Weighted Sobolev Spaces and Degenerate Elliptic Equations

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By

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This book first published 2023

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

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To my parents, Simone and Larissa

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Introduction

The goal of this book it to introduce the reader to different topics of the theory of degenerate elliptic partial differential equations.

Let ω be a *weight* on $\mathbb{R}^N$, i.e., a locally integrable function on $\mathbb{R}^N$ such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be open, $1 \leq p < \infty$ and k a nonnegative integer. The weighted Sobolev space $W^{k,p}(\Omega, \omega)$ consists of all functions u with weak derivatives $D^\alpha u$, $|\alpha| \leq k$, satisfying

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, \omega \, dx \right)^{1/p} < \infty.$$

In the case $\omega = 1$, this space is denoted $W^{k,p}(\Omega)$ (classic Sobolev spaces). Sobolev spaces without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see, e.g., [32], [35], [43], [56]).

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B. Muckenhoupt (see [50]). These classes have found many useful applications in harmonic analysis (see [55]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^N$ often belong to A_p (see [46]). There are, in fact, many in-

teresting examples of weights (see [43] for p-admissible weights).

In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. This bad behavior can be caused by the coefficients of the corresponding differential operator as well as by the solution itself. There are several very concrete problems form practice which lead to such differential equations, e.g form glaciology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, reaction-diffusion problems, etc (see some examples of applications of degenerate elliptic equations in [8], [27] [57]).

Let us start with some well-known facts. We will consider a nonlinear differential operator of order $2k$ in the divergence form

$$(Au)(x) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha a_\alpha(x, u, \nabla u, ..., \nabla^k u) \qquad (0.1)$$

for $x \in \Omega \subset \mathbb{R}^N$, where the coefficients $a_\alpha = a_\alpha(x, \xi)$ are defined on $\Omega \times \mathbb{R}^m$, and where $\nabla^j u = \{D^\gamma u \, : \, |\gamma| = j\}$ (for $j = 0, 1, ..., k$) is the gradient of the j-th order. Here we suppose that

(i) $a_\alpha(x, \xi)$ satisfy the Carathéodory condition, i.e., $a_\alpha(., \xi)$ is measurable in Ω for every $\xi \in \mathbb{R}^m$ and $a_\alpha(x, .)$ is continuous in $\mathbb{R}^m$ for a.e. $x \in \Omega$;

(ii) $a_\alpha(x, \xi)$ satisfy the *growth condition*

$$|a_\alpha(x, \xi)| \leq C_\alpha \left(g_\alpha(x) + \sum_{|\beta| \leq k} |\xi_\beta|^{p-1} \right)$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^m$, $C_\alpha > 0$ and $g_\alpha \in L^{p'}(\omega)$ ($\frac{1}{p} + \frac{1}{p'} = 1$);

(iii) $a_\alpha(x, \xi)$ satisfy the *ellipticity condition*

$$\sum_{|\alpha| \leq k} a_\alpha(x, \xi) \xi_\alpha \geq C \sum_{|\alpha| \leq k} |\xi_\alpha|^p,$$

for every $\xi \in \mathbb{R}^m$ with constant $C > 0$ independent of ξ;

(iv) $a_\alpha(x, \xi)$ satisfy the *monotonicity condition*

$$\sum_{|\alpha| \leq k} (a_\alpha(x, \xi) - a_\alpha(x, \eta))(\xi_\alpha - \eta_\alpha) > 0$$

for every $\xi, \eta \in \mathbb{R}^m$, $\xi \neq \eta$.

A typical example of a differential operator A satisfying all the foregoing conditions is the so-called p-Laplacian Δ_p defined for $p > 1$ by

$$\begin{aligned}
\Delta_p u &= \text{div}(|\nabla u|^{p-2}\nabla u) \\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left[\left(\frac{\partial u}{\partial x_1} \right)^2 + ... + \left(\frac{\partial u}{\partial x_N} \right)^2 \right]^{(p-2)/2} \frac{\partial u}{\partial x_i} \right),
\end{aligned}$$

or its modification, the operator $\tilde{\Delta}_p$ defined by

$$\tilde{\Delta}_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right),$$

or the little more complicated operator

$$Au = -\tilde{\Delta}_p u + |u|^{p-2}u.$$

Let us slightly change the foregoing operator into

$$(Au)(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a_i(x) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + a_0(x)|u|^{p-2}u, \quad (0.2)$$

with given functions (coefficients) $a_i(x)$ $(i = 1, ..., N)$ satisfying

$$a_i \in L^\infty \ (i = 0, 1, ..., N), \quad\quad\quad (0.3)$$

and $a_i(x) \geq C_1 > 0 \ (i = 0, 1, ..., N)$ for a.e. $x \in \Omega.$ $\quad (0.4)$

Then all conditions (i), (ii), (iii) and (iv) remain satisfied and we look for a weak solution in the Sobolev space $W^{1,p}(\Omega)$. However, the situation changes dramatically if some of the coefficients $a_i(x)$ violate conditions (0.3) and/or condition (0.4) (i.e., with coefficients which are singular ($a_i(x)$ are unbounded) and/or degenerated ($a_i(x)$ are only positive a.e.)). The situation can be saved using the weighted Sobolev space $W^{1,p}(\Omega, \omega)$ instead of the classical Sobolev space $W^{1,p}(\Omega)$.

