, analytic continuation, residue calculus, Jordan’s lemma, Laplace transformation, gamma function and its properties, psi function, and Stirling’s number, among others. I also use generalized functions (such as the Mittag-Leffler function and its variants), in other chapters. The first chapter deals with the concept of mixed differentiation and integration, product rules and chain rules, etc., of classical calculus; that I used subsequently for concept generalization. In all other chapters, I carry out detailed derivations in their totality, with each step elaborated and explained. Thus, I am not economical with page space and the descriptive language used. Purists may term this as a verbose treatment, but students nevertheless appreciate this unconventional approach of opening out each step in the derivation process with descriptive explanations, especially those studying Engineering and Applied Sciences. This type of detailed treatment is missing in most of the existing literature as details of derivations are skipped, which students and users usually find difficult to absorb. I wrote this book in this unconventional way mainly because of demand from the student community in my part of the globe.

In 2008, the organizers of the ‘International Mathematics Olympiad’ (INMO) at Mumbai asked me to introduce this subject of fractional calculus to a select group of class XI and XII school students. That was a challenge. I started with the idea of tossing several coins. I then arrived at a formula for a Probability Generating Function (PGF) - for one coin, for two coins and then extended for n - coins. Then, I said to the students, “let us put n equal to ½ and see how ‘a half-coin’ should behave”. Mathematically I demonstrated to the students that we have constructed this half-coin, but I stated, “we have our limitations today, because we are presently unable to attach much physical sense (especially in terms of the notion of ‘negative’ probability) to this new construct”. Hence, we are in a paradoxical situation, and I made it clear to the students that “the paradox is because mathematics goes far beyond our physical understanding”. This is how I began the concept generalization.

With this example, I stated, “as we have one whole differentiation and two whole differentiations, and n-whole differentiation, we can have ½- differentiations or ½ - integrations”, and proceeded in a very simple way to develop the concept, which was once considered a paradox. Today however, we have a physical and engineering idea of it. This physical and engineering sense I have already dealt with in my previous books and other publications, listed in the Bibliography section. Thus, we can say fractional calculus is concept generalization for the existing classical calculus theory; it is also termed as generalized calculus and sometimes even called on-Newtonian calculus. It is not a paradox anymore.

This introduction of the subject to the students gave me confidence that, if I develop the paradoxical, complex-looking mathematics in a different, easy and interesting way, perhaps the students will be hooked on this subject. To some extent I was successful as I delivered detailed classes at Jadavpur University and Calcutta University, and some short classes at Pune University, Mumbai University, IIT- Kharagpur, VNIT- Nagpur, and at several other places. The result is that I see the growth of this subject in this part of globe. Students of Mathematics, Physics, and Engineering have taken up this subject for further research. Some students are presently working, or have previously worked with me in such fields on this subject. Here, in this book I aim to deliver detailed derivations, in a simplified though unconventional way, of the various aspects of the beautiful mathematics of fractional calculus.

Recalling the simple classical integration process that is viewed as the area under the curve (that is, the area under the original function being integrated), we take into account all of the values of the function from the present point of interest to the start of the function. We recall that we memorize all the past points (in a causal sense), and sum them up in order to carry out the classical integration process. The ‘fractional integration’ that we will learn is also the area under the curve, but it is an ‘area under a shape-changing curve’, which keeps on changing as we move ahead. Fractional integration, too, has to account for all the previous values of the function, but here different weights (with decreasing value) are multiplied by each previous and past value of the function, giving a real ‘fading memory’ effect. This is reality as the past memory always fades as we move ahead.

Differentiation, as we know classically, is a slope at a particular point in a curve. The ‘fractional differentiation’ that we will learn will be a slope at a particular point of a function, which is the ‘area under the shape changing curve’. Therefore, in order to evaluate fractional differentiation at a desired point, I should consider all the past values of the function. That is the fractional derivative, which has an in-built fractional integration process. We will be calling this the ‘differ-integration’ process. We will also see the embedded ‘memory’ in the concept of fractional differentiation This means that all the past points with decreasing weights are considered in order to evaluate fractional derivatives at the point of interest; unlike classical differentiation, which is a point property or local property.

