Particle Kinetics and Laser-Plasma Interactions

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Foreword

This book addresses graduate and postgraduate students, scientists, and engineers interested in high-intensity laser-plasma interactions and high-energy-density physics. It is based on the lectures that I and my colleagues are teaching master students at the University of Bordeaux.

With progress in high-power laser technology, understanding the physics of laser interaction with plasma is indispensable for many applications, including inertial confinement nuclear fusion and laboratory astrophysics. Our primary interest concerns the plasmas created with high energy, high power lasers with intensities in the range of $10^{12} - 10^{16}$ W/cm^2 and pulse duration ranging from several picoseconds to a few nanoseconds. Such lasers can heat plasmas to temperatures of tens of millions of Kelvin (or to a few kiloelectronvolts), so the particles are hot but non-relativistic. Under special conditions, a part of laser energy through wave-particle interactions can be transferred to a relatively small number of particles that can attain relativistic energies. Particle heating and acceleration to high energies are of great importance for applications of high-energy-density physics, particularly for inertial confinement fusion and fusion energy production. Reading this book requires general knowledge of plasma physics, electromagnetism, and statistical mechanics. This information can be found in the standard textbooks. I would recommend the books by Lev Landau and Evgeny Lifshitz [93, 94, 100], John David Jackson [77], Frank Chen [35], Paul Bellan [13], Jean-Marcel Rax [140], I. P. Shkarofsky, T. W. Johnston and M. P. Bachynskii [154] and Hideaki Takabe [163]. This book contains three chapters: plasma kinetics, waves in plasma, and laser-plasma interactions. Plasma kinetics is based on the microscopic formulation of the distribution function of particles and provides a theoretical description of the collective dynamics of charged particles driven by electromagnetic interactions. Two classes of collective phenomena are discussed: first, collisionless, relatively low-density plasma, where self-consistent electric and magnetic fields control particle dynamics, the binary collisions can be neglected, and particle distribution could be far from thermal equilibrium. The second class of phenomena involves denser plasmas, which are closer to equilibrium, where collisions dominate, and hydrodynamic description is appropriate. Reduction of the fully kinetic description to a more simple hydrodynamic model is considered, and dynamic and transport properties are discussed. These features are of great importance for high-energy-density physics. A detailed description of hydrodynamic processes in the application to high energy density plasmas can be found in books by Yakov Zeldovich and Yuri Raizer [188], Stefano Atzeni and Jürgen Meyer-ter-Vehn [7] and Paul Drake [50].

The second chapter addresses collective excitations – plasma waves – that can propagate far away from the source and efficiently interact with particles. The properties of these waves and the characteristics of their interaction with particles are of great importance for understanding their nonlinear interaction, which is the subject of the third chapter. Waves in plasmas can often be unstable. An excitation of waves that grow to large amplitudes is a phenomenon of great importance in plasma physics. These unstable waves may transform plasma into a turbulent state. This chapter also includes a section describing the emission of electromagnetic waves by charged particles, which is important for many applications and, in particular, for remote plasma diagnostics.

Two domains of plasma physics – particle kinetics and electromagnetic waves – provide the background to the third chapter of this book dedicated to the physics of laser-plasma interactions. Two major phenomena are discussed: laser propagation and energy deposition in a spatially inhomogeneous plasma and laser-driven plasma instabilities. Although these processes are studied already more than 50 years, there are only very few comprehensive reviews and books: well-known review by William Kruer [88] published almost 40 years ago, more recent books by Paul Gibbon [66], Peter Mulser and Dieter Bauer [131] and recently published books by Guy Bonnaud [23] and Pierre Michel [122]. This book is complementary. It provides a theoretical description of plasma processes and laser-plasma interactions with particular attention to simplified models that have analytical solutions that provide a comprehensive illustration of the key processes that appear in more complicated combinations in more realistic situations. The theoretical models are illustrated with the results of numerical simulations and representative experiments.

An appropriate mathematical description is provided throughout the text. Readers with a sufficient background at the level of standard university courses will be capable of reproducing the key results. Formal mathematical descriptions are supported by qualitative explanations that guide intuition and provide a basic framework. The theoretical material is completed with practical exercises at the end of each section.

The international system of units is used throughout the text. Conversion to the Gaussian system and convenient expressions for the basic plasma parameters can be found in the plasma formulary [72]. Bold symbols are used to denote vector and tensor quantities.

I am grateful to my colleagues Guillaume Duchateau and Eugene Gamaly for reading the text and providing multiple comments and suggestions and Jean-Luc Feugeas for providing his artistic work for the cover page. I am also thankful to all my collaborators for sharing their interest in many joint research projects, some of which are referred to in this work.

Chapter 1

Particle kinetics

1.1 Kinetic description of a plasma

This section provides the background of the plasma kinetic theory by introducing such fundamental quantity as the particle distribution function and considering the interaction of charged particles with the electric and magnetic fields. Microscopic and macroscopic plasma descriptions are introduced here and used throughout this book.

1.1.1 Distribution function of particles

The complete microscopic description of a gas, an N-particle system confined in a volume V, is given by the coordinates $\mathbf{r}_i(t)$ and the momenta $\mathbf{p}_i(t)$ of all particles at all time. This detailed information allows one to determine how many particles are in a given domain of momenta $d\mathbf{p}$ and in a given volume of the coordinate space $d\mathbf{r}$. The *microscopic distribution function*, $f_{\rm micro}(t, \mathbf{r}, \mathbf{p})$, characterizes the number of particles at a moment t in a volume of the phase space $d\mathbf{r} \times d\mathbf{p}$:

$$dN = f_{\rm micro}(t, \mathbf{r}, \mathbf{p})\, d\mathbf{r}\, d\mathbf{p}. \tag{1.1.1}$$

In classical physics with point-like particles, the function $f_{\rm micro}$ is discontinuous. It is non-zero only in the points in the phase space that coincide precisely with the positions and the momenta, $\mathbf{r}_i$ and $\mathbf{p}_i$, of particles. The function $f_{\rm micro}$ can be written as a product of Dirac delta-functions corresponding to the ensemble of particles:

$$f_{\rm micro}(t, \mathbf{r}, \mathbf{p}) = \sum_{i=1}^{N} \delta[\mathbf{r} - \mathbf{r}_i(t)]\, \delta[\mathbf{p} - \mathbf{p}_i(t)], \tag{1.1.2}$$

where $\delta(\mathbf{r}) \equiv \delta(x)\, \delta(y)\, \delta(z)$ is the Dirac delta-function in three dimensions. This microscopic distribution function has an exact expression if the positions and momenta of all particles are known. However, such a definition of the distribution function for systems containing a large number of particles is formal. It represents essentially the ensemble of all coordinates and momenta of all particles. Knowing that there are more than 10^{25} molecules in one cubic meter of air at the normal pressure, it is easy to understand that such a microscopic distribution function cannot be calculated, and it is unpractical because the average distance between particles is smaller than the available spatial resolution.

Consequently, the objective of the kinetic theory is not to remain at the level of description of discrete particles but to introduce a *continuous distribution function* $f(t, \mathbf{r}, \mathbf{p})$ defined everywhere in the phase space. Instead of determining an exact number of particles, as the microscopic function does, the continuous distribution function defines the *probability* to find a certain number of particles in a given phase volume. Such a continuous distribution function can be obtained by making an *average* of the microscopic distribution function over a particular phase volume $W_a = V_a \times P_a$:

$$f(t, \mathbf{r}, \mathbf{p}) = \langle f_{\rm micro}(t, \mathbf{r}, \mathbf{p}) \rangle_{W_a}, \tag{1.1.3}$$

where V_a is an elementary volume in the coordinate space and P_a is the elementary volume in the momentum space. This function provides an approximate description of the number of particles in a given phase space volume because it differs from $f_{\rm micro}$. However, this difference $\delta f_{\rm micro} = f_{\rm micro} - f$ is a random quantity with a zero average. This random deviation is called a *fluctuation*. The statistical theory aims to minimize the fluctuations to have the most exact possible description of the system. But fluctuations cannot be neglected completely: they describe correlations between the motion of neighboring particles and manifest themselves in the collisional processes. In the first crude approximation, one may neglect fluctuations and consider only the average function f. This approximation corresponds to *collisionless plasmas*.