A typical example is the degenerate p-Laplacian

$$- \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u),$$

with $p \neq 2$.

In the linear case, we consider the second order, linear, elliptic equation with divergence structure

$$\operatorname{div}(A(x)\,\nabla u(x)) = 0 \qquad (0.5)$$

where $A(x) = [a_{ij}(x)]_{i,j=1,\dots,N}$ is a symmetric matrix with measurable coefficients, defined in a domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$). We assume the following ellipticity condition

$$0 < \omega(x)|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq |\xi|^2 v(x), \qquad (0.6)$$

for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, where ω and v are measurable functions, finite and positive a.e. $x \in \Omega$. The equation (0.5) is *degenerate* if ω^{-1} is unbounded, and the equation (0.5) is *singular* if v is unbounded.

Below, there follows a short outline of the book

Chapter 1 is of an introductory character. In the first section, we present the notation and conventions that will be used in the book and we summarize some facts from the theory of weighted Sobolev spaces.

In the Chapter 2 we summarize some facts from the theory of A_p-weights and an approximation result for A_p-weights.

The theme of Chapter 3 is degenerate linear elliptic equations. In this chapter we study the existence and uniqueness of solution, the approximation of solutions and the concept of entropy solution for degenerate linear elliptic equations.

Chapter 4 is devoted to the study of some problems with degenerate semilinear elliptic equations. In Chapter 5 we study some degenerate quasilinear elliptic equations (existence, uniqueness and entropy solutions). Finally, in Chapter 6 we investigate the existence and uniqueness of solutions for some degenerate nonlinear elliptic problems using the theory of monotone operators.

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List of symbols

Here we introduce the basic notation which will be observed throughout this book.

$\mathbb{N}$ - the set of natural numbers.
$\mathbb{R}$ - the set of real numbers.
$\mathbb{R}^N$ - the N-dimensional Euclidean space of points $x = (x_1, x_2, ..., x_N)$.
$\alpha = (\alpha_1, ..., \alpha_N)$ - an (N-dimensional) multiindex (i.e., with components α_j which are nonnegative integers).
$|\alpha| = \alpha_1 + ... + \alpha_N$ - the length of the multiindex α.
$D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}....\partial x_N^{\alpha_N}}$ - partial derivative of order $|\alpha|$.
sup E - lowest upper bound (supremum) of the set E.
inf E - greatest lower bound (infimum) of the set E.
$s^+ = \max\{s, 0\}$ - the positive part of $s \in \mathbb{R}$.
$s^- = \max\{-s, 0\}$ - the negative part of $s \in \mathbb{R}$.
X^* - the adjoint (dual) space to the space X.
$\langle ., . \rangle$ - inner product in $\mathbb{R}^N$.
$(.|.)_X$ - duality between the Banach spaces X and X^*, $(f|x) = f(x)$, $x \in X$ and $f \in X^*$.
$\|x\|_X$ - norm of x in a Banach space X.
$x_n \to x$ - convergence in norm.
$x_n \rightharpoonup x$ - weak convergence.
X^N - product space of X ($N \in \mathbb{N}$), i.e., X^N consists of all $u = (u_1, u_2, ..., u_N)$, where $u_j \in X$ for all j. If X is a Banach space over $\mathbb{R}$, then X^N is also a Banach space over $\mathbb{R}$ with the norm

$$\|u\| = \sum_{j=1}^N \|u_j\|.$$

$\mathcal{D}(f)$ - domain of f.
$\mathcal{R}(f)$ - range of f.
supp$f$ - support of the function f.

$\underline{\lim}$, $\overline{\lim}$ - lower, upper limit.

$\partial\Omega$ - boundary of a set Ω.

$\partial\Omega \in C^{0,1}$ - piecewise smooth boundary.

$\overline{\Omega} = \Omega \cup \partial\Omega$ - the closure of Ω.

$C^{\infty}(\Omega)$ - space of infinitely continuously differentiable functions $f : \Omega \to \mathbb{R}$.

$C_0^{\infty}(\Omega)$ - the space of infinitely differentiable functions with compact support.

$X \hookrightarrow Y$ - the space X is continuously and compactly embedded into the space Y.

∇u - gradient of the map u, $\nabla u = \left(\dfrac{\partial u}{\partial x_1}, ..., \dfrac{\partial u}{\partial x_N} \right)$.

$\operatorname{div} f$ - divergence of the map f, $\operatorname{div} f = \displaystyle\sum_{j=1}^{N} \dfrac{\partial f}{\partial x_j}$.

Δf - Laplace operator of the map f, $\Delta f = \displaystyle\sum_{j=1}^{N} \dfrac{\partial^2 f}{\partial x_j^2}$.

$\operatorname{osc}(u, E)$ - oscillation of the function $u : E \to \mathbb{R}$,

$\quad \operatorname{osc}(u, E) = \sup\limits_{E} u - \inf\limits_{E} u$.

$|E|$ - measure of the measurable set E with respect to the N-dimensional Lebesgue measure.

$\mu(E)$ - measure of the set E with respect to the measure μ.

$\displaystyle\int_{\Omega} f \, d\mu$ - integral with respect to measure μ.

$\displaystyle\int_{\partial\Omega} f \, d\sigma(x)$ - surface integral.

$L^p(\Omega)$ - Lebesgue space.

$L^p(\Omega, \omega)$ - weighted Lebesgue space.

$W^{k,p}(\Omega, \omega)$ - weighted Sobolev space.

$\mathcal{M}^p(\Omega, \omega)$ - weighted Marcinkiewicz space.

Chapter 1

Weighted Sobolev Spaces

In this first chapter we introduce the *weighted Sobolev spaces* $W^{k,p}(\Omega, \omega)$ and investigate their basic properties which are needed in chapters to come. Throughout this book Ω will denote an open subset of $\mathbb{R}^N$, $N \geq 2$.

1.1   Preliminaries

Let $\mathbb{R}^N$ denote Euclidean N-space. The norm of a point $x = (x_1, ..., x_N)$ in $\mathbb{R}^N$ is given by $|x| = (x_1^2 + ... + x_N^2)^{1/2}$. If $E \subset \mathbb{R}^N$, the boundary, the closure and the complement of E with respect to $\mathbb{R}^N$ are denoted by ∂E, $\overline{E}$ and $E^c = \mathbb{R}^N \setminus E$. Let χ_E the characteristic function of a set E, i.e.,

$$\chi_E(x) = \left\{ \begin{array}{ll} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{array} \right.$$

The set $B(a, r) = \{x \in \mathbb{R}^N : |x - a| < r\}$ is an open ball in $\mathbb{R}^N$ with radius r and center a. If $\lambda > 0$, then $\lambda B(a, r) = B(a, \lambda r)$.

Let $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{Z}_+^N$ be a multi-index. Then $|\alpha| = \alpha_1 + ... + \alpha_N$, $\alpha! = \alpha_1!...\alpha_N!$, $x^\alpha = x_1^{\alpha_1}...x_N^{\alpha_N}$ for $x \in \mathbb{R}^N$, and $D^\alpha = \dfrac{\partial_1^\alpha}{\partial x_1^{\alpha_1}}...\dfrac{\partial_N^\alpha}{\partial x_N^{\alpha_N}}$.

We denote by $C(\Omega)$ the set of all continuous functions $\varphi : \Omega \to \mathbb{R}$. Moreover, supp$\varphi$ (support of φ) is the closure (in the Euclidean norm of $\mathbb{R}^N$) of the set $\{x \in \Omega : \varphi(x) \neq 0\}$, i.e., supp$\varphi = \overline{\{x \in \Omega : \varphi \neq 0\}}$. For $k \in \mathbb{N}$, then $C^k(\Omega)$ is the set of k times continuously differentiable functions $\varphi : \Omega \to \mathbb{R}$ and $C^\infty(\Omega) = \bigcap_{k=1}^{\infty} C^k(\Omega)$.

The notation $E \Subset \Omega$ means that $\overline{E}$ is a compact subset of Ω. Then, if $k \in \mathbb{N}$, $C_0^k(\Omega) = \{\varphi \in C^k(\Omega) : \text{supp}\varphi \Subset \Omega\}$ and $C_0^\infty = \{\varphi \in C^\infty(\Omega) : \text{supp}\varphi \Subset \Omega\}$. The gradient of $\varphi \in C^1(\Omega)$ if $\nabla \varphi = (\partial_1 \varphi, ..., \partial_N \varphi)$.

If $1 \leq p \leq \infty$, then p' is the conjugate exponent to p given by $\frac{1}{p} + \frac{1}{p'} = 1$ with the usual conventions when $p = 1$ or $p = \infty$.

Let $\Omega \subset \mathbb{R}^N$ be open. We denote the Lebesgue space by

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{1/p} < \infty \right\},$$

and

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup} |u(x)| < \infty \right\}.$$

The space $L^p(\Omega)$ equipped with the norm $\|u\|_{L^p(\Omega)}$ ($1 \leq p \leq \infty$) is a Banach space. By $L^p_{\text{loc}}(\Omega)$ we will denote the set of all functions $\varphi \in L^p(K)$ for every compact set $K \subset \Omega$.