Today, we have several engineered systems based on fractional calculus. These engineered systems, are based on the use of the ‘fractional Laplace variable’, which appears when we conduct Laplace transforms of fractional derivatives and fractional integration operators. We will also learn this here in this book. With the fractional Laplace variable, we have developed analog and digital electronic circuits allowing fractional differentiation and fractional integration. We will also study the use of the Laplace transform method to solve fractional differential equations here.

The picture in Figure-P1 below shows a magnetic levitation system, where the metal ball is floating in the air, and is being controlled by current in the coil of a magnet. This current is being governed proportionately by fractional derivatives and fractional integration of the error signal for the ball’s position (Courtesy: BRNS funded joint project of VNIT Nagpur and BARC, which developed a digital fractional order controller for industrial applications. This system also demonstrates that by using fractional calculus in control, we are getting efficient controls, as compared to the classical schemes using classical calculus. This is depicted in Figure-P2.

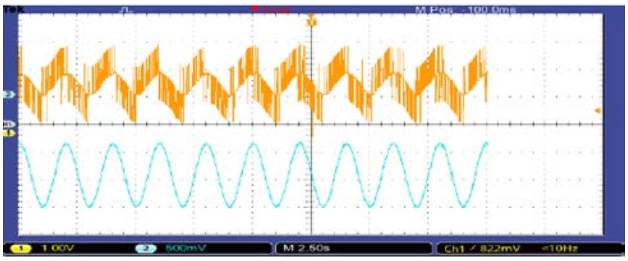
Comparing these two pictures, we see that, in order to do the same job (that is, to position the floating ball and slowly make it follow sinusoidal command), the voltage output of the controller in the case of the second case is fluctuating severely. Many critics will say it is ‘noise’, but why is the ‘so called noisy’ output absent in fractional calculus-based systems? We are using the same electronics and only changing the program of the processor (in this case, the micro-controller); once for classical control governed via classical differentiation and integration, and in the next case for control governed via fractional calculus. The justification for having better efficient control via fractional calculus is that the fractional differentiation and integration operations possess inherent memory. This memory in the system works in order to govern the ball’s position based on its previous or past experience. Therefore, this fractional differentiation and integration process gives us an ideal filtering action, whereas classical differentiation is a point (or local) property that does not therefore have memory, and which acts instantly with no previous experience. Also, classical integration is the summation of all the previous values of function when all are equally weighted; whereas fractional integration is also the summation of previous values, but with decreasing weights. Thus, fractional differentiation and fractional integration give us a memory action where memory fades as we move on. We will study how memory is inherently embedded in fractional calculus.

Figure-P1: Picture showing magnetic levitation where fractional calculus is used for controls

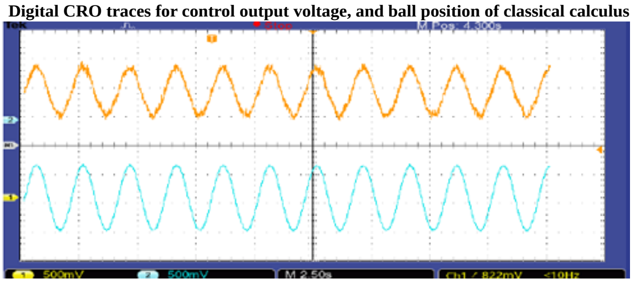
Thus, in the second case, the manoeuvring signal is instantaneously oscillating very dramatically, again giving us the notion that some ‘noise’ is being injected into the system. We also infer in the second case that the controller (with classical calculus) is making a lot of effort to control this ball and keep it afloat, while in the first case the controller’s action (based on fractional calculus) is smooth and effortless. So we can see that the fractional calculus-based system does the control action with less effort than the conventional classical calculus based controllers, and that it is, therefore, better and more efficient. Figure-P3 gives us the experimental records for CRO traces of control voltage (upper trace) and ball position (lower trace) in detail.