The choice of the phase volume W_a of averaging the distribution function in Eq. (1.1.3) is not arbitrary. It depends on the physical problem that one is dealing with. The volume V_a must be larger than the mean volume attributed to each particle $V_a \gg V/N$, where V is the plasma volume. If V_a is too small, one may have a precise spatial resolution. Still, because of a very small number of particles N_a in the volume V_a, the average function will be defined with insufficient precision. The general statistical theory shows that the relative amplitude of fluctuations around the mean value is on the order of $N_a^{-1/2}$. So N_a should be sufficiently large. At the same time, the volume V_a must be small compared to the total volume, $V \gg V_a$, to describe the system with sufficient precision. Correspondingly, N_a should be much smaller than the total number of particles N. Similar arguments apply to the choice of the phase volume P_a.

Practically, the phase volume $W_a = V_a \times P_a$ is defined by the resolution of measurements in experiments, the number of macro-particles available, and the computer performance in numerical simulations. For example, the mean distance between particles in air at normal conditions is about 10 nm. So, choosing the resolution distance $d_a = V_a^{1/3}$ on the order of ten microns would be sufficient. That cubic volume contains $N_a \sim 10^9$ particles. The characteristic momentum of the air molecules at room temperature is $p_t \sim (m_a T)^{1/2} \sim 1.5 \times 10^{-23}$ N·s. Here, m_a is the mass of a molecule, and T is the temperature measured in energetic units (electronvolts); the Boltzmann constant is included in the definition of temperature. Thus, choosing the size of an elementary cubic cell in the phase space $\Delta p \sim 0.1\, p_t$ would be sufficient. Assuming no particles may have momentum ten times larger than p_t, there would be 10^6 elementary cells in the phase volume with $N_p \sim 10^3$ particles on average in each cell. Consequently, measuring the mean values with a precision of $N_p^{-1/2} \sim 10\%$ will be possible.

The characteristic distance of spatial averaging in plasmas is defined by the distance of screening of the potential of a charged particle, the Debye length λ_{D}. The phenomenon of electric field screening in plasma is discussed in Sec. 1.2.2. It is essential to mention that if the number of charged particles in the Debye sphere is sufficiently large, one can perform an average, and the fluctuations induced by particle binary collisions are small. So, the plasma evolution is controlled by self-consistent large-scale electromagnetic fields.

1.1.2 Klimontovich equation

A kinetic equation describes the evolution of the distribution function. It can be derived from the expression for the microscopic function. According to the definition (1.1.2), the temporal evolution of f_{micro} is due to the movements of particles in the phase space. The temporal derivative of the distribution function f_{micro} can be written as:

$$\partial_t f_{\mathrm{micro}} = \mathrm{d}_t \sum_{i=1}^{N} \delta[\vec{r} - \vec{r}_i(t)]\, \delta[\vec{p} - \vec{p}_i(t)]$$

where $\partial_t = \partial/\partial t$ is the partial temporal derivative, and $\mathrm{d}_t = d/dt$ is the total temporal derivative. As the coordinates and the momenta of particles are the quantities depending on time according to the equation of motion, we can write the derivative as follows:

$$\begin{aligned}\partial_t f_{\mathrm{micro}} = &-\sum_{i=1}^{N} \mathrm{d}_t\mathbf{r}_i \cdot \boldsymbol{\nabla}\, \delta[\mathbf{r} - \mathbf{r}_i(t)]\, \delta[\mathbf{p} - \mathbf{p}_i(t)] \\ &-\sum_{i=1}^{N} d_t\mathbf{p}_i \cdot \partial_{\mathbf{p}}\, \delta[\mathbf{p} - \mathbf{p}_i(t)]\, \delta[\mathbf{r} - \mathbf{r}_i(t)] \end{aligned} \tag{1.1.4}$$

where $\boldsymbol{\nabla} = \partial/\partial\mathbf{r}$ is the spatial gradient and $\partial_{\mathbf{p}} = \partial/\partial\mathbf{p}$ is the partial derivative with respect to the momentum.

Let us consider first the derivative of the particle orbits, $\mathrm{d}_t\mathbf{r}_i$. It is, by definition, equal to the particle velocity, $\mathrm{d}_t\mathbf{r}_i = \mathbf{v}_i$, which is related to the momentum. In classical mechanics $\mathbf{v}_i = \mathbf{p}_i/m$, and in relativistic mechanics $\mathbf{v}_i = \mathbf{p}_i/m\gamma$, where the relativistic factor $\gamma = (1 + \mathbf{p}_i^2/m^2c^2)^{1/2}$, m is the particle mass and c is the light velocity. Similarly, the derivative $\mathrm{d}_t\mathbf{p}_i$ is related to the force according to the Newton's law,

$$\mathrm{d}_t\mathbf{p}_i = \mathbf{F}_i,$$

where the force $\mathbf{F}_i$ applied to the i-th particle; for charged particles it is the Lorentz force. In the most general case, it depends on time, the position, and the speed of the particle; $\mathbf{F}(t, \mathbf{r}_i, \mathbf{p}_i)$. It can be separated into two parts: the external force, $\mathbf{F}^{\mathrm{ext}}$, and a microscopic force, $\mathbf{F}_{i\,\mathrm{micro}}$ produced by other ($j \neq i$) particles, $\mathbf{F}^{\mathrm{ext}} + \mathbf{F}_{i\,\mathrm{micro}}$. The first one is produced by the sources out of plasma (capacitors, coils, laser beams, etc.) and varies slowly in space and time. In contrast, the second one is a sum of the Lorentz forces produced by all other particles in plasma. It varies strongly in space and time because of particles' random motion. This internal, self-consistent Lorentz force is at the origin of the interaction between the particles and the collective behavior of plasma.

One can remove the subscripts of speed $\mathbf{v}_i$ and of the force $\mathbf{F}_i$ in Eq. (1.1.4) thanks to the specific property of the Dirac delta-function: $\phi(a)\,\delta(x - a) = \phi(x)\,\delta(x - a)$, where ϕ is an arbitrary function. So, one can take the velocities and the forces out of the summation and write Eq. (1.1.4) as an equation for the microscopic distribution function:

$$\partial_t f_{\mathrm{micro}} = -\mathbf{v} \cdot \boldsymbol{\nabla} f_{\mathrm{micro}} - \mathbf{F}_{\mathrm{micro}} \cdot \partial_{\mathbf{p}} f_{\mathrm{micro}}\,. \tag{1.1.5}$$

This relation is called the *Klimontovich kinetic equation*. It was proposed for the first time in the book by Yuri Klimontovich [83] in 1967 and further developed by many authors. Although this microscopic equation represents an ensemble of discrete particles, individual particles' coordinates and momenta are not formally present. The differential operators are applied to the point in the phase space, $(\mathbf{r}, \mathbf{p})$, but not to the coordinates of the particles. This allows us to develop an equation for the average distribution function.

1.1.3 Vlasov kinetic equation

To obtain a regular equation for the continuous distribution function, we follow the prescription given in Eq. (1.1.3). As explained in the previous section, we must average the Klimontovich equation (1.1.5) over a phase volume W_a. The averaging of the derivatives does not pose a problem because these operators are applied to a point in the phase space but not to the positions of individual particles:

$$\partial_t f + \mathbf{v}\cdot\boldsymbol{\nabla} f + \langle \mathbf{F}_{\rm micro}\cdot\partial_{\mathbf{p}} f_{\rm micro}\rangle = 0. \tag{1.1.6}$$

The last term on the left-hand side needs special consideration. This is a product of two microscopic quantities. To simplify this term, we need to introduce an additional hypothesis of *weak correlations*, that is, to assume that an average distance between the particles is sufficiently large, so each particle moves freely as if there are no other particles around. These free trajectories are regular. The average forces define them, and the perturbations induced by other particles are sufficiently small and can be considered second-order terms.

Correspondingly, the microscopic quantities, $f_{\rm micro}$ and $\mathbf{F}_{\rm micro}$ can be presented as a sum of average quantities and corresponding fluctuations:

$$f_{\rm micro} = f + \delta f_{\rm micro}, \qquad \mathbf{F}_{\rm micro} = \mathbf{F} + \delta\mathbf{F}_{\rm micro}. \tag{1.1.7}$$

This allows us to represent the last term in Eq. (1.1.6) as a sum of four terms:

$$\begin{aligned}\mathbf{F}_{\rm micro}\cdot\partial_{\mathbf{p}} f_{\rm micro} = {}& \mathbf{F}\cdot\partial_{\mathbf{p}} f + \delta\mathbf{F}_{\rm micro}\cdot\partial_{\mathbf{p}} f \\ & + \mathbf{F}\cdot\partial_{\mathbf{p}}\delta f_{\rm micro} + \delta\mathbf{F}_{\rm micro}\cdot\partial_{\mathbf{p}}\delta f_{\rm micro}.\end{aligned} \tag{1.1.8}$$

With appropriately chosen averaging volume W_a, the amplitudes of fluctuations of the microscopic distribution function, $\delta f_{\rm micro}$ and the microscopic force, $\delta\mathbf{F}_{\rm micro}$, are small with zero mean value, $\langle\delta f_{\rm micro}\rangle = 0$ and $\langle\delta\mathbf{F}_{\rm micro}\rangle = 0$. Then, the second and third terms in the right-hand side of Eq. (1.1.8) are averaged to zero, and the fourth term is of second order. So, only the first term must be considered in the first approximation.

Moreover, we have to account for the fact that there are several particle species in plasma, at least two: electrons and ions. So, we define the distribution functions for each species α. Then, in the first order, a kinetic equation for the average distribution function takes the following standard form:

$$\partial_t f_\alpha + \mathbf{v}\cdot\nabla f_\alpha + \mathbf{F}_\alpha\cdot\partial_{\mathbf{p}} f_\alpha = 0. \tag{1.1.9}$$

The terms on the left-hand side describe the evolution of the average distribution function due to spatial gradients (second term) and a self-consistent force (third term). In plasma, this is the Lorentz force that accounts for the action of electric and magnetic fields on a particle with a charge q_α:

$$\mathbf{F}_\alpha = q_\alpha(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{1.1.10}$$

This kinetic equation with the *self-consistent* force, that is, the force produced by all plasma particles, is called *Vlasov equation*. It was empirically introduced by the Russian scientist Anatoly Vlasov in 1938 to give a theoretical explication of plasma oscillations [177]. It is widely used in all kinetic models for particles interacting through long-range electromagnetic and/or gravitational forces.

The Vlasov equation describes the evolution of the distribution function on a large scale, larger than the Debye length, and it can be applied to low-density plasmas. By contrast, it does not account for the fluctuations induced by correlated particle motion. In the simplest form, these correlations can be considered as binary collisions. The collisional effects become important when the particles approach closely to each other. Interaction at short distances, smaller than the Debye length, is due to binary collisions. Therefore, the Vlasov equation can be used under the conditions where plasma can be considered collisionless, that is, on the time scales smaller than the average collisional time and on the spatial scales smaller than the particle mean free path.

1.1.4 Collision integral

Let us consider now the contribution of the fluctuation terms $\delta f_{\rm micro}$ and $\delta \mathbf{F}_{\rm micro}$ in the kinetic equation (1.1.6). It originates from the fourth term on the right-hand side of Eq. (1.1.8):

$$\partial_t f_\alpha + \mathbf{v}\cdot\boldsymbol{\nabla} f_\alpha + \mathbf{F}_\alpha\cdot\partial_{\mathbf{p}} f_\alpha = \\ -\left\langle \delta\mathbf{F}_{{\rm micro}\,\alpha}\cdot\partial_{\mathbf{p}}\sum_\beta \delta f_{{\rm micro}\,\beta}\right\rangle \equiv \sum_\beta C_{\alpha\beta}. \tag{1.1.11}$$

Compared to the Vlasov equation, the right-hand side is nonzero, called the *collision integral*. To find its explicit form, we need to solve equations for the fluctuations of electromagnetic fields and the distribution function. The explicit form of the collision integral is discussed in Sec. 1.3.1. Here, we present some general comments.

First, the hypothesis of weak fluctuations allows us to account only for binary collisions between the particles of species α and β in an additive way. That means that collision integral Eq. (1.1.11) is a sum of binary collision terms, while triple collisions are excluded since they contribute to higher-order terms. Thus, weak correlations correspond to small perturbations of the particle orbits. In each collision, the particle does not change much in its direction of propagation and its momentum. So, the right-hand side of Eq. (1.1.11) describes the pitch angle particle scattering.

A general kinetic equation that takes into account the scattering of particles at small angles was developed by Adriaan Fokker and Max Planck in 1917. It is demonstrated in Sec. 1.3.2 that Eq. (1.1.11) is a particular case of the Fokker-Planck equation.

It is important to notice that the collision integral is local. There are no operators containing a spatial or a temporal derivative. So, the plasma is supposed to be homogeneous and stationary at the correlation scales. The Debye length defines the spatial scale, $\lambda_{{\rm D}\alpha}$. The temporal scale depends on the characteristic (thermal) speed of particles, $v_{{\rm T}\alpha} = (T_\alpha/m_\alpha)^{1/2}$. The ratio $\lambda_{{\rm D}\alpha}/v_{{\rm T}\alpha} = 1/\omega_{{\rm p}\alpha}$ is the inverse of plasma frequency.

The collision integral in the form (1.1.11) satisfies the conservation laws. It preserves the number of particles of each species, the total momentum of motion of the system, and its total energy. The mathematical demonstration of these properties is presented in Sec. 1.4.1, but it is easy to understand them qualitatively. The collisional term in Eq. (1.1.11) takes into account the binary collisions. These are elastic collisions – in every collision, the number of particles, the momentum, and the energy are preserved. So, it is unsurprising that the collision integral has the same properties as its microscopic origin.

1.1.5 Maxwell's equations for macroscopic fields

Kinetic equations (1.1.9) and (1.1.11) must be completed with equations defining the average Lorentz force, that is, with equations for the mean fields, $\mathbf{E} = \langle \mathbf{E}_{\rm micro}\rangle$ and $\mathbf{B} = \langle \mathbf{B}_{\rm micro}\rangle$. The microscopic fields verify Maxwell's equations at the microscopic scale:

$$\boldsymbol{\nabla}\times\mathbf{E}_{\rm micro} = -\partial_t\mathbf{B}_{\rm micro}, \quad \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm micro} = \rho_{\rm micro} + \rho^{\rm ext},$$
$$\mu_0^{-1}\boldsymbol{\nabla}\times\mathbf{B}_{\rm micro} = \mathbf{j}_{\rm micro} + \mathbf{j}^{\rm ext} + \epsilon_0\partial_t\mathbf{E}_{\rm micro}, \quad \boldsymbol{\nabla}\cdot\mathbf{B}_{\rm micro} = 0.$$

Here, ϵ_0 and μ_0 are the dielectric permittivity and magnetic permeability of vacuum, respectively. These electric and magnetic fields are generated by the plasma particles or by external sources, ρ^{ext} and $\mathbf{j}^{\text{ext}}$. The microscopic density

$$\rho_{\text{micro}}(t,\mathbf{r}) = \sum_\alpha q_\alpha \sum_{i=1}^{N} \delta[\mathbf{r} - \mathbf{r}_{i\alpha}(t)]$$

is a sum of all charges multiplied by the corresponding Dirac delta function defining the particle position. In the same way, the density of the microscopic current is a sum of all currents produced by particles:

$$\mathbf{j}_{\text{micro}}(t,\mathbf{r}) = \sum_\alpha q_\alpha \sum_{i=1}^{N} \mathbf{v}_{i\alpha}(t)\,\delta[\mathbf{r} - \mathbf{r}_{i\alpha}(t)].$$