For $k \in \mathbb{N}$ and $1 \leq p < \infty$ we denote by

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \leq k\}.$$

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space if equipped with the norm $\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)} \right)^{1/p}$. We denote by

$W^{k,p}_{\mathrm{loc}}(\Omega)$ the set of all functions $u \in W^{k,p}(K)$ for every compact set $K \subset \Omega$. Further, the space $W^{k,p}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|.\|_{W^{k,p}(\Omega)}$.

We will use the abbreviation "a.e." for "*almost everywhere* " or "*almost every*" with respect to Lebesgue measure. Similarly, "*measurable*" and "*locally integrable*" always mean Lebesgue measure and locally integrable with respect to Lebesgue measure, respectively. If Ω is an open subset of $\mathbb{R}^N$, then the set of locally integrable functions on Ω will denoted by $L^1_{\mathrm{loc}}(\Omega)$.

The Lebesgue N-measure of a measurable subset $E \subset \mathbb{R}^N$ is denoted $|E|$.

Throughout, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. Two quantities A and B are said *equivalent* or *comparable* if there exists two constant C_1 and C_2 such that $C_1\,A \le B \le C_2\,A$ (and we use the notation $A \approx B$).

1.2   Weighted Sobolev Spaces

In this section, we intend to recall some concepts and results which will play an important role in our further considerations. The present section explores some properties of weighted Sobolev spaces and investigate their basic properties which are need in chapters to come. These spaces are only a slight modification of the classical Sobolev spaces $W^{k,p}(\Omega)$ but their property are much more complicated and less transparent. Complete expositions can be found in [38], [43], [46] and [56].

Definition 1.2.1. *Let Ω be an open set in $\mathbb{R}^N$. We denote by $\mathcal{W}(\Omega)$ the set of all measurable a.e. in Ω positive, finite and locally integrable functions $\omega = \omega(x)$, $x \in \Omega$ ($0 < \omega(x) < \infty$ a.e. in Ω). Elements of $\mathcal{W}(\Omega)$ will be called weight functions.*

Every weight $\omega \in \mathcal{W}(\Omega)$ gives rise to a measure on the mea-

surable subsets of Ω through integration. This measure will also be denoted by ω. Thus, $\omega(E) = \int_E \omega \, dx$ for measurable set $E \subset \Omega$ (or $\mu(E) = \int_E \omega(x) \, dx$).

Definition 1.2.2. *Let $\Omega \subset \mathbb{R}^N$ an open set and $\omega \in \mathcal{W}(\Omega)$. For $1 \leq p < \infty$, we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that*

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f|^p \, \omega \, dx \right)^{1/p} < \infty. \tag{1.1}$$

For $\omega(x) \equiv 1$ we obtain the usual Lebesgue space $L^p(\Omega)$.

Theorem 1.2.1. *The space $L^p(\Omega, \omega)$ equipped with the norm (1.1) is a Banach space.*

Proof. See [33], Theorem III.6.6 □

We now determine conditions on the weight ω that guarantee that functions in $L^p(\Omega, \omega)$ are locally integrable on Ω.

Definition 1.2.3. *Let $1 \leq p < \infty$ and let ω be a weight. We say that $\omega \in \mathcal{B}_p(\Omega)$ if*
    (a) $\omega^{-1/(p-1)}$ is locally integrable, when $p > 1$;
    (b) ess $\sup\limits_{x \in B} \dfrac{1}{\omega(x)} < \infty$ for all ball B, when $p = 1$.

Now, suppose that $f \in L^p(\Omega, \omega)$, $\omega \in \mathcal{B}_p(\Omega)$ and let $B \subset \Omega$ be a ball.
    (i) If $1 < p < \infty$, by Hölder's inequality (with $1/p + 1/p' = 1$), we obtain

$$\int_B |f| \, dx = \int_B |f| \, \omega^{1/p} \, \omega^{1/p'} \, \omega^{-1} \, dx$$

$$\leq \left(\int_B |f|^p \, \omega \, dx \right)^{1/p} \left(\int_B \omega^{(1-p')} \, dx \right)^{1/p'}$$

$$\leq \;\; \|f\|_{L^p(\Omega,\omega)}\left(\int_B \omega^{-1/(p-1)}\,dx\right)^{1/p'}.$$

(ii) If $p = 1$, we have

$$\int_B |f|\,dx = \int_B |f|\,\omega\,\omega^{-1}\,dx \leq \left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{\omega(x)}\right)\left(\int_\Omega |f|\,\omega\,dx\right).$$

Its follows that $L^p(\Omega,\omega)\subset L^1_{\mathrm{loc}}(\Omega)$ and that the convergence in $L^p(\Omega,\omega)$ implies local convergence in $L^1(\Omega)$.

Hence, we have the following result.

Theorem 1.2.2. *Let $\Omega\subset\mathbb{R}^N$ be an bounded open set and a weight $\omega\in\mathcal{B}_p(\Omega)$ $(1\leq p < \infty)$. Then $L^p(\Omega,\omega)$ is continuously embedded in $L^1(\Omega)$*
(i.e., $L^p(\Omega,\omega)\hookrightarrow L^1_{loc}(\Omega)$).

If $\omega\notin\mathcal{B}_p(\Omega)$, the conclusion of Theorem 1.2.2 need not hold.

Example 1.2.1. *Let $\Omega = (-1/2, 1/2)\subset\mathbb{R}$, $p = 2$ and $\omega(x) = |x|$. We have that $\omega\notin\mathcal{B}_2(\Omega)$, since $\omega^{-1/(2-1)}(x) = |x|^{-1}$. Let us take*
$f(x) = |x|^{-1}|ln(x)|^{-2/3}$. We have that $f\in L^2(\Omega,\omega)$ because

$$\begin{aligned}\|f\|^2_{L^2(\Omega,\omega)} &= \int_{-1/2}^{1/2}\left(|x|^{-1}|ln(x)|^{-2/3}\right)^2|x|\,dx\\ &= 2\int_0^{1/2}|x|^{-1}|ln(x)|^{-4/3}\,dx\\ &= 2\int_{ln(2)}^\infty t^{-4/3}\,dt < \infty,\end{aligned}$$

but $f\notin L^1_{loc}(\Omega)$, since $(-1/4, 1/4)\subset(-1/2, 1/2)$ and

$$\int_{-1/4}^{1/4}|f|\,dx = \int_{-1/4}^{1/4}|x|^{-1}|ln(x)|^{-2/3}\,dx$$

$$= 2 \int_{ln(4)}^{\infty} t^{-2/3}\, dx = \infty.$$

If $\omega \in \mathcal{B}_p(\Omega)$ then $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω. It thus make sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 1.2.4. *Let $\Omega \subset \mathbb{R}^N$ be open set, $1 \leq p < \infty$ and k a nonnegative integer. Suppose that the weight $\omega \in \mathcal{B}_p(\Omega)$. We define the weighted Sobolev space $W^{k,p}(\Omega, \omega)$ as the set of functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$ for $|\alpha| \leq k$. The norm of $u \in W^{k,p}(\Omega, \omega)$ is given by*

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|\, \omega\, dx \right)^{1/p}. \qquad (1.2)$$

Theorem 1.2.3. *Let $\omega \in \mathcal{B}_p(\Omega)$, $1 \leq p < \infty$. Then $W^{k,p}(\Omega, \omega)$ is a Banach space if equipped with the norm (1.2).*

Proof. For simplicity, we will deal with the case $k = 1$, i.e., with space $W^{1,p}(\Omega, \omega)$.
Step 1. If $\omega \in \mathcal{B}_p(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Then, if $u \in L^p(\Omega, \omega)$, we have

$$L_0(u) = \int_\Omega u\,\varphi\, dx, \text{ and}$$

$$L_j(u) = \int_\Omega u\, D_j\varphi\, dx,$$

defines a continuous linear functional L_j $(j = 0, 1, ..., n)$ on $L^p(\Omega, \omega)$. In fact, if we denote $\Omega_\varphi = supp(\varphi)$ (support of φ), then $\Omega_\varphi = \overline{\Omega}_\varphi \subset \Omega$ and

$$|L_j(u)| \leq \int_\Omega |u|\, |D_j\varphi|\, dx = \int_\Omega |u|\, \omega^{1/p}\, |D_j\varphi|\, \omega^{-1/p}\, dx$$

$$\leq \left(\int_\Omega |u|^p \, \omega \, dx \right)^{1/p} \int_{\Omega_\varphi} |D_j\varphi|^{p'} \omega^{-p'/p} \, dx \bigg)^{1/p'}$$

$$= \|u\|_{L^p(\Omega,\omega)} \left(\int_{\Omega_\varphi} |D_j\varphi|^{p/(p-1)} \, \omega^{-1/(p-1)} \, dx \right)^{(p-1)/p}$$

$$\leq \|u\|_{L^p(\Omega,\omega)} \left(\sup_{\Omega_\varphi} |D_j\varphi| \right) \left(\int_\Omega \omega^{-1/(p-1)} \, dx \right)^{(p-1)/p}$$

$$= C \, \|u\|_{L^p(\Omega,\omega)},$$

where

$$C = \left(\sup_{\Omega_\varphi} |D_j\varphi| \right) \left(\int_\Omega \omega^{-1/(p-1)} \, dx \right)^{(p-1)/p} < \infty,$$

since $\varphi \in C_0^\infty(\Omega)$ and $\omega \in \mathcal{B}_p(\Omega)$.
Step 2. Let $\{u_n\}$ be a Cauchy sequence in $W^{1,p}(\Omega,\omega)$, i.e.,

$$\|u_m - u_n\|_{W^{1,p}(\Omega,\omega)}^p$$

$$= \int_\Omega |u_m - u_n|^p \, \omega \, dx + \sum_{j=1}^N \int_\Omega |D_ju_m - D_ju_n|^p \, \omega \, dx$$