Figure-P2: Picture showing magnetic levitation where classical calculus is used for controls

Now we pose a question- what is the implication of ‘lesser effort’ by a control system doing the same job? We have carried out another development where we use classical and fractional calculus to regulate (control) the speed of a DC motor. Figure-P4 gives the picture of the full setup of the DC motor speed control system. Figure-P5 gives us the record of armature voltage and current, while the DC motor is controlled by classical calculus. The multiplication of armature voltage and armature current gives 231.07 Watts as input power to the armature of the DC motor, while regulating the speed at 1000 RPM. Figure-P6 gives the same motor running at the same speed of 1000 RPM but controlled by fractional calculus, which shows an intake power by an armature of 181.61 Watts. Thus, to run a DC motor at a speed of 1000 RPM, the classical calculus-based system takes 17.3% more power than the fractional calculus-based control system. The experiment is done at several speed settings from 500RPM to 1300RPM and at all speeds we observe lesser power intake, when control is based on fractional calculus. This gives us a clue that by using fractional calculus, we are achieving ‘energy/fuel efficiency’. Maybe in future, the industry will adapt this new mathematics to make a fuel/energy efficient control system; at least, I hope so!



Digital CRO traces for control output voltage, and ball position of classical calculus



Digital CRO traces for control output voltage, and ball position of fractional calculus

Figure-P3: Picture showing CRO traces for control voltage and ball position for both systems with classical calculus and with fractional calculus

Figure-P4: DC motor speed control setup

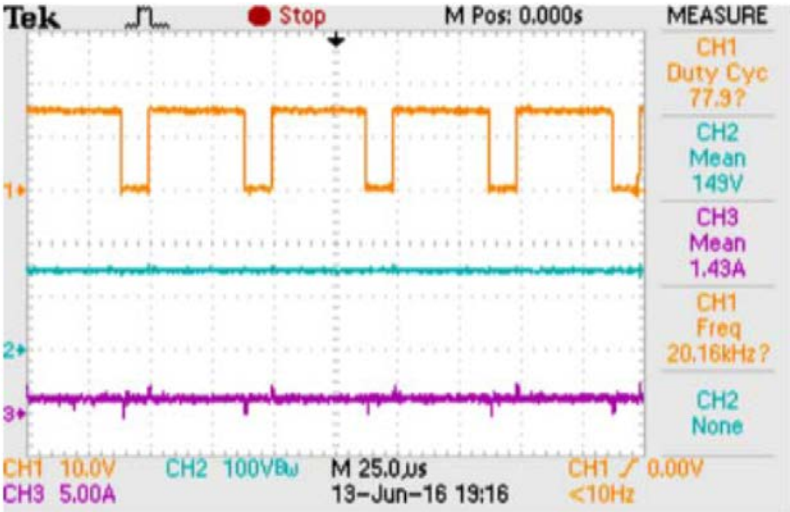


Figure-P5: Armature voltage and current at 1000 RPM DC motor with classical calculus

Figure-P6: Armature voltage and current at 1000 RPM DC motor with fractional calculus

Anyway, how is the efficiency achieved? Is it because by using fractional calculus in my control action, I am interacting with the actual plant (i.e. the DC motor or the magnetic levitation system) in a better and more efficient way? Therefore, can I say that fractional calculus is the mathematics with which natural dynamic systems operate better? This thought came to my mind while I was using logarithmic logics and ratio control and doing derivative operations on these logarithmic domains, for nuclear reactor control. The usage of logarithmic logic gave me better and more efficient nuclear reactor control than conventional use of linear logics, and these 'new' formulas were implemented in nuclear plants, in 2002. When asked why, I advised that the reason was that the way to govern 'a natural, exponential system' (i.e. the nuclear reactor) was using logarithmic power error and its derivative in logarithmic domains which closely match the language of the system. Therefore, maybe this hypothesis is increasing the efficiency of the control action. In the servo-system, the servant, i.e. the plant (the nuclear reactor), is efficiently able to understand the master that is the controller's command (language, in a logarithmic and exponential domain). Perhaps, in the logarithmic case, 'we are talking to the plant to be controlled (i.e. the nuclear reactor)', which is naturally exponential in the language of the process. This delivers an efficient way of achieving better control.

The point, which is emphasized here, is that if we communicate in the language of the dynamic system then we will be communicating better. Thus for efficient communication, 'communicate in French with persons in France'! The above experiments do point out that perhaps fractional calculus is the language that dynamic systems understand better! I had an opportunity to talk on the topic of 'fuel efficient controls', in Beijing, China, at the International Conference on Nuclear Engineering (ICONE- 13), in 2005; for the first time worldwide. At that point, it was my conjecture or hypothesis that we should apply non-Newtonian calculus to achieve fuel efficiency. Today, we have energy/fuel efficient controls practically realized, via non-Newtonian calculus. I feel blessed that I could see, within my life span, the original conjecture or hypothesis of mine regarding energy/fuel efficient controls realized practically and become a reality. Therefore, this is one motivation to provide a course on fractional calculus for Science and Engineering students.

Here, I may also mention that tuning the controller of a plant to either classical calculus or fractional calculus (say DC motor, magnetic levitation system, nuclear reactor, etc.) uses minimization of the chosen performance index (PI). The PI is a function of the controller's parameters that we set, and the plant to be controlled is described by plant's transfer function. By using the minimization technique, we obtain the values of the controller's parameters. This minimization technique, for minimizing the PI is like minimizing the least square error (LSE), call it 'E' for the curve fitting the polynomial. Here I give one example that we can get a better fit with a lower value of E if the curve is composed of fractional power monomials.

Let us say we have n data points $x_1, x_2, x_3, ..., x_n$ and that the corresponding values are $y_1, y_2, y_3, ..., y_n$. Assume by examination of the plotted points; we say linear fitting is good here for this set. Therefore, we need a linear function; $y = f(x) = ax + b$ is to be fitted so that LSE, that is, E is a function of a and b (i.e. $\mathrm{E}(a,b) = \sum_{i=1}^{n}\left(y_i - (ax_i + b)\right)^2$) and is minimized. Following simple mathematics:

$$
\begin{aligned}
&\frac{\partial}{\partial a}\left[\mathrm{E}(a,b)\right] = -2\sum_{i=1}^{n} x_i\left(y_i - (ax_i + b)\right) = 0 \\
&a\left(x_1^2 + x_2^2 + ... + x_n^2\right) + b\left(x_1 + x_2 + ... + x_n\right) = x_1y_1 + x_2y_2 + ...x_ny_n \\
&\frac{\partial}{\partial b}\left[\mathrm{E}(a,b)\right] = -2\sum_{i=1}^{n}\left(y_i - (ax_i + b)\right) = 0 \\
&a\left(x_1 + x_2 + ... + x_n\right) + \overbrace{b + b + ...b}^{n} = y_1 + y_2 + ...y_n
\end{aligned}
\tag{P1}
$$

Compactly we use a set of equations, i.e.

$$
\begin{bmatrix} \sum_{i=1}^{n} x_i & n \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}
\tag{P2}
$$

to get the values of a and b, which minimise LSE, i.e. $\mathrm{E}(a,b)$. The above expressions ((P1) and (P2)) can be used for the linear fitting which has two degrees of freedom, namely a and b. Now, if $\mathrm{E}(a,b) \neq 0$, then we try another function ($y = ax^{\alpha} + b$), and search for the value of, which is close to one (i.e. $\alpha \sim 1.00$) to make the obtained E from the linear fit still lower.

This we demonstrate with an example. Take six data points with $n = 6$ as follows where we want to fit a linear function, $y = ax + b$: