Recent Advances in Nonlinear Analysis and Optimization with Applications

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Preface

This book focuses on recent advances in nonlinear analysis and optimization with important applications drawn from various fields, such as artificial intelligence, genetic algorithms, optimization problems under uncertainty, and fuzzy logic. Specifically, it is devoted to nonlinear problems associated with optimization which have some connection with applications. The ideas and techniques developed here will serve to stimulate further research in this dynamic field, and, in this way, the book will become a valuable reference for researchers, engineers, and students in the field of mathematics, management science, operations research, optimal control science, and economics.

The book is structured into nine chapters. In **Chapter** 1, a class of E-differentiable E-invex multiobjective programming problems is considered. For E-differentiable multicriteria optimization problems, two E-Lagrange functions and their E-saddle points are defined. Also, the E-saddle point criteria are established for the considered E-differentiable multiobjective programming problems with both inequality and equality constraints under E-invexity hypotheses. In **Chapter** 2, a modified interval-valued variational control problem involving first-order partial differential equations (PDEs) and inequality constraints is investigated. Specifically, under some generalized convexity assumptions, LU-optimality conditions are formulated and proved for the considered interval-valued variational control problem. Also, to illustrate the main results and their effectiveness, an application is provided. Moreover, there are studied the connections between the LU-optimal solutions of the interval-valued variational control problem and the saddle-points associated with the interval-valued Lagrange functional corresponding to the modified interval-valued variational control problem. **Chapter** 3 presents a study on the partial differential equations (PDE) and partial differential inequations (PDI) constrained multi-time variational optimization problem (MVOP) of a curvilinear functional by converting it into an equivalent unconstrained multi-time variational optimization problem ($\text{MVOP}_{\infty}\rho$) with the help of the exact minimax penalty function method. Also, the saddle point criteria for (MVOP) and the relationships between a saddle point for (MVOP) and a minimizer of ($\text{MVOP}_{\infty}\rho$) are established under convexity assumption. Further, the theoretical results developed in this study are accompanied by suitable examples. **Chapter** 4, presents an adapted negative selection algorithm related to the security of swarm systems. In **Chapter** 5, the exactness property of the absolute value exact penalty function method used for solving a new class of nonconvex nonsmooth constrained optimization problems with both inequality and equality constraints is analyzed. The threshold of the penalty parameter is given such that, for all penalty parameters, there is the equivalence between the sets of optimal solutions for the nonsmooth constrained optimization problem and its associated penalized optimization problem with the absolute value exact penalty function as the objective function. This result is established for nonsmooth constrained optimization problems involving locally Lipschitz b-invex functions. **Chapter** 6 addresses a multi-objective optimization problem wherein all the objectives and constraints are interval-valued functions. Necessary and sufficient optimality conditions for the problem are established. Additionally, the weak and strong duality relationship between the primal and the corresponding dual problem is deliberated. Furthermore, counterexamples are provided to justify the theoretical developments in the paper. The main goal of **Chapter** 7 is to introduce strongly pseudomonotone and strongly quasimonotone maps of higher order in terms of set-valued maps. Solutions associated with the strong Minty type variational inequality are obtained with the help of these maps. Also, an existence theorem is established to obtain the solution for a given complementarity problem over a certain cone in which the underlying map is a strongly pseudomonotone map of a higher order. In **Chapter** 8, a Newton method is proposed to locate non-dominated solutions for a fuzzy optimization problem under the consideration of gH-differentiability, which extends and improves a recent existing method, which generalizes others existing in the recent literature. The efficiency of the considered Newton method is shown and illustrated through practical examples. **Chapter** 9 deals with a semi-infinite programming problem with real-valued Lipschitz continuous nonconvex nonsmooth objective function and an infinite number of inequality and equality constraints. Sufficient optimality conditions are derived for a feasible point under generalized convexity assumptions in terms of Michel-Penot subdifferentials. Also, Wolfe and Mond-Weir type dual models are formulated for the primal nonsmooth semi-infinite optimization problem and weak, strong, and strict converse duality results are established under generalized convexity assumptions.

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$\boldsymbol{E}$-saddle point criteria for a class of $\boldsymbol{E}$-differentiable $\boldsymbol{E}$-invex multiobjective programming problems

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Abstract. In this work, a class of E-differentiable E-invex multiobjective programming problems is considered. For E-differentiable multicriteria optimization problems, two E-Lagrange functions and their E-saddle points are defined. Then the E-saddle point criteria are established for the considered E-differentiable multiobjective programming problems with both inequality and equality constraints under E-invexity hypotheses.

Keywords: E-invex set; E-invex function; E-differentiable function; E-saddle point; E-Lagrange function; E-saddle point criteria.

1 Introduction

In this work, we consider the following (not necessarily differentiable) multiobjective programming problem (MOP) with both inequality and equality constraints:

$$\begin{aligned} &\text{minimize } f(x) = (f_1(x), ..., f_p(x)) \\ &\text{subject to } g_j(x) \leqq 0, \ \ j \in J = \{1, ..., m\}, \\ &\qquad h_t(x) = 0, \ \ t \in T = \{1, ..., q\}, \\ &\qquad x \in R^n, \end{aligned} \tag{MOP}$$

where $f_i : R^n \to R$, $i \in I = \{1, ..., p\}$, $g_j : R^n \to R$, $j \in J$, $h_t : R^n \to R$, $t \in T$, are real-valued functions defined on R^n. We shall write $g := (g_1, ..., g_m) : R^n \to R^m$ and $h := (h_1, ..., h_q) : R^n \to R^q$ for convenience. Let

$$\Omega := \{x \in X : g_j(x) \leqq 0, \ \ j \in J, \ h_t(x) = 0, \ \ t \in T\}$$

be the set of all feasible solutions of (MOP). Further, we denote by $J(x)$ the set of inequality constraint indices that are active at a feasible solution x, that is, $J(x) = \{j \in J : g_j(x) = 0\}$.

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in this work. For any vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \cdots, n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \cdots, n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, \cdots, n$;
(iv) $x \geq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, \cdots, n$ but $x \neq y$;

(v) $x \ngtr y$ is the negation of $x > y$.

For such multi-criterion optimization problems, the following concepts of a weak Pareto solution and a Pareto solution are defined as follows:

Definition 1. *A feasible point $\overline{x}$ is said to be a weak Pareto (weakly efficient) solution of (MOP) if and only if there is no other feasible point x such that*

$$f(x) < f(\overline{x}).$$

Definition 2. *A feasible point $\overline{x}$ is said to be a Pareto (efficient) solution of (MOP) if and only if there is no other feasible point x such that*

$$f(x) \leq f(\overline{x}).$$

In most real-life problems, decisions are made taking into account several conflicting criteria, rather than by optimizing a single objective. Such a problem is called a vector optimization problem (or a multiobjective programming problem). Many different approaches have been designed to characterize the solvability of such optimization problems. One of them is using saddle point criteria. There has been an increasing interest in developing saddle point criteria in optimization theory. Due to its wide application in vector optimization problems, saddle point criteria are investigated by many authors (see, for example, [4], [5], [6], [7], [8], [9], [10], [11], [13], [14], [17], [18], [19], [20], [21], [22], [23], [24] [26], and others).

One of the notions of generalized convexity introduced into optimization theory is the concept of E-convexity. The definitions of E-convex sets and E-convex functions were introduced by Youness [25]. This kind of generalized convexity is based on the effect of an operator $E : R^n \to R^n$ on the sets and the domain of functions. Megahed et al. [15] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator $E : R^n \to R^n$. Recently, Abdulaleem [1] introduced a new concept of generalized convexity as a generalization of the notion of E-differentiable E-convexity and the notion of differentiable invexity. Namely, he defined the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions.

In this work, the class of E-differentiable vector optimization problems with both inequality and equality constraints is considered. Namely, we extend the class of E-differentiable vector optimization problems introduced by Antczak and Abdulaleem [4] to a new class of E-differentiable vector optimization problems, that is, a new class of E-differentiable multiobjective programming problems under E-invexity hypotheses is wider than the class of E-differentiable multiobjective programming problems under E-convexity hypotheses. For such (not necessarily) differentiable multiobjective programming problems with E-differentiable functions, characterizations of their saddle points are presented. The so-called scalar and vector-valued E-Lagrange functions and their E-saddle points are defined for the considered E-differentiable vector optimization problem. Then, the E-saddle point criteria are established for E-differentiable multiobjective programming problems under E-invexity hypotheses. Hence, the equivalence between E-saddle points and (weak E-Pareto solutions) E-Pareto solutions are proved for the considered E-differentiable vector optimization problems with E-invex functions. In this way, tools of differentiable analysis are used in proving saddle point criteria for (not necessarily) differentiable vector optimization problems.

2 Preliminaries

The definition of an E-invex set and the definition of an E-invex function were introduced by Abdulaleem [1]. Now, we recall these definitions.

Definition 3. *Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set if and only if there exists a vector-valued function $\eta : M \times M \to R^n$ such that the relation*

$$E(u) + \lambda\eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Remark 1. If η is a vector-valued function defined by $\eta(z, y) = z - y$, then the definition of an E-invex set reduces to the definition of an E-convex set (see Youness [25]).

Remark 2. If $E(a) = a$, then the definition of an E-invex set with respect to the function η reduces to the definition of an invex set with respect to η (see Mohan and Neogy [16]).

We now give the definition of an E-differentiable function introduced by Megahed et al. [15].

Definition 4. *Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is*

$$(f \circ E)(x) = (f \circ E)(u) + \nabla(f \circ E)(u)(x - u) + \theta(u, x - u)\|x - u\|, \quad (1)$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Definition 5. *Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R$ be an E-differentiable function on M. It is said that f is an E-invex function with respect to η if, for all $x \in M$,*

$$f(E(x)) - f(E(u)) \geqq \nabla f(E(u))\eta(E(x), E(u)). \quad (2)$$

If inequality (2) holds for any $u \in M$, then f is E-invex with respect to η on M.

Remark 3. Taking into account Definition 5, we mention the following special cases:

a) If f is a differentiable function and $E(x) \equiv x$ (E is an identity map), then the definition of an E-invex function reduces to the definition of an invex function introduced by Hanson [12] in the scalar case.
b) If $\eta : M \times M \to R^n$ is defined by $\eta(x, u) = x - u$, then we obtain the definition of an E-differentiable E-convex function introduced by Megahed et al. [15].
c) If f is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$, then the definition of an E-invex function reduces to the definition of a differentiable convex function.
d) If f is E-differentiable and $\eta(x, u) = x - u$, then we obtain the definition of a differentiable E-convex function introduced by Youness [25].

Definition 6. *Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R$ be an E-differentiable function on M. It is said that f is a strictly E-invex function with respect to η if, for all $x \in M$ with $E(x) \neq E(u)$, the inequalities*

$$f(E(x)) - f(E(u)) > \nabla f(E(u))\eta(E(x), E(u)), \tag{3}$$

hold. If inequality (3) is fulfilled for any $u \in M$ $(E(x) \neq E(u))$, then f is strictly E-invex with respect to η on M.

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Throughout the work, we shall assume that the functions constituting the considered multiobjective programming problem (MOP) are E-differentiable at any feasible solution.

Now, for the considered multiobjective programming problem (MOP), we define its associated differentiable vector optimization problem as follows:

$$\begin{aligned} &\text{minimize } f(E(x)) = (f_1(E(x)), ..., f_p(E(x))) \\ &\text{subject to } g_j(E(x)) \leqq 0, \ \ j \in J = \{1, ..., m\}, \\ &\qquad h_t(E(x)) = 0, \ \ t \in T = \{1, ..., q\}, \\ &\qquad x \in R^n. \end{aligned} \qquad (\mathrm{VP}_E)$$

Let $\Omega_E := \{x \in R^n : g_j(E(x)) \leqq 0, \ j \in J, \, h_t(E(x)) = 0, \ t \in T\}$ be the set of all feasible solutions of (VP_E). Since the functions constituting the problem (MOP) are assumed to be E-differentiable at any feasible solution of (MOP), by Definition 4, the functions constituting the E-vector optimization problem (VP_E) are differentiable at any its feasible solution (in the usual sense). Further, we denote by $J_E(x)$ the set of inequality constraint indices that are active at a feasible solution $x \in \Omega_E$, that is, $J_E(x) = \{j \in J : (g_j \circ E)(x) = 0\}$.

Now, we give the definitions of a weak Pareto (a weakly efficient) solution and a Pareto (an efficient) solution of the vector optimization problem (VP_E), which are, at the same time, a weak E-Pareto solution (a weakly E-efficient solution) and an E-Pareto solution (an E-efficient solution) of the considered multiobjective programming problem (MOP).

Definition 7. *A feasible point $E(\overline{x})$ is said to be a weak E-Pareto solution (weakly E-efficient solution) of (MOP) if and only if there is no other feasible point $E(x)$ such that*

$$f(E(x)) < f(E(\overline{x})).$$

Definition 8. *A feasible point $E(\overline{x})$ is said to be an E-Pareto solution (E-efficient solution) of (MOP) if and only if there is no other feasible point $E(x)$ such that*

$$f(E(x)) \leq f(E(\overline{x})).$$

Lemma 1. *([3]) Let $E : R^n \to R^n$ be a one-to-one and onto. Then $E(\Omega_E) = \Omega$.*

Lemma 2. *([3]) Let $\overline{x} \in \Omega$ be a weak Pareto solution (a Pareto solution) of the considered multiobjective programming problem (MOP). Then, there exists $\overline{z} \in \Omega_E$ such that $\overline{x} = E(\overline{z})$ and $\overline{z}$ is a weak Pareto (a Pareto) solution of the E-vector optimization problem (VP_E).*

Lemma 3. *([3]) Let $\overline{z} \in \Omega_E$ be a weak Pareto (a Pareto) solution of the E-vector optimization problem (VP_E). Then $E(\overline{z})$ is a weak Pareto solution (a Pareto solution) of the considered multiobjective programming problem (MOP).*

Now, we give the Karush-Kuhn-Tucker necessary optimality conditions for the differentiable constrained E-vector optimization problem (VP$_E$) and, thus, the so-called E-Karush-Kuhn-Tucker necessary optimality conditions for not necessarily differentiable constrained multiobjective programming problem (MOP) in which the involved functions are E-differentiable established by Abdulaleem [1].

Theorem 1. *(E-Karush–Kuhn–Tucker necessary optimality conditions). Let $\overline{x} \in \Omega_E$ be a weak Pareto solution of the constrained E-vector optimization problem (VP$_E$) (and, thus, $E(\overline{x})$ be a weak E-Pareto solution of the considered constrained multiobjective programming problem (MOP)). Further, let f, g, h be E-differentiable at $\overline{x}$ and the E-Guignard constraint qualification [1] be satisfied at $\overline{x}$. Then there exist Lagrange multipliers $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^s$ such that*

$$\sum_{i=1}^{p} \overline{\lambda_i} \nabla \left(f_i \circ E\right)(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla \left(g_j \circ E\right)(\overline{x}) + \sum_{t=1}^{s} \overline{\xi}_t \nabla \left(h_t \circ E\right)(\overline{x}) = 0, \tag{4}$$

$$\overline{\mu}_j \left(g_j \circ E\right)(\overline{x}) = 0, \;\; j \in J\left(E\left(\overline{x}\right)\right), \tag{5}$$

$$\overline{\lambda} \geq 0, \, \overline{\mu} \geqq 0. \tag{6}$$

We now give the definitions of a Karush–Kuhn–Tucker point for the E-vector optimization problem (VP$_E$) and an E-Karush–Kuhn–Tucker point for the considered multiobjective programming problem (MOP).

Definition 9. $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p \times R^m \times R^s$ *is said to be a Karush–Kuhn–Tucker point for the E-vector optimization problem (VP$_E$) if the Karush–Kuhn–Tucker necessary optimality conditions (4)-(6) are satisfied at $\overline{x}$ with Lagrange multiplier $\overline{\lambda}$, $\overline{\mu}$, $\overline{\xi}$.*

Definition 10. *Let $\overline{x}$ be a feasible solution of the E-vector optimization problem (VP$_E$). Then $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p \times R^m \times R^s$ is said to be an E-Karush–Kuhn–Tucker point for the considered multiobjective programming problem (MOP) if the E-Karush–Kuhn–Tucker necessary optimality conditions (4)-(6) are satisfied at $\overline{x}$ with Lagrange multiplier $\overline{\lambda}$, $\overline{\mu}$, $\overline{\xi}$.*

3 Scalar *E*-saddle point criteria

For the E-vector optimization problem (VP$_E$), we give the definition of the scalar Lagrange function $L_E : \Omega_E \times R^p_+ \times R^m_+ \times R^s \to R$ as follows

$$L_E(x, \lambda, \mu, \xi) := \sum_{i=1}^{p} \lambda_i f_i(E(x)) + \sum_{j=1}^{m} \mu_j g_j(E(x)) + \sum_{t=1}^{s} \xi_t h_t(E(x)). \tag{7}$$

Then, we give the definition of a saddle point of the Lagrange function L_E defined for the problem (VP$_E$).

Definition 11. *A point $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p_+ \times R^m_+ \times R^s$ is said to be a saddle point of the E-vector optimization problem (VP$_E$) if*

1) $L_E\left(\overline{x}, \overline{\lambda}, \mu, \xi\right) \leqq L_E\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \;\; \forall \mu \in R^m_+, \;\; \forall \xi \in R^s$,
2) $L_E\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \leqq L_E\left(x, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \;\; \forall x \in \Omega_E$.

Now, for the considered multiobjective programming problem (MOP), we define its scalar Lagrange function $L : \Omega \times R^p_+ \times R^m_+ \times R^s \to R$ as follows

$$L(z, \lambda, \mu, \xi) := \sum_{i=1}^{p} \lambda_i f_i(z) + \sum_{j=1}^{m} \mu_j g_j(z) + \sum_{t=1}^{s} \xi_t h_t(z). \tag{8}$$

For the scalar Lagrange function L defined above, we now give the definition of its E-saddle point.

Definition 12. *Let $\overline{x}$ be a feasible solution of the E-vector optimization problem (VP$_E$). A point $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p_+ \times R^m_+ \times R^s$ is said to be an E-saddle point for the considered multiobjective programming problem (MOP) if*

i) $L\left(E\left(\overline{x}\right), \overline{\lambda}, \mu, \xi\right) \leqq L\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \quad \forall \mu \in R^m_+,\ \forall \xi \in R^s,$
ii) $L\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \leqq L\left(z, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \quad \forall z \in \Omega.$

Now, under E-differentiable E-invexity assumptions, we prove the sufficient condition for a saddle point of the Lagrange function L_E defined for the E-vector optimization problem (VP$_E$) which we use in proving the sufficient condition for an E-saddle point of the Lagrange function L defined for the original multicriteria optimization problem (MOP).

Theorem 2. *Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p_+ \times R^m_+ \times R^s$ be a Karush–Kuhn–Tucker point of the E-vector optimization problem (VP$_E$). Further, assume that f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+\left(\overline{x}\right) = \{t \in T : \overline{\xi}_t > 0\}$ and $-h_t$, $t \in T^-\left(\overline{x}\right) = \{t \in T : \overline{\xi}_t < 0\}$ are E-invex at $\overline{x}$ on Ω_E with respect to the same function η. Then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a saddle point of the Lagrange function L_E defined for the E-vector optimization problem (VP$_E$).*

Proof. First, we prove the inequality i) in Definition 11. By assumption, we have $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p_+ \times R^m_+ \times R^s$ is a Karush–Kuhn–Tucker point for the E-vector optimization problem (VP$_E$). Hence, by $\overline{x} \in \Omega_E$, it follows that the relations

$$\xi_t h_t(E(\overline{x})) = \overline{\xi}_t h_t(E(\overline{x})), \quad t \in T \tag{9}$$

hold for all $\xi = (\xi_1, ..., \xi_s) \in R^s$. Using again the feasibility of $\overline{x}$ in the problem (VP$_E$) together with the E-Karush-Kuhn-Tucker necessary optimality condition (5), we obtain that the inequalities

$$\mu_j g_j(E(\overline{x})) \leqq \overline{\mu}_j g_j(E(\overline{x})), \quad j \in J \tag{10}$$

hold for all $\mu = (\mu_1, ..., \mu_m) \in R^m_+$. Adding both sides of (9) and (10), it follows that the inequality

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \mu_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \xi_t h_t(E(\overline{x})) \leqq$$

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(\overline{x}))$$

holds for all $\mu \in R^m_+$ and for all $\xi \in R^s$. Then, by the definition of the Lagrange function L_E (see (8)), the inequality

$$L_E\left(\overline{x}, \overline{\lambda}, \mu, \xi\right) \leqq L_E\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \tag{11}$$

holds for all $\mu \in R_+^m$ and $\xi \in R^s$. This means that the inequality i) in Definition 11 is fulfilled.

Now, we prove the second inequality in Definition 11. By assumption, f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+(\overline{x})$, $-h_t$, $t \in T^-(\overline{x})$, are E-invex with respect to η on Ω_E. Then, by Definition 5, the following inequalities

$$f_i(E(x)) - f_i(E(\overline{x})) \geqq \nabla f_i(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ i \in I, \tag{12}$$

$$g_j(E(x)) - g_j(E(\overline{x})) \geqq \nabla g_j(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ j \in J(E(\overline{x})), \tag{13}$$

$$h_t(E(x)) - h_t(E(\overline{x})) \geqq \nabla h_t(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{14}$$

$$-h_t(E(x)) + h_t(E(\overline{x})) \geqq -\nabla h_t(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ \ t \in T^-(E(\overline{x})) \tag{15}$$

hold, respectively, for all $x \in \Omega_E$. Multiplying each inequality (12)-(15) by the corresponding Lagrange multipliers, respectively, we obtain

$$\overline{\lambda}_i f_i(E(x)) - \overline{\lambda}_i f_i(E(\overline{x})) \geqq \overline{\lambda}_i \nabla f_i(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ i \in I, \tag{16}$$

$$\overline{\mu}_j g_j(E(x)) - \overline{\mu}_j g_j(E(\overline{x})) \geqq \overline{\mu}_j \nabla g_j(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ j \in J(E(\overline{x})), \tag{17}$$

$$\overline{\xi}_t h_t(E(x)) - \overline{\xi}_t h_t(E(\overline{x})) \geqq \overline{\xi}_t \nabla h_t(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{18}$$

$$\overline{\xi}_t h_t(E(x)) - \overline{\xi}_t h_t(E(\overline{x})) \geqq \overline{\xi}_t \nabla h_t(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ \ t \in T^-(E(\overline{x})). \tag{19}$$

Then adding both sides of each inequality (16)-(19), we get that the inequality

$$\begin{aligned}&\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(x)) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(x)) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(x)) \\ &- \sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) - \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) - \sum_{t=1}^{s} \overline{\xi}_t h_t(E(\overline{x})) \\ &\geqq \left[\overline{\lambda}_i \nabla f_i(E(\overline{x})) + \overline{\mu}_j \nabla g_j(E(\overline{x})) + \overline{\xi}_t \nabla h_t(E(\overline{x}))\right] \eta(E(x), E(\overline{x}))\end{aligned} \tag{20}$$

By (20) and the E-Karush-Kuhn-Tucker necessary optimality condition (4), it follows that

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(x)) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(x)) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(x)) \geqq$$
$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(\overline{x}))$$

holds for all $x \in \Omega_E$ Then, by the definition of Lagrange function L_E, the inequality

$$L_E\left(x, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \geqq L_E\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \tag{21}$$

holds for all $x \in \Omega_E$. Hence, by (11) and (21), it follows that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a saddle point of the Lagrange function L_E defined for the E-vector optimization problem (VP$_E$). Thus, the conclusion of this theorem is established.

From Theorem 2, it follows directly the following result:

Corollary 1. *Let $\overline{x} \in \Omega_E$ be a (weak) Pareto optimal solution of the E-vector optimization problem (VP$_E$). Further, assume that all hypotheses of Theorem 2 are satisfied. Then, there exist Lagrange multipliers $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, and $\overline{\xi} \in R^s$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a saddle point of the scalar Lagrange function L_E defined for the problem (VP$_E$).*

By using Theorem 2 and Corollary 1, we now prove the sufficient condition for an E-saddle point of the Lagrange function L defined for the original multicriteria optimization problem (MOP).

Theorem 3. *Let $\overline{x}$ be a feasible solution of the E-vector optimization problem (VP$_E$) such that $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p_+ \times R^m_+ \times R^s$ is an E-Karush-Kuhn-Tucker point of the considered multiobjective programming problem (MOP). Further, assume that f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+\left(\overline{x}\right)$ and $-h_t$, $t \in T^-\left(\overline{x}\right)$ are E-invex at $\overline{x}$ on Ω_E with respect to the same function η. Then $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p_+ \times R^m_+ \times R^s$ is an E-saddle point of the Lagrange function L defined for the multiobjective programming problem (MOP).*

Proof. By assumption, $\overline{x}$ is a feasible solution of the E-vector optimization problem (VP$_E$) such that $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p_+ \times R^m_+ \times R^s$ is an E-Karush-Kuhn-Tucker point of the considered multiobjective programming problem (MOP). Then, by Definition 10, the E-Karush-Kuhn-Tucker necessary optimality conditions are fulfilled for the problem (MOP) with Lagrange multipliers $\overline{\lambda}, \overline{\mu}, \overline{\xi}$. Since all hypotheses of Theorem 2 are fulfilled, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p_+ \times R^m_+ \times R^s$ is a saddle point of the Lagrange function L_E defined for the E-vector optimization problem (VP$_E$). This means, by Definition 11, that the following conditions

$$L_E\left(\overline{x}, \overline{\lambda}, \mu, \xi\right) \leqq L_E\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \ \forall \mu \in R^m_+, \ \forall \xi \in R^s,$$

$$L_E\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \leqq L_E\left(x, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \ \forall x \in \Omega_E$$

hold. By the definition of the Lagrange function L_E, it follows that the inequalities

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \mu_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \xi_t h_t(E(\overline{x})) \leqq$$

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(\overline{x})) \ \forall \mu \in R^m_+, \ \forall \xi \in R^s \tag{22}$$

and

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(\overline{x})) \leqq$$

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(x)) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(x)) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(x)) \quad \forall x \in \Omega_E \tag{23}$$

hold. Since $E : R^n \to R^n$ is an one-to-one and onto operator, this means that, for any $x \in \Omega_E$, there exists $z \in \Omega$, such that $z = E\left(x\right)$. Hence, (22) and (23) yield, respectively,

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \mu_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \xi_t h_t(E(\overline{x})) \leqq$$

$$\sum_{i=1}^{p}\overline{\lambda}_i f_i(E(\overline{x}))+\sum_{j=1}^{m}\overline{\mu}_j g_j(E(\overline{x}))+\sum_{t=1}^{s}\overline{\xi}_t h_t(E(\overline{x}))\ \forall \mu\in R_+^m,\ \forall \xi\in R^s \tag{24}$$

and

$$\sum_{i=1}^{p}\overline{\lambda}_i f_i(E(\overline{x}))+\sum_{j=1}^{m}\overline{\mu}_j g_j(E(\overline{x}))+\sum_{t=1}^{s}\overline{\xi}_t h_t(E(\overline{x}))\leqq$$

$$\sum_{i=1}^{p}\overline{\lambda}_i f_i\left(z\right)+\sum_{j=1}^{m}\overline{\mu}_j g_j\left(z\right)+\sum_{t=1}^{s}\overline{\xi}_t h_t\left(z\right)\quad \forall z\in \Omega. \tag{25}$$

By Definition 12, (24) and (25) imply that $\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\in \Omega\times R_+^p\times R_+^m\times R^s$ is an E-saddle point of the Lagrange function L defined for the multiobjective programming problem (MOP).

As it follows from the above proof, the definition of an E-saddle point of the Lagrange function L can be formulated as follows:

Definition 13. *Let $\overline{x}$ be a feasible solution of the E-vector optimization problem (VP$_E$). A point $\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\in \Omega\times R_+^p\times R_+^m\times R^s$ is said to be an E-saddle point for the considered multiobjective programming problem (MOP) if*

i) $L\left(E\left(\overline{x}\right),\overline{\lambda},\mu,\xi\right)\leqq L\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\quad \forall \mu\in R_+^m,\ \forall \xi\in R^s,$
ii) $L\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\leqq L\left(E\left(x\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\quad \forall E\left(x\right)\in \Omega.$

In order to illustrate the E-saddle point criteria established for the Lagrange function defined for the considered multiobjective programming problem (MOP), we give the example of E-differentiable vector optimization problem with E-invex functions.

Example 1. Consider the following nonconvex nondifferentiable vector optimization problem

$$\begin{aligned} \text{minimize } f\left(x_1,x_2\right)&=(e^{\sqrt[3]{x_2}}+\sqrt[3]{x_1}\ ,\ e^{\sqrt[3]{x_1}}+\sqrt[3]{x_2})\\ \text{s.t. } g\left(x_1,x_2\right)&=e^{\sqrt[3]{x_2}}-e^{\sqrt[3]{x_1}}\leqq 0, \qquad\qquad \text{(MOP1)}\\ h\left(x_1,x_2\right)&=e^{\sqrt[3]{x_2}}-e^{\sqrt[3]{x_1^2}}=0. \end{aligned}$$

Note that $\Omega=\left\{(x_1,x_2)\in R^2: e^{\sqrt[3]{x_2}}-e^{\sqrt[3]{x_1}}\leqq 0\wedge e^{\sqrt[3]{x_2}}-e^{\sqrt[3]{x_1^2}}=0\right\}$. Let $E:R^2\to R^2$ be an one-to-one and onto mapping defined by $E\left(x_1,x_2\right)=\left(x_1^3,x_2^3\right)$. Now, for the considered nonconvex nondifferentiable multiobjective programming problem (MOP1), we define its associated E-vector optimization problem (VP1$_E$) as follows:

$$\begin{aligned} \text{minimize } f(E\left(x_1,x_2\right))&=(e^{x_2}+x_1\ ,\ e^{x_1}+x_2)\\ \text{s.t. } g(E\left(x_1,x_2\right))&=e^{x_2}-e^{x_1}\leqq 0, \qquad\qquad (\text{VP1}_E)\\ h(E\left(x_1,x_2\right))&=e^{x_2}-e^{x_1^2}=0. \end{aligned}$$

Note that $\Omega_E=\left\{(x_1,x_2)\in R^2: e^{x_2}-e^{x_1}\leqq 0\wedge e^{x_2}-e^{x_1^2}=0\right\}$ and $\overline{x}=(0,0)$ is a feasible solution of the problem (VP1$_E$). Let η be defined by $\eta(E(x),E(\overline{x}))=(e^{x_1},-e^{x_2})$. Further, note that all functions constituting the considered vector optimization problem (MOP1) are

E-differentiable at $\overline{x} = (0,0)$. Then, it can also be shown that the E-Karush–Kuhn–Tucker necessary optimality conditions (4)-(6) are fulfilled at $\overline{x} = (0,0)$ with Lagrange multipliers $\overline{\lambda}_1 = \frac{2}{3}$, $\overline{\lambda}_2 = \frac{1}{3}$, $\overline{\mu} = 1$ and $\overline{\xi} = 0$. Since all hypotheses of Theorem 2 are fulfilled, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a saddle point of the Lagrange function L_E defined for the E-vector optimization problem (VP1$_E$). Furthermore, by Theorem 3, it follows that $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is an E-saddle point of the Lagrange function L of the considered multiobjective programming problem (MOP1). Moreover, it is not possible to use the saddle point criteria from [4] since that the functions constituting (VP1$_E$) are not E-convex at $\overline{x}$ on Ω_E. However, it is possible to use E-saddle point criteria established in the paper since that the functions constituting (VP1$_E$) are E-invex at $\overline{x}$ on Ω_E with respect to η.

4 Vector *E*-saddle point criteria

We now, give a definition of the vector-valued Lagrange function $L_{p_E} : \Omega_E \times R^p \times R^m \times R^s \to R^p$ for the constrained E-vector optimization problem (VP$_E$), as follows:

$$L_{p_E}(x, \lambda, \mu, \xi) := diag\, \lambda f(E(x)) + \frac{1}{p}\left[\sum_{j=1}^{m} \mu_j g_j(E(x)) + \sum_{t=1}^{s} \xi_t h_t(E(x))\right] e \tag{26}$$

where $e = [1, ..., 1] \in R^p$ and, moreover

$$diag\, \lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_p \end{bmatrix}.$$

Now, we give the definition of a saddle point of the vector-valued Lagrange function defined for the problem (VP$_E$).

Definition 14. *A point* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p_+ \times R^m_+ \times R^s$ *is said to be a saddle point of the vector-valued Lagrange function* L_{p_E} *defined in the E-vector optimization problem (VP$_E$) if,*

a) $L_{p_E}\left(\overline{x}, \overline{\lambda}, \mu, \xi\right) \leqq L_{p_E}\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \quad \forall \mu \in R^m_+, \quad \forall \xi \in R^s,$
b) $L_{p_E}\left(x, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \nleq L_{p_E}\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \quad \forall x \in \Omega_E.$

For the considered multiobjective programming problem (MOP), we define its vector-valued Lagrange function $L_p : \Omega \times R^p_+ \times R^m_+ \times R^s \to R$ as follows

$$L_p(z, \lambda, \mu, \xi) := diag\, \lambda f(z) + \frac{1}{p}\left[\sum_{j=1}^{m} \mu_j g_j(z) + \sum_{t=1}^{s} \xi_t h_t(z)\right] e, \tag{27}$$

For the vector-valued Lagrange function L_p defined above, we now give the definition of its vector E-saddle point.

Definition 15. *Let* $\overline{x}$ *be a feasible solution of the E-vector optimization problem (VP$_E$). A point* $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p_+ \times R^m_+ \times R^s$ *is said to be a saddle point for the considered multiobjective programming problem (MOP) if* *i)* $L_p\left(E\left(\overline{x}\right),\overline{\lambda},\mu,\xi\right) \leqq L_p\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right) \quad \forall \mu \in R^m_+, \forall \xi \in R^s,$

ii) $L_p\left(z,\overline{\lambda},\overline{\mu},\overline{\xi}\right) \nleq L_p\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right) \quad \forall z \in \Omega.$

Now, we prove the sufficient condition for $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right) \in \Omega_E \times R^p \times R^m \times R^s$ to be a vector saddle point of the Lagrange function.

Theorem 4. *Let $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right) \in \Omega_E \times R^p \times R^m \times R^s$ be a Karush–Kuhn–Tucker point of the Lagrange function for the considered E-vector optimization problem (VP$_E$). Further, assume that f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+\left(\overline{x}\right)$ and $-h_t$, $t \in T^-\left(\overline{x}\right)$ are E-invex at $\overline{x}$ on Ω_E with respect to the same function η. Then $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a saddle point of the vector-valued Lagrange function L_{p_E} defined for the E-vector optimization problem (VP$_E$).*

Proof. First, we prove the inequality a) in Definition 14. By assumption, we have $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right) \in \Omega_E \times R^p \times R^m \times R^s$ is a Karush-Kuhn-Tucker point of the Lagrange function L_{p_E} defined for the E-vector optimization problem (VP$_E$). Hence, from the feasibility of $\overline{x}$, it follows that the inequalities

$$\xi_t h_t(E(\overline{x})) = \overline{\xi}_t h_t(E(\overline{x})), \quad t \in T \tag{28}$$

hold for all $\xi = (\xi_1, ..., \xi_s) \in R^s$. Using the feasibility of $\overline{x} \in \Omega_E$ in the problem (VP$_E$) together with E-Karush-Kuhn-Tucker necessary optimality condition (5), we obtain that the inequalities

$$\mu_j g_j(E(\overline{x})) \leqq \overline{\mu}_j g_j(E(\overline{x})), \quad j \in J \tag{29}$$

hold for all $\mu = (\mu_1, ..., \mu_m) \in R^m_+$. Combining (28) and (29), we get that the inequality

$$diag\,\overline{\lambda} f(E(\overline{x})) + \frac{1}{p}\left[\sum_{j=1}^{m}\mu_j g_j(E(\overline{x})) + \sum_{t=1}^{s}\xi_t h_t(E(\overline{x}))\right]e$$
$$\leq diag\,\overline{\lambda} f(E(\overline{x})) + \frac{1}{p}\left[\sum_{j=1}^{m}\overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{s}\overline{\xi}_t h_t(E(\overline{x}))\right]e$$

holds for all $\mu \in R^m_+$ and $\xi \in R^s$. Then, by the definition of the Lagrange function L_{p_E} (see (26), the inequality

$$L_{p_E}\left(\overline{x},\overline{\lambda},\mu,\xi\right) \leqq L_{p_E}\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right) \tag{30}$$

holds for all $\mu \in R^m_+$ and $\xi \in R^s$.

We now prove the second inequality in Definition 14. We proceed by contradiction. Suppose, contrary to the result, that there exists $\tilde{x} \in \Omega_E$ such that $L_{p_E}\left(\tilde{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right) \leq L_{p_E}\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$. Then, by the definition of the vector-valued Lagrange function L_{p_E} (see (26)), it follows that

$$\sum_{i=1}^{k}\overline{\lambda}_i f_i(E(\tilde{x})) + \sum_{j=1}^{m}\overline{\mu}_j g_j(E(\tilde{x})) + \sum_{t=1}^{s}\overline{\xi}_t h_t(E(\tilde{x}))$$
$$< \sum_{i=1}^{k}\overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m}\overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{s}\overline{\xi}_t h_t(E(\overline{x})). \tag{31}$$

By assumption, f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+$, $-h_t$, $t \in T^-$, are E-invex with respect to η at $\overline{x}$ on Ω_E. Then, by Definition 5, the following inequalities

$$f_i\left(E\left(\widetilde{x}\right)\right) - f_i\left(E\left(\overline{x}\right)\right) \geqq \nabla f_i\left(E\left(\overline{x}\right)\right)\eta\left(E\left(\widetilde{x}\right), E\left(\overline{x}\right)\right), \, i \in I, \tag{32}$$

$$g_j(E(\widetilde{x})) - g_j(E(\overline{x})) \geqq \nabla g_j(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ j \in J(E(\overline{x})), \tag{33}$$

$$h_t(E(\widetilde{x})) - h_t(E(\overline{x})) \geqq \nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{34}$$

$$-h_t(E(\widetilde{x})) + h_t(E(\overline{x})) \geqq -\nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ \ t \in T^-(E(\overline{x})) \tag{35}$$

hold, respectively. Multiplying inequalities (32)-(35) by the corresponding Lagrange multipliers, respectively, we obtain

$$\overline{\lambda}_i f_i(E(\widetilde{x})) - \overline{\lambda}_i f_i(E(\overline{x})) \geqq \overline{\lambda}_i \nabla f_i(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ i \in I, \tag{36}$$

$$\overline{\mu}_j g_j(E(\widetilde{x})) - \overline{\mu}_j g_j(E(\overline{x})) \geqq \overline{\mu}_j \nabla g_j(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ j \in J(E(\overline{x})), \tag{37}$$

$$\overline{\xi}_t h_t(E(\widetilde{x})) - \overline{\xi}_t h_t(E(\overline{x})) \geqq \overline{\xi}_t \nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{38}$$

$$\overline{\xi}_t h_t(E(\widetilde{x})) - \overline{\xi}_t h_t(E(\overline{x})) \geqq \overline{\xi}_t \nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ \ t \in T^-(E(\overline{x})). \tag{39}$$

Combining (36)-(39), we get

$$\sum_{i=1}^{k} \overline{\lambda}_i (f_i \circ E)(\tilde{x}) + \sum_{j\in J(E(\overline{x}))} \overline{\mu}_j (g_j \circ E)(\tilde{x}) + \sum_{t\in T^+(E(\overline{x}))}^{s} \overline{\xi}_t (h_t \circ E)(\tilde{x}) +$$

$$\sum_{t\in T^-(E(\overline{x}))}^{s} \overline{\xi}_t (h_t \circ E)(\tilde{x}) - \left[\sum_{i=1}^{k} \overline{\lambda}_i (f_i \circ E)(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j (g_j \circ E)(\overline{x}) + \right.$$

$$\left. \sum_{t\in T^+(E(\overline{x}))}^{s} \overline{\xi}_t (h_t \circ E)(\overline{x}) + \sum_{t\in T^-(E(\overline{x}))}^{s} \overline{\xi}_t (h_t \circ E)(\overline{x}) \right]$$

$$\geqq \left[\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(E(\overline{x})) + \sum_{j\in J(E(\overline{x}))} \overline{\mu}_j \nabla g_j(E(\overline{x})) + \sum_{t\in T^+(E(\overline{x}))}^{s} \overline{\xi}_t \nabla h_t(E(\overline{x})) \right.$$

$$\left. + \sum_{t\in T^-(E(\overline{x}))}^{s} \overline{\xi}_t \nabla h_t(E(\overline{x})) \right] \eta(E(\widetilde{x}), E(\overline{x})).$$

By the E-Karush-Kuhn-Tucker necessary optimality condition (4) and taking Lagrange multipliers $\overline{\mu}_j$, $j \notin J(\overline{x})$, and $\overline{\xi}_t$, $t \notin T^+(\overline{x}) \cup T^-(\overline{x})$, we get that the inequality

$$\sum_{i=1}^{k} \overline{\lambda}_i f_i(E(\tilde{x})) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\tilde{x})) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(\tilde{x}))$$

$$\geqq \sum_{i=1}^{k} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(\overline{x})). \tag{40}$$

holds, contradicting (31). Thus, the proof of this theorem is completed.

From Theorem 4, it follows directly the following result:

Corollary 2. *Let $\overline{x} \in \Omega_E$ be a (weak) Pareto solution of the E-vector optimization problem (VP$_E$) and the Karush–Kuhn–Tucker necessary optimality conditions (4)-(6) be satisfied with Lagrange multiplier $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, and $\overline{\xi} \in R^s$. Further, assume that all hypotheses of Theorem 4 are satisfied. Then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a vector saddle point of the vector-valued Lagrange function L_{p_E} defined for the E-vector optimization problem (VP$_E$).*

Using the above results established for the E-vector optimization problem (VP$_E$), now under E-differentiable E-invexity assumptions, we prove the sufficient condition for $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p \times R^m \times R^s$ to be a vector E-saddle point of the Lagrange function L_p defined for the problem (MOP).

Theorem 5. *Let $\overline{x}$ be a feasible solution of the E-vector optimization problem (VP$_E$) such that $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p_+ \times R^m_+ \times R^s$ is an E-Karush–Kuhn–Tucker point of the considered multiobjective programming problem (MOP. Further, assume that f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+\left(\overline{x}\right)$ and $-h_t$, $t \in T^-\left(\overline{x}\right)$ are E-invex at $\overline{x}$ on Ω_E with respect to the same function η. Then $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p_+ \times R^m_+ \times R^s$ is a vector E-saddle point of the Lagrange function L_p defined for the multiobjective programming problem (MOP).*

Proof. By assumption, $\overline{x}$ is a feasible solution of the E-vector optimization problem (VP$_E$) such that $\left(E\left(\overline{x}\right), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p_+ \times R^m_+ \times R^s$ is an E–Karush–Kuhn-Tucker point of the considered multiobjective programming problem (MOP). Then, by Definition 10, the E-Karush–Kuhn–Tucker necessary optimality conditions are fulfilled for the problem (MOP) with Lagrange multipliers $\overline{\lambda}, \overline{\mu}