$$\to 0,$$

when $m, n \to \infty$. Then $\{u_n\}$ and $\{D_ju_n\}$ $(j = 1, ..., n)$ are Cauchy sequences in $L^p(\Omega,\omega)$. By Theorem 1.2.1 there exist functions v, Ψ_j such that $v = \lim u_n$ and $\Psi_j = \lim D_ju_n$ in $L^p(\Omega,\omega)$. We have that $\Psi_j = D_jv$ $(j = 1, ..., N)$. In fact, by Step 1 the functional L_j is a continuous linear functional on $L^p(\Omega,\omega)$, consequently

$$L_j(u_n) \to L_j(v) \text{ for } n \to \infty.$$

We also have that $L_0(u) = \displaystyle\int_\Omega u \, \varphi \, dx$ $(\varphi \in C_0^\infty(\Omega))$ define a con-

tinuous linear functional on $L^p(\Omega, \omega)$. We have

$$
\begin{aligned}
L_j(u_n) &= \int_\Omega u_n \, D_j\varphi \, dx \\
&= -\int_\Omega \varphi \, D_j u_n \, dx \\
&= -L_0(D_j u_n).
\end{aligned}
$$

By a limiting process we obtain $L_j(v) = -L_0(\Psi_j)$, i.e.,

$$
\int_\Omega v \, D_j\varphi \, dx = -\int_\Omega \Psi_j \, \varphi \, dx,
$$

for every $\varphi \in C_0^\infty(\Omega)$. Therefore Ψ_j is the distributional derivative of v, i.e., $\Psi_j = D_j v$ $(j = 1, 2, .., N)$. Moreover, we have

$$
\begin{aligned}
\|u_n - v\|_{W^{1,p}(\Omega, \omega)}^p &= \int_\Omega |u_n - v|^p \, \omega \, dx \\
&+ \sum_{j=1}^N \int_\Omega |D_j u_n - D_j v|^p \, \omega \, dx \to 0
\end{aligned}
$$

for $n \to \infty$. Therefore, the Cauchy sequence $\{u_n\}$ converges to v in $W^{1,p}(\Omega, \omega)$, i.e., $W^{1,p}(\Omega, \omega)$ is a Banach space. $\qquad \square$

Theorem 1.2.4. *Let $\omega \in \mathcal{W}(\Omega)$. Then $C_0^\infty(\Omega) \subset W^{1,p}(\Omega, \omega)$ if and only if $\omega \in L_{loc}^1(\Omega)$.*

Proof. ($\Leftarrow$) If $\omega \in L_{loc}^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ ($\overline{\Omega}_\varphi = \Omega_\varphi$ the support of φ), we obtain

$$
\int_\Omega |\varphi|^p \, \omega \, dx \le \max_{\overline{\Omega}_\varphi}(|\varphi|^p)\left(\int_{\Omega_\varphi} \omega \, dx\right) < \infty,
$$

and

$$
\int_\Omega |D_j\varphi|^p \, \omega \, dx \le \max_{\overline{\Omega}_\varphi}(|D_j\varphi|^p)\left(\int_{\Omega_\varphi} \omega \, dx\right) < \infty.
$$

Hence, $C_0^\infty(\Omega) \subset W^{1,p}(\Omega,\omega)$.

($\Rightarrow$) If $C_0^\infty(\Omega) \subset W^{1,p}(\Omega,\omega)$. Let $K \subset \Omega$ be a compact set. Then there exists a function $\varphi \in C_0^\infty(\Omega)$ such that $D_j\varphi(x) = 1$ for $x \in K$ and we have

$$
\begin{aligned}
0 \;\leq\; & \int_K \omega\,dx = \int_K |D_j\varphi|^p \,\omega\,dx \\
\leq\; & \int_\Omega |D_j\varphi|^p \,\omega\,dx \\
\leq\; & \|\varphi\|_{W^{1,p}(\Omega,\omega)} < \infty.
\end{aligned}
$$

Hence, $\omega \in L^1_{\mathrm{loc}}(\Omega)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 1.2.5. *Let $\Omega \subset \mathbb{R}^N$ be open and $\omega \in L^1_{loc}(\Omega)$. We define the space $W_0^{1,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega,\omega)$ with respect to the norm $\|.\|_{W^{1,p}(\Omega,\omega)}$ (i.e., $W_0^{1,p}(\Omega,\omega) = \overline{C_0^\infty(\Omega)}$).*

1.3   Examples of weights

Next we give some examples of weights.

1.3.1   Power-type weights

(i) Let Ω be a bounded domain in $\mathbb{R}^N$. Let M be a nonempty subset of $\overline{\Omega} = \Omega \cup \partial\Omega$, and denote $d_M(x) = \mathrm{dist}(x, M)$ for $x \in \Omega$ (the distance of the point x from the set M). Let $\varepsilon \in \mathbb{R}$ and let us denote the power-type weight

$$
\omega(x) = [d_M(x)]^\varepsilon.
$$

So the singularity ($\varepsilon < 0$) or degenerations ($\varepsilon > 0$) can appear on the boundary $\partial\Omega$ of Ω as well as in the interior of the domain. The set M is very often a closed part of the boundary $\partial\Omega$ (i.e., $M \subset \partial\Omega$).

(ii) Let $s = s(t)$ be a continuous positive function defined for $t > 0$ and such that

$$\text{either} \quad \lim_{t \to 0} s(t) = 0 \quad \text{or} \quad \lim_{t \to 0} s(t) = +\infty.$$

Let us denote $\omega(x) = s\left(d_M(x)\right)$. We have that:

(a) if $\lim\limits_{t \to 0} s(t) = 0$ then $W^{k,p}(\Omega) \hookrightarrow W^{k,p}(\Omega, s(d_M))$;

(b) if $\lim\limits_{t \to 0} s(t) = \infty$ then $W^{k,p}(\Omega, s(d_M)) \hookrightarrow W^{k,p}(\Omega)$.

For more information on properties of spaces with theses weights see [46].

1.3.2 A_p - weights

The class of A_p-weight was introduced by B. Muckenhoupt (see [50], where he showed that the A_p weights are precisely those weights ω for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^N, \omega)$ to $L^p(\mathbb{R}^N, \omega)$ $(1 < p < \infty)$, that is

$$M : \ L^p(\mathbb{R}^N, \omega) \to L^p(\mathbb{R}^N, \omega)$$
$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy,$$

is bounded if and only if $\omega \in A_p$ $(1 < p < \infty)$, i.e., there exists a positive constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega \, dx\right)\left(\frac{1}{|B|} \int_B \omega^{-1/(p-1)} \, dx\right)^{p-1} \leq C,$$

for every ball $B \subset \mathbb{R}^N$.

We say that a weight ω belongs to A_1 if there is a constant $C = C_{1,\omega}$ such that

$$\frac{1}{|B|}\int_B \omega\,dx \le C \operatorname*{ess\,inf}_B \omega$$

for all balls B in $\mathbb{R}^N$.

The union of all Muckenhoupt classes A_p is denoted by A_∞, i.e., $A_\infty = \bigcup_{p>1} A_p$.

Example of A_p-weights

(i) If ω is a weight and there exist two positive constants C_1 and C_2 such that $C_1 \le \omega(x) \le C_2$ for a.e. $x \in \mathbb{R}^N$, then $\omega \in A_p$ for $1 \le p < \infty$.

(ii) If $x \in \mathbb{R}^N$, $\omega(x) = |x|^\alpha$ is in A_p if and only if $-N < \alpha < N(p-1)$ (see Corollary 4.4 in [55]).

(iii) $\omega(x) = e^{\lambda\varphi(x)} \in A_2$, with $\varphi \in W^{1,N}(\Omega)$ (where Ω is a open set in $\mathbb{R}^N$) and λ is sufficiently small (see Corollary 2.18 in [38]).

(iv) $\omega(x) = |x|^\alpha(\max\{1, -\ln(|x|)\})^\beta$, $x \in \mathbb{R}^n$, is an A_1-weight if and only if $\alpha < 0$ or $\alpha = 0 \le \beta$ (see Proposition 7.2 in [4]).

For more information about A_p-weights see [35], [38], [55] and [56].

1.3.3 p - admissible weights

Let ω be a locally integrable, nonnegative function in $\mathbb{R}^N$ and $1 < p < \infty$. We say that ω is *p-admissible* if the following four conditions are satisfied:

(I) $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^N$ and ω is *doubling* , i.e., there is a constant $C_1 > 0$ such that $\omega(2B) \le C_1\,\omega(B)$, whenever $B = B(x,r)$ is a ball in $\mathbb{R}^N$ (where $2B(x,r) = B(x,2r)$).

(II) If Ω is an open set and $\varphi_k \in C^\infty(\Omega)$ is a sequence of functions such that

$$\int_\Omega |\varphi_k|^p \omega\, dx \to 0 \quad \text{and} \quad \int_\Omega |\nabla \varphi_k - \vartheta|^p \omega\, dx \to 0$$

as $k \to \infty$, where $\vartheta = (\vartheta_1, ..., \vartheta_N)$ is a vector-valued measure function with $\vartheta_j \in L^p(\Omega, \omega)$, then $\vartheta = 0$.

(III) There are constants $\theta > 1$ and $C_2 > 0$ such that

$$\left(\frac{1}{\omega(B)} \int_B |\varphi|^{\theta p} \omega\, dx \right)^{1/\theta p} \leq C_2 R \left(\frac{1}{\omega(B)} \int_B |\nabla \varphi|^p \omega\, dx \right)^{1/p}.$$

whenever $B = B(x_0, R)$ is a ball in $\mathbb{R}^N$ and $\varphi \in C_0^\infty(B)$.

(IV) There is a constant $C_3 > 0$ such that

$$\int_B |\varphi - \varphi_B|^p \omega\, dx \leq C_3 R^p \int_B |\nabla \varphi|^p \omega\, dx$$

whenever $B = B(x_0, R)$ is a ball in $\mathbb{R}^N$ and $\varphi \in C^\infty(B)$ is bounded and $\varphi_B = \dfrac{1}{\omega(B)} \displaystyle\int_B \varphi\, \omega\, dx$.

It follows immediately from condition (I) that the measure ω and the Lebesgue measure are mutually absolutely continuous. Condition (II) guarantees that the gradient of a Sobolev function is well defined. Condition (III) is the *weighted Sobolev inequality* and condition (IV) is the *weighted Poincaré inequality*.

Examples of p-admissible weights

(1) If $\omega \in A_p$ $(1 < p < \infty)$ then ω is a p-admissible weight.

(2) $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, $\alpha > -N$, is a p-admissible weight for all $p > 1$.

(3) If $f : \mathbb{R}^N \to \mathbb{R}^N$ is a K - quasiconformal mapping and $J_f(x)$ is the determinant of its jacobian matrix, then $\omega(x) = J_f(x)^{1-p/N}$ is p-admissible for $1 < p < N$.

A continuous mapping $f : \mathbb{R}^N \to \mathbb{R}^N$, $f = (f_1, ..., f_N)$, is K-quasiconformal if:

(a) the coordinate functions $f_j \in W^{1,N}_{\text{loc}}(\Omega)$;

(b) f is a homeomorphism onto $f(\mathbb{R}^N)$;

(c) there is $K \geq 1$ such that the inequality $|f'(x)|^N \leq K \, J_f(x)$ is satisfied for a.e. x in Ω (here $f'(x)$ denotes the formal derivative of f at x, i.e., the $N \times N$ matrix $\left(\dfrac{\partial f_i}{\partial x_j}(x)\right)$ of the partial derivatives of the coordinate functions f_i of f, $|f'(x)| = \max\limits_{|h|=1} |f'(x) \, h|$ and $J_f(x) = \det(f'(x))$ is the Jacobian determinant of f at x.

(4) See [5] for non-A_p examples of p-admissible weights.

For more information about p-admissible weights see [43].

Remark 1.3.1. *Recently P.Hajłasz and P.Koskela (see [41]) showed that conditions (I) - (IV) can be reduced to only two: ω is a p-admissible weights ($1 < p < \infty$) if and only if ω is doubling and there are constants $C > 0$ and $\lambda \geq 1$ such that*

$$\frac{1}{\omega(B)} \int_B |\varphi - \varphi_B| \, \omega \, dx \leq C \, R \left(\frac{1}{\omega(\lambda B)} \int_{\lambda B} |\nabla \varphi|^p \, \omega \, dx \right)^{1/p},$$

(the weak $(1, p)$ -Poincaré inequality).

1.3.4 Regular weights

Let $\omega(x) \geq 0$ be a weight, with $\omega \in L^1_{\text{loc}}(\Omega)$. We consider the

set of functions

$$X = \left\{ u \in W^{1,1}_{\text{loc}}(\Omega) \text{ such that } \|u\|_\omega^2 = \int_\Omega (u^2 + |\nabla u|^2)\, \omega\, dx < \infty \right\}.$$

A weighted Sobolev space can be defined, in general, in two ways:

(a) $W = W(\Omega, \omega)$ is the completion of the set X with respect to the norm $\|.\|_\omega$;

(b) $H = H(\Omega, \omega)$ is the completion of the $\{u \in C^\infty(\Omega) : \|u\|_\omega < \infty\}$ with respect to the norm $\|.\|_\omega$ (the energy norm).

We have that $H \subseteq W$. By definition, functions smooth in the interior of Ω are dense in H, while the space W is known to contain all functions of finite well-defined " energy".

The classical Sobolev space corresponds to the weight $\omega(x) \equiv 1$ and is uniquely defined since the spaces $W(\Omega)$ and $H(\Omega)$ are the same for each domain Ω. Of course, if ω is bounded above and away from zero $(0 < c_1 \leq \omega(x) \leq c_2 < \infty)$, the spaces W and H are also the same. However, condition $\omega \in L^1_{\text{loc}}(\Omega)$ do not in general ensure the equality $H = W$.

Example 1.3.1. *Let $N = 2$, $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ |x| < 1\}$ and let*

$$\omega(x) = \begin{cases} \left(\ln(2/|x|) \right)^{\alpha}, & \text{for } x_1 x_2 > 0, \\ \left(\ln(2/|x|) \right)^{-\alpha}, & \text{for } x_1 x_2 < 0, \end{cases}$$

with $\alpha > 1$. Then $H \neq W$. In fact, in the polar variables $r = |x|$ and $\theta = \arccos(x_1/r)$, we set