One can present these microscopic sources by using the microscopic distribution function, adding contributions of all species, and taking an integral over the momentum:

$$\rho_{\text{micro}}(t,\mathbf{r}) = \sum_\alpha q_\alpha \int f_{\text{micro}\,\alpha}(t,\mathbf{r},\mathbf{p})\,d\mathbf{p}, \tag{1.1.12}$$

$$\mathbf{j}_{\text{micro}}(t,\mathbf{r}) = \sum_\alpha q_\alpha \int \mathbf{v}\, f_{\text{micro}\,\alpha}(t,\mathbf{r},\mathbf{p})\,d\mathbf{p}. \tag{1.1.13}$$

This form of presentation of the internal sources allows for the direct average of microscopic Maxwell's equations. As Maxwell's equations are linear, their macroscopic form is the same as their microscopic counterpart:

$$\boldsymbol{\nabla}\times\mathbf{E} = -\partial_t\mathbf{B}, \qquad \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E} = \rho^{\text{int}} + \rho^{\text{ext}}, \tag{1.1.14}$$

$$\mu_0^{-1}\boldsymbol{\nabla}\times\mathbf{B} = \mathbf{j}^{\text{int}} + \mathbf{j}^{\text{ext}} + \epsilon_0\partial_t\mathbf{E}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0. \tag{1.1.15}$$

The internal sources, ρ^{int} and $\mathbf{j}^{\,\text{int}}$, are defined as an average of microscopic sources (1.1.12) and (1.1.13):

$$\rho^{\text{int}}(t,\mathbf{r}) = \sum_\alpha q_\alpha \int f_\alpha(t,\mathbf{r},\mathbf{p})\,d\mathbf{p}, \tag{1.1.16}$$

$$\mathbf{j}^{\text{int}}(t,\mathbf{r}) = \sum_\alpha q_\alpha \int \mathbf{v}\, f_\alpha(t,\mathbf{v},\mathbf{p})\,d\mathbf{p}. \tag{1.1.17}$$

Since electromagnetic fields operate on a long-range, every particle moves in a collective field created by many neighboring particles. This means that plasma exhibits a collective behavior, which is different from a neutral gas, where the effective radius of particle interaction is much smaller than the average distance between two particles. Microscopic electromagnetic fields related to the fluctuations of the distribution function are discussed in Sec. 2.1.11.

1.1.6 Plasma macroscopic characteristics

Every microscopic physical quantity can find a correspondent average counterpart. For example, we already demonstrated a relation between the microscopic charge density (1.1.12) and the mean charge density (1.1.16). The density of particles of species α is defined as:

$$n_\alpha(t,\mathbf{r}) = \int f_\alpha(t,\mathbf{r},\mathbf{p})\,d\mathbf{p}. \tag{1.1.18}$$

Similarly a *mean plasma velocity* $\mathbf{u}_\alpha$ and a *mean energy* $\mathcal{E}_\alpha$ are defined as:

$$\mathbf{u}_\alpha(t,\mathbf{r}) = \frac{1}{n_\alpha(t,\mathbf{r})}\int \mathbf{v}_\alpha\, f_\alpha(t,\mathbf{r},\mathbf{p})\, d\mathbf{p}, \tag{1.1.19}$$

$$\mathcal{E}_\alpha(t,\mathbf{r}) = \frac{1}{n_\alpha(t,\mathbf{r})}\int \varepsilon_\alpha f_\alpha(t,\mathbf{r},\mathbf{p})\, d\mathbf{p}, \tag{1.1.20}$$

where $\mathbf{v}_\alpha$ and $\varepsilon_\alpha(\mathbf{p})$ are the speed and the energy of the particle. In the classical mechanics $\mathbf{v}_\alpha = \mathbf{p}/m_\alpha$ and $\varepsilon_\alpha = \mathbf{p}^2/2m_\alpha$. The definitions (1.1.19) and (1.1.20) also apply to the relativistic or quantum plasmas, where the relation of the speed, $\mathbf{v}_\alpha(\mathbf{p})$ and the energy of the particle, $\varepsilon_\alpha(\mathbf{p})$, with the momentum $\mathbf{p}$ are more complicated. For example, in a relativistic plasma, where $p \gtrsim m_\alpha c$, the speed, $\mathbf{v}_\alpha = \mathbf{p}/\gamma_\alpha m_\alpha$, and the kinetic energy of particle, $\varepsilon_\alpha = (\gamma_\alpha - 1)\, m_\alpha c^2$, depend of the relativistic factor $\gamma_\alpha(\mathbf{p}) = (1 + \mathbf{p}^2/m_\alpha^2 c^2)^{1/2}$.

The tensor of *momentum flux* and the vector of *energy flux*:

$$P_{\alpha\, ij}(t,\mathbf{r}) = \int p_i v_j\, f_\alpha(t,\mathbf{r},\mathbf{p})\, d\mathbf{p}, \tag{1.1.21}$$

$$\mathbf{Q}_\alpha(t,\mathbf{r}) = \int \varepsilon_\alpha(\mathbf{p})\, \mathbf{v}\, f_\alpha(t,\mathbf{r},\mathbf{p})\, d\mathbf{p}, \tag{1.1.22}$$

describe the *transport processes* in plasma – transfer of the momentum and energy in a plasma.

It is often convenient to separate the mean motion of particles of the species α with a mean speed $\mathbf{u}_\alpha$ and a chaotic (thermal) motion with a relative velocity $\mathbf{w} = \mathbf{v} - \mathbf{u}_\alpha$. Then the distribution function in the local reference frame, $F_\alpha(\mathbf{k}) = f_\alpha(\mathbf{p} - m_\alpha \mathbf{u}_\alpha)$, where $\mathbf{k} = m_\alpha(\mathbf{v} - \mathbf{u}_\alpha)$ is the particle momentum in the proper reference frame such that

$$\int \mathbf{k}\, F_\alpha(\mathbf{k})\, d\mathbf{k} = 0.$$

That propriety allows us to divide the energy (1.1.20) and the fluxes (1.1.21) and (1.1.22) of all particles into the mean and the chaotic parts. For the average particle energy, according to Eq. (1.1.20), we have:

$$\mathcal{E}_\alpha = \frac{1}{2m_\alpha n_\alpha}\int (\mathbf{k} + m_\alpha \mathbf{u}_\alpha)^2 F_\alpha(\mathbf{k})\, d\mathbf{k}.$$

This integral can be presented as

$$\mathcal{E}_\alpha = \frac{1}{2} m_\alpha \mathbf{u}_\alpha^2 + \mathcal{W}_\alpha, \tag{1.1.23}$$

where the internal energy $\mathcal{W}_\alpha$ is a measure of thermal motion. It is defined by the following equation:

$$\mathcal{W}_\alpha = \frac{1}{2m_\alpha n_\alpha}\int \mathbf{k}^2 F_\alpha(\mathbf{k})\, d\mathbf{k}. \tag{1.1.24}$$

In the same way, the momentum flux can be presented in the form $P_{\alpha\, ij} = m_\alpha n_\alpha u_{\alpha\, i} u_{\alpha\, j} + p_{\alpha\, ij}$, where the second term corresponds to the plasma pressure [1]

$$p_{\alpha\, ij} = m_\alpha^{-1}\int k_i k_j F_\alpha(\mathbf{k})\, d\mathbf{k}. \tag{1.1.25}$$

[1] Note the difference between the particle momentum $\mathbf{p}$ and the pressure p_α.

The energy flux is divided into three terms:

$$Q_{\alpha\, i} = \frac{1}{2} m_\alpha n_\alpha u_\alpha^2 u_{\alpha\, i} + n_\alpha \mathcal{W}_\alpha u_{\alpha\, i} + p_{\alpha\, ij} u_{\alpha\, j} + q_{\alpha\, i}. \tag{1.1.26}$$

They describe the convective energy transport, $\frac{1}{2} m_\alpha n_\alpha u_\alpha^2 u_{\alpha\, i}$, transport of the enthalpy, $n_\alpha \mathcal{W}_\alpha u_{\alpha\, i} + p_{\alpha\, ij} u_{\alpha\, j}$, and the diffusive heat flux:

$$\mathbf{q}_\alpha = \frac{1}{2m_\alpha^2} \int \mathbf{k}^2 \mathbf{k}\, F_\alpha(\mathbf{k})\, d\mathbf{k}. \tag{1.1.27}$$

In a plasma in thermal equilibrium, the local distribution function is a Maxwellian function depending on the momentum:

$$F_\alpha(\mathbf{k}) = \frac{n_\alpha}{(2\pi m_\alpha T_\alpha)^{3/2}} \exp\left(-\frac{k^2}{2m_\alpha T_\alpha}\right). \tag{1.1.28}$$

This is an isotropic distribution function, the kinetic pressure is a scalar, $p_{\alpha\, ij} = p_\alpha \delta_{ij}$, and there are the following relations between the thermal particle energy, temperature, and pressure: $\varepsilon_\alpha = \frac{3}{2} T_\alpha$ and $p_\alpha = n_\alpha T_\alpha$. It is also convenient to define the thermal speed of particles as $v_{\mathrm{T}\alpha} = (T_\alpha / m_\alpha)^{1/2}$. (The temperature is measured in units of energy throughout the book.)

1.1.7 Numerical solution of kinetic equations

Kinetic equations (1.1.9) or (1.1.11) coupled to Maxwell's equations (1.1.14) and (1.1.15) provide the most detailed description of the plasma properties. Some particular solutions of these equations for idealized conditions are described in the following sections of this book. However, analytical solutions are only available for some practical situations, and numerical methods are widely used. The large number of dimensions in the phase space – three components of momentum and three coordinates – makes numerical solutions very challenging even with the best available high-performance computers. Often, problems are resolved in the phase space with a reduced number of dimensions: one or two in the coordinate space and one, two, or three in the momentum space.

Two numerical methods are used for solving the kinetic equation. One consists of discretizing the phase space and interpolating the convective and force terms of Eq. (1.1.9) on the grid using the finite differences. Collision integrals are calculated by developing the distribution over the spherical harmonics in the phase space using the method of the potentials of Rosenbluth [149]. Examples of such Vlasov-Fokker-Planck (VFP) codes developed for high-energy-density physics applications are described in Refs. [175, 176, 166]. The VFP codes are very accurate and can resolve detailed features of the distribution function in the phase space but are relatively slow and time-consuming. They are characterized by low numerical noise and used for studies such kinetic processes as particle trapping in the wave potential or resonance wave-particle interaction. However, they are less suitable for investigation of processes on a large scale.

Another numerical method, Particle-in-Cell (PIC), is based on the probabilistic approach. The distribution function in the phase space is sampled with finite-size particles (macro-particles) that are moving in the coordinate space according to the Lorentz force (1.1.10), which is interpolated from the grid on the particle position at each time step. In return, the charge density and the current are interpolated to the grid from the particle positions and are used as sources for solving Maxwell's equations. Since the phase space is sampled only at the particles' positions, this method is much faster and more robust than the VFP method. It is widely used in numerical simulations of laser-plasma interactions. The two open-access PIC codes are EPOCH [6] and SMILEI [44]. Unfortunately, since each macro-particle represents a large number of real particles, PIC codes suffer from high numerical noise and numerical heating, which require special care while describing physical results.

In what follows, we show examples of processes calculated with VFP and PIC codes.

1.1.8 Exercises

1. Using the Vlasov equation (1.1.9), find a relation between the charge density and the density of current in the plasma. Take a divergence of the Ampere equation (1.1.15) and show that it is equivalent to the Poisson equation without external sources.

2. Analog of the Maxwellian distribution function for the relativistic particles is the Maxwell-Jüttner distribution function

$$f_{\rm MJ}(\mathbf{p}) = C_{\rm MJ} \exp(-\alpha\gamma),$$

   where $\gamma = (1 + p^2/m^2c^2)^{1/2}$ is the particle relativistic factor and $\alpha = mc^2/T$. Find a relation of the constant $C_{\rm MJ}$ to the particle density n and temperature T. Find relations between the average energy $\mathcal{E}_\alpha$, pressure p and temperature.
   Response:
   $C_{\rm MJ} = n\alpha/4\pi m^3c^3 K_2(\alpha)$, where K_2 is the Bessel function, $p = nT$ and $\mathcal{E}_\alpha = [3 - \alpha + \alpha K_1(\alpha)/K_2(\alpha)]T$.

1.2 Collisionless plasmas

In this section, we consider equilibrium solutions of the Vlasov kinetic equation first in a homogeneous plasma and then in plasmas with more complicated geometries where the self-consistent electric and magnetic fields modify the particle trajectories. The particular problems presented blow aim on the demonstration of the basic properties of collisionless plasmas such as screening of electric field, trapping particles in the electrostatic potential and formation of collisionless solitons and shocks.

1.2.1 Equilibrium of a homogeneous plasma

At equilibrium, the distribution function of particles is independent of time. It can be found as a time-independent solution of a system of the Vlasov equation (1.1.9) and Maxwell's equations. Let us consider first the case of a homogeneous plasma, $\nabla f_\alpha = 0$. The term $\mathbf{E} \cdot \partial_{\mathbf{p}} f_\alpha$ in the Vlasov equation equals to zero if the electric field is zero, $\mathbf{E} = 0$. The term with the magnetic field, $(\mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{p}} f_\alpha$, equals to zero in two cases: (i) the magnetic field is zero, $\mathbf{B} = 0$, or (ii) the particle distribution function is isotropic, $f_\alpha(p)$. (In the latter case, the derivative $\partial_{\mathbf{p}} f_\alpha$ is parallel to the particle velocity, and the scalar product with the Lorentz force is zero.)

Maxwell's equations for zero electric and constant magnetic fields are satisfied if the sources are zero, $\rho = 0$ et $\mathbf{j} = 0$. These conditions lead to three important consequences:

- The plasma is electrically neutral: for two species of opposite charges – the ions of a positive charge $q_i = Ze$ and the electrons of a negative charge $q_e = -e$, the neutrality condition reads $Zn_i = n_e$.
- The electric current in plasma is zero:

$$\mathbf{j} = \sum_\alpha q_\alpha n_\alpha \mathbf{u}_\alpha = q_i n_i (\mathbf{u}_i - \mathbf{u}_e) = 0.$$

  That condition means that both species have the same mean velocities, $\mathbf{u}_i = \mathbf{u}_e = \mathbf{u}$. Therefore, in the reference frame moving with this velocity, the distribution functions of both species are isotropic, $f_\alpha(\mathbf{p}) = f_\alpha(p)$.
- The distribution function has only one maximum and is decreasing as a function of momentum, $\partial_p f_\alpha < 0$. This third condition is not evident. It comes from the condition of *stability* of the distribution function with respect to small amplitude perturbations. That issue is discussed in Sec. 2.1.8.

Within these limitations, the particle distribution function in a collisionless plasma is arbitrary and depends essentially on the way the plasma is created. Account for binary collisions strongly reduces the choice of equilibrium distribution functions. As discussed in Sec. 1.3, after a few collision times, the distribution function of all particles relaxes to a Maxwellian distribution function (1.1.28).

1.2.2 Plasma in a capacitor

Let us consider a plasma in a capacitor that is characterized by the width L and potential Φ_0. We want to find the particle density and the potential distribution inside the capacitor, assuming that the temperature T is uniform. The plasma, $x \in (-L/2, L/2)$, is made of protons ($q_i = e$) and electrons ($q_e = -e$), and the distribution function of ions and electrons is a Maxwellian function in the point $x = -L/2$ where the potential is zero.

The electric field in the capacitor: $E = -d_x\Phi$ depends on the spatial distribution of electrical potential $\Phi(x)$. The distribution function of particles follows from the Vlasov equation (1.1.9), which, in our case of a stationary plasma and electric field directed along the x axis, reads:

$$v_x\, \partial_x f - q\, d_x\Phi\, \partial_{p_x} f = 0. \tag{1.2.1}$$

Here, we omitted particle species' index α as it applies to electrons and ions. This equation involves only the x-component of the particle momentum. The distribution function on two other components, p_y and p_z, is defined by the initial condition: a Maxwellian function.

The solution of this differential equation, depending on two variables, x and p_x, can be constructed using the method of characteristics. The total differential of f is: $\mathrm{d}f = dp_x\, \partial_{p_x} f + dx\, \partial_x f$. To satisfy Eq. (1.2.1), one should find a line in the phase space (x, p_x), where $\mathrm{d}f$ is zero. Equation (1.2.1) takes the form of a full differential if the following relation is satisfied:

$$\frac{dx}{v_x} = -\frac{dp_x}{q\, d_x\Phi}.$$

Writing this equation as $v_x\, dp_x + q\, d\Phi = 0$, one can see that it is a full differential $\mathrm{d}W = 0$ of the total energy of the particle,

$$W = p_x^2/2m + q\Phi(x).$$

This relation describes the conservation of the mechanical energy of a particle in a stationary potential Φ. As a consequence, the general solution to the Vlasov equation reads:

$$f(x, p_x) = F[W(x, p_x)], \tag{1.2.2}$$

where F is an arbitrary function. According to the boundary condition at zero potential, $\Phi(-L/2) = 0$, the distribution function F is a Maxwellian function (1.1.28). Then, the solution to Eq. (1.2.1) verifying the boundary condition is:

$$F(W, p_y, p_z) = \frac{n_0}{(2\pi mT)^{3/2}} \exp\left(-\frac{W}{T} - \frac{p_y^2 + p_z^2}{2mT}\right).$$

The density of particles can be calculated according to the definition (1.1.18):

$$n(x) = n_0 \exp\left(-\frac{q\Phi(x)}{T}\right) \tag{1.2.3}$$

where n_0 is the density at $x = -L/2$, where the potential is zero.

The constant n_0 is defined using the condition of conservation of the total number of particles in the capacitor N_0. Without an external potential, the densities of unperturbed ions and the electrons are equal, $n_{i0} = n_{e0} = n_0$, and the total number par unity of the surface is $N_0 = n_0 L$. Knowing potential, we obtain n_0 by using the condition $N_0 = \int_{-L/2}^{L/2} dx\, n(x)$. To simplify calculations, we consider the case $e\Phi \ll T$, where the plasma is sufficiently hot, and we can develop exponential function in the Boltzmann law (1.2.3) into Taylor series, $\mathrm{e}^{\Phi} \approx 1+\Phi$. This allows us to write the density of particles in the potential Φ as:

$$n(x) = n_0 \left[1 - \frac{q}{T}(\Phi(x) - \bar{\Phi})\right] \tag{1.2.4}$$

where $\bar{\Phi} = (1/L) \int_{-L/2}^{L/2} dx\, \Phi(x)$ is the mean potential.

Knowing the density distribution, we can now solve the Poisson equation (1.1.14) and find the distribution of potential Φ and field E. Introducing the dimensionless potential $\varphi = e\,(\Phi - \bar{\Phi})/T$ and the Debye length $\lambda_{\mathrm{D}} = (\epsilon_0 T/e^2 n_0)^{1/2}$, we can write the Poisson equation (1.1.14) as:

$$d_x^2 \varphi = 2\lambda_{\mathrm{D}}^{-2} \varphi\,. \tag{1.2.5}$$

This equation must be solved with the condition that the potential difference at the capacitor plates is Φ_0: $\varphi(L/2) - \varphi(-L/2) = e\,\Phi_0/T$. Moreover, because of the symmetry of the problem, the function $\varphi(x)$ is an odd function, $\varphi(-x) = -\varphi(x)$, and therefore, $\int_{-L/2}^{L/2} dx\, \varphi(x) = 0$. The first integral of the Poisson equation (1.2.5) can be calculated by multiplying this equation by φ' and integrating over x:

$$(\varphi')^2 = 2\lambda_{\mathrm{D}}^{-2}\,(\varphi^2 + \varphi_0^2)$$

where φ_0 is a constant of integration that is determined later. The solution of this equation is:

$$\varphi(x) = \varphi_0 \sinh(x\sqrt{2}/\lambda_{\mathrm{D}}).$$

The constant of integration is set to zero because $\varphi(x)$ is an odd function. Another constant φ_0 is defined by the condition $\varphi(\pm L/2) = \pm e\,\Phi_0/2T$. This gives the following expression for the potential φ:

$$\varphi(x) = \frac{e\,\Phi_0}{2T}\,\frac{\sinh(x\sqrt{2}/\lambda_{\rm D})}{\sinh(L/\sqrt{2}\lambda_{\rm D})}\,.$$

The mean potential $\bar{\Phi}$ is deduced from the boundary condition, which reads $\Phi(-L/2) = 0$. The expression for the potential inside the capacitor reads:

$$\Phi(x) = \frac{\Phi_0}{2} + \frac{\Phi_0}{2}\,\frac{\sinh(x\sqrt{2}/\lambda_{\rm D})}{\sinh(L/\sqrt{2}\lambda_{\rm D})}\,.$$

This electrostatic potential distribution is shown in Fig. 1.1(a). It increases monotonously from the left plate of the capacitor to the right plate.

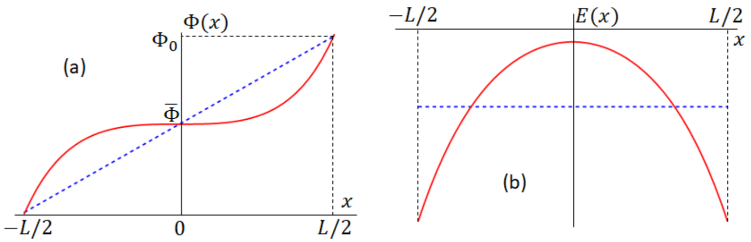

Figure 1.1: (a) Distribution of the potential in a capacitor in the presence (red solid line) and in the absence of plasma (blue dotted line). The plasma is in the interval $x \in (-L/2, L/2)$. (b) Distribution of the electric field inside a capacitor in the presence (red solid line) and in the absence of plasma (blue dotted line).

By taking the derivative of Φ, we obtain an expression for the electric field:

$$E(x) = -\frac{\Phi_0}{\sqrt{2}\lambda_{\rm D}}\,\frac{\cosh(x\sqrt{2}/\lambda_{\rm D})}{\sinh(L/\sqrt{2}\lambda_{\rm D})}\,.$$

It is shown in Fig. 1.1(b). There are two characteristic limits in this formula. We can develop the hyperbolic function in a Taylor series in a low-density plasma where the Debye length is larger than the capacitor width, $\lambda_{\rm D} \gg L$. Then, the electric field $E \approx -\Phi_0/L$ is constant, same as in the case without plasma (dotted line in Fig. 1.1(b)). However, if $\lambda_{\rm D} \ll L$, the electric field is localized near the plates of the capacitor at a distance of a few Debye lengths:

$$E(x) \approx -\frac{\Phi_0}{\sqrt{2}\lambda_{\rm D}}\,\exp\left(-\frac{L-2|x|}{\sqrt{2}\lambda_{\rm D}}\right)\,.$$

In this case, the plasma polarization has a strong effect. The electric field is enhanced near the plates and is suppressed inside; see Fig. 1.1(b). The plasma produces screening of the electric field in a capacitor. Utilizing (1.2.4), we can find the density of electrons and ions and demonstrate that the plasma acts as a dielectric. The plasma has a positive charge near the left plate of the capacitor and a negative one near the right plate. However, inside the capacitor, the plasma is quasi-neutral at a distance of a few Debye lengths.

Knowing how much time the plasma needs to reach this stationary state is interesting. In the stationary state, the term with the time derivative in the Vlasov equation (1.1.9) is small compared to the other two terms: $\partial_t f_\alpha \ll q_\alpha E\, \partial_p f_\alpha$. Estimating the partial derivative $\partial_p f_\alpha$ as $f_\alpha / m_\alpha v_{\mathrm{T}\alpha}$, we can evaluate the characteristic relaxation time to the stationary state as $\Delta t_\alpha \approx m_\alpha v_{\mathrm{T}\alpha}/|q_\alpha E|$. This time differs for electrons and ions because of the large mass ratio: $m_i/m_e \gg 1$ (1836 for the hydrogen plasma). The lighter electrons reach the equilibrium in a time $(m_i/m_e)^{1/2} \sim 40$ shorter than the ions. Therefore, the stationary state is established in two steps: first, the electrons are set to equilibrium, then ions approach it later.

1.2.3 Particles in an external electric potential

Charged particles in a non-monotonous external electric potential exhibit a complex motion, and their distribution functions may deviate from a Maxwellian function. In particular, in some cases, the particles can be trapped in a potential well. We consider here a simple representative case: the distribution function of particles of a charge q and mass m in an external potential $\Phi(x)$. We assume that the particles move just in one direction along the x-axis under the action of the electric field $E_x = -\partial_x \Phi$. The distribution function, $f(p_x, x)$, is a solution to the stationary Vlasov equation (1.2.1). The general solution to this equation is given in the previous section 1.2.2 as Eq. (1.2.2). Assuming that for $|x| \to \infty$, the potential is zero and the distribution function is a Maxwellian function (1.1.28), the solution to Eq. (1.2.2) is:

$$\begin{aligned} f(x, p_x) &= \frac{n_0}{(2\pi m T)^{1/2}}\, \mathrm{e}^{-W/T} \mathcal{H}(W) \\ &= \frac{n_0}{(2\pi m T)^{1/2}} \exp\left(-\frac{p_x^2}{2mT} - \frac{q\Phi(x)}{T}\right), \end{aligned} \tag{1.2.6}$$

where $\mathcal{H}(W)$ is the Heaviside function. This expression represents the Maxwell-Boltzmann distribution function of charged particles in an electrostatic potential.

Knowing the distribution function f, we can calculate the density and pressure of particles at position x following the definitions (1.1.18) and

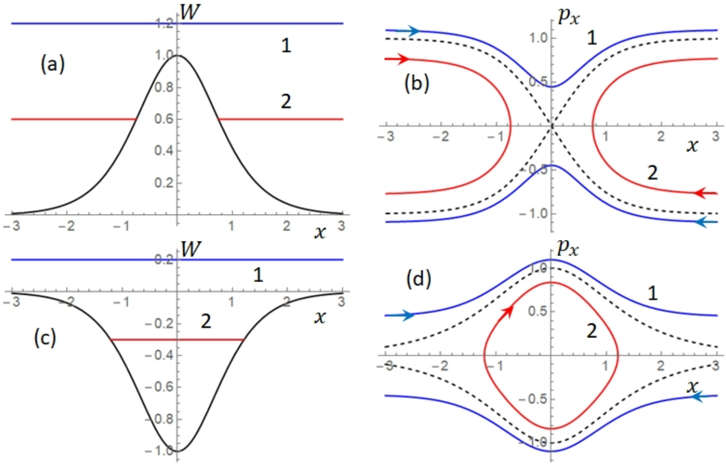

Figure 1.2: Energy W, left column, (a) and (c) and the phase space $p_x(x)$ of particles, right column, (b) and (d) in an electrostatic potential $q\Phi = \pm\cosh^{-2} x$: (a) and (b) a potential bump: all the particles can go to infinity – the distribution function is defined uniquely. 1 – total energy of a free particle, 2 – particles with energy less than $q\Phi_{\max}$ are reflected; (c) and (d) a potential well: particles with a negative energy are trapped (3). Their distribution function depends on the temporal evolution, and their trajectories are closed. Dashed lines show the separatrix $p_s(x)$. Arrows show the direction of particle motion.

(1.1.21):

$$n(x) = n_0 \mathrm{e}^{-q\Phi(x)/T}, \qquad P_{xx} = n_0 T\, \mathrm{e}^{-q\Phi(x)/T} \tag{1.2.7}$$

where n_0 is the density of particles at infinity where the potential is zero. Formula (1.2.7) represents the Boltzmann distribution. For a positive potential, $q\Phi > 0$, the density and pressure of particles decrease where the potential increases. This is due to a partial reflection of particles from the potential bump (see Fig. 1.2(a)).

The case of a potential well or a negative potential has to be considered separately. As shown in Fig. 1.2(b), particles with negative energies are trapped and oscillate in the well. The distribution function of these particles is not defined uniquely. It depends on how the potential has been created. If, for example, we first created a potential in an empty space and after that launched particles from infinity, then there will be no particles with a negative energy, that is, $f = 0$ for $W < 0$, and no particles are trapped. However, some particles can be trapped if we proceed another way and create a potential in an already existing plasma.

We consider this latter case supposing that the potential is created slowly, and thus, the particles follow an adiabatic evolution in time. When the potential evolves, the energy of particles is not conserved. However, according to classical mechanics, another quantity is conserved: that is the *adiabatic invariant* – the integral of the particle momentum between the stopping points:

$$I = 2\int_{x_1}^{x_2} p_x(x,t)\,dx \equiv 2\int_{x_1}^{x_2} \sqrt{2m\,[W - q\,\Phi(x,t)]}\,dx \qquad (1.2.8)$$

where $q\Phi < 0$, the particle energy is negative, $W < 0$ and $W > q\Phi_{\rm min}$, the integral is taken between the turning points $x_{1,2}(t)$ where $q\Phi(x_{1,2}) = W$.

The distribution function of trapped particles must depend on I but not explicitly on the coordinate x. Moreover, $F(W)$ must be a continuous function of W, particularly for the energy $W = 0$, that separates the free and trapped particles. As W changes in time, the only possibility to satisfy this condition of continuity is to make $F(W < 0)$ constant: $f_{\rm tr} = F(W < 0) = F(0) = n_0(2\pi m\,T)^{-1/2}$. This choice corresponds to the maximum number of trapped particles. By contrast, the distribution function of free particles, $f_{\rm free}$, with $W > 0$, is a Maxwellian function. These two parts of the total distribution are connected at the separatrix $p_s(x)$, which is defined as $W(p_s) = 0$ or $p_s^2/2m + q\Phi(x) = 0$:

$$f(x,p_x) = \frac{n_0}{(2\pi m\,T)^{1/2}} \times \begin{cases} 1 & |p_x| < p_s \\ \exp\left(-\frac{p_x^2}{2m\,T} - \frac{q\,\Phi(x)}{T}\right) & |p_x| < p_s. \end{cases} \qquad (1.2.9)$$

The difference in the behavior of free and trapped particles can be better understood while looking at the orbits of particles in the phase space, (x, p_x), presented in Fig. 1.2, panels (c) and (d). In the case of a positive potential, the orbits of all the particles are connected to infinity. The distribution function is completely defined for these asymptotic values because they are conserved along the trajectory. In the case of a negative potential, the separatrix, $p_s(x)$, divides the phase space into free particles ($|p_x| > p_s$) and trapped particles ($|p_x| < p_s$). The period of oscillations of the trapped particles depends on their energy. Because of that, particles mix up with time, and the distribution function becomes constant, equal to its value along the separatrix, independently of the initial condition.

Using expression (1.2.9), we can calculate the density and pressure of particles:

$$n(x) = \frac{2}{\sqrt{\pi}} n_0 \sqrt{\frac{-q\,\Phi}{T}} + n_0 \,\mathrm{erfc}\left(\sqrt{\frac{-q\,\Phi}{T}}\right) \mathrm{e}^{-q\Phi/T}, \tag{1.2.10}$$

$$P_{xx}(x) = \frac{2}{\sqrt{\pi}} n_0 T \sqrt{\frac{-q\,\Phi}{T}} + \frac{4}{3\sqrt{\pi}} n_0 T \left(\frac{-q\,\Phi}{T}\right)^{3/2} + n_0 \,\mathrm{erfc}\left(\sqrt{\frac{-q\,\Phi}{T}}\right) \mathrm{e}^{-q\Phi/T}, \tag{1.2.11}$$

where $\mathrm{erfc}(\xi) = 1 - \mathrm{erf}(\xi)$ is the complementary error function. The error function is defined as $\mathrm{erf}(\xi) = (2/\sqrt{\pi}) \int_0^\xi \mathrm{e}^{-t^2} dt$. We can evaluate expressions (1.2.10) and (1.2.11) in two limits:

- For a weak potential, $|q\,\Phi| \ll T$, developing these expressions in Taylor series, we obtain:

$$n \approx n_0 \left[1 - \frac{q\,\Phi}{T} - \frac{4}{3\sqrt{\pi}} \left(\frac{-q\,\Phi}{T}\right)^{3/2}\right], \tag{1.2.12}$$

$$P_{xx} \approx n_0 T \left[1 - \frac{q\,\Phi}{T} + \frac{1}{2} \left(\frac{q\,\Phi}{T}\right)^2\right]. \tag{1.2.13}$$

  The difference from the Boltzmann distribution is in the last term in the expression for the density. The contribution of trapped particles is small.

- For a strong potential, $|q\,\Phi| \gg T$, the contribution of free particles is negligible, and we find:

$$n \approx \frac{2}{\sqrt{\pi}} n_0 \sqrt{\frac{-q\,\Phi}{T}}, \qquad P_{xx} \approx \frac{4}{3\sqrt{\pi}} n_0 T \left(\frac{-q\,\Phi}{T}\right)^{3/2}. \tag{1.2.14}$$

  In difference from the Boltzmann distribution, the density increases as a square root of the potential and the pressure as the potential in a power 3/2.

1.2.4 Solitons and collisionless shocks

Knowing the particle distribution in a prescribed potential, we can take one step forward and find self-consistent potential distributions in a collisionless plasma. Here, we consider two such structures: solitons and shocks. Soliton corresponds to a localized structure with a zero potential at infinity, which moves with a constant velocity u_0. Collisionless shock is a structure moving with a constant speed and where the potential increases from zero upstream of the shock to some higher value Φ_0 downstream.

Electrostatic soliton

Let us consider a soliton with an amplitude $\Phi_{\max} > 0$ propagating in a plasma from the right to the left with a velocity u_0. The plasma electrons are characterized by a charge $q_e = -e$, density Zn_0, and temperature T_e far from the soliton. The ions with a charge $q_i = Ze$ are considered as a cold fluid. Then, in the reference frame associated with the soliton, the ions move from the left to the right with a velocity u_0. Their velocity at the position x is defined by conservation of the total energy $W = m_i u_i^2/2 + Ze\Phi$ in the potential Φ:

$$u_i(x) = \sqrt{u_0^2 - 2Ze\Phi(x)/m_i}. \tag{1.2.15}$$

The ion density is related to the ion velocity by the condition of continuity $n_i(x) = n_0 u_0 / u_i(x)$. Electrons coming from the infinity are not trapped in a positive potential; see Fig. 1.2(c). However, electrons with negative total energies $W = m_e v_e^2/2 - e\Phi$ larger than $-e\Phi_{\max}$ can be trapped. Assuming that the potential well is filled with trapped particles, we use expression (1.2.10) for the electron density distribution. (We suppose that electron temperature is sufficiently high, so the electron thermal velocity is much larger than the ion flow velocity, $v_{\mathrm{Te}} \gg u_0$, and the latter can be neglected in the electron distribution function.) Then, the equation for the electric potential follows from the Poisson equation:

$$\begin{aligned} d_x^2\Phi &= -\frac{e}{\epsilon_0}(Zn_i - n_e) \\ &= \frac{Zen_0}{\epsilon_0}\left[2\sqrt{\frac{e\Phi}{\pi T_e}} + \mathrm{e}^{e\Phi/T_e}\mathrm{erfc}\left(\sqrt{\frac{e\Phi}{T_e}}\right) - \frac{u_0}{\sqrt{u_0^2 - 2Ze\Phi/m_i}}\right]. \end{aligned}$$

The coordinate x is not entered explicitly in this differential equation. It can be, therefore, integrated similarly as we integrated Eq. (1.2.5). Introducing a dimensionless electric potential $\varphi = e\Phi/T_e$, dimensionless coordinate $\xi = x/\lambda_{\mathrm{De}}$, where $\lambda_{\mathrm{De}} = (\epsilon_0 T_e / Ze^2 n_0)^{1/2}$ is the electron Debye length, and the Mach number $M_s = u_0/c_s$, where $c_s = (ZT_e/m_i)^{1/2}$ is the ion acoustic velocity, this equation reads:

$$\varphi'' = 2\sqrt{\frac{\varphi}{\pi}} + \mathrm{erfc}(\sqrt{\varphi})\,\mathrm{e}^{\varphi} - \frac{M_s}{\sqrt{M_s^2 - 2\varphi}} = -\partial_\varphi U. \tag{1.2.16}$$

The right-hand side of this equation represents a difference of electron and ion dimensionless densities. It can be considered as a derivative of the effective potential $U(\varphi)$.

By multiplying this equation by φ' and integrating it assuming that $\varphi(-\infty) = 0$ we have:

$$\frac{1}{2}(\varphi')^2 + U = \mathcal{E}. \tag{1.2.17}$$

Here, the effective energy $\mathcal{E}$ is a constant, the effective potential $U(\varphi)$ is defined as follows

$$U = M_s^2 - M_s\sqrt{M_s^2 - 2\varphi} + 1 - 2\sqrt{\frac{\varphi}{\pi}} - \frac{4}{3\sqrt{\pi}}\varphi^{3/2} - \mathrm{e}^{\varphi}\mathrm{erfc}(\sqrt{\varphi}), \qquad (1.2.18)$$

with a constant of integration $U(0)$ set to zero. The first two terms in U represent the ion contribution, and the four others describe the contribution of electrons. The second term represents a ratio of the ion ram pressure, $n_i m_i u_i^2$, to the ion thermal pressure $n_0 T_e$, and the last three terms represent the electron thermal pressure:

$$p_e = 2\sqrt{\frac{\varphi}{\pi}} + \frac{4}{3\sqrt{\pi}}\varphi^{3/2} + \mathrm{e}^{\varphi}\mathrm{erfc}(\sqrt{\varphi}). \qquad (1.2.19)$$

Consequently, effective potential U represents the plasma pressure acting on the soliton.

Equation (1.2.17) describes the energy conservation in a mechanical system of a particle of a unit mass moving with a velocity φ' in a potential U shown in Fig. 1.3(a). Mach number is the only free parameter in the problem. Since $U(0) = 0$, solutions of this equation exist only for $U \leq 0$. Solutions connecting to $\varphi = 0$ at $\xi \to \pm\infty$ correspond to the horizontal line $U = 0$. The point where the effective potential crosses abscissa defines the soliton amplitude: $U(\varphi_{\max}) = 0$. For a given M_s, only one such solution exists in a limited range of the Mach numbers, $1 < M_s < M_{\max} = 3.08$. For $M_s \leq 1$, the potential is positive, and there are no real solutions for $M_s > M_{\max}$. The maximum value of the Mach number is defined by the condition $M_{\max}^2 = 2\,\varphi_{\max}(M_{\max})$. As shown in Fig. 1.3(b), the soliton amplitude $\varphi_{\max}(M_s)$ increases from zero for $M_s = 1$ the value of 4.75 at $M_{\max} = 3.08$.

The solution shown in Fig. 1.3(a) with a red dashed line, $U = \mathrm{const} < 0$, corresponds to a periodic nonlinear wave of a final amplitude. The potential oscillates between the minimum and maximum value defined by the points where the dashed line crosses the potential curve. In particular, the minimum potential curve (shown with a red dot in the figure) corresponds to a solution with a constant density $n_e = n_i$ and a potential $\varphi_1(M_s)$.

The spatial shape of the soliton and the periodic waves can be obtained from Eq. (1.2.17) by plotting the inverse function $x(\varphi)$. The soliton solution is shown in Fig. 1.4 for three values of $M_s = 1.5$, 2.5 and 3.0. The potential (panel a) has a bell-like structure with amplitude of the order of electron temperature and width of the order of the Debye length. As the Mach number increases, the amplitude increases and the width decreases. Solitons do not carry a net charge: as shown in Fig. 1.3(b), the positive charge is concentrated near the tip of the soliton, surrounded by a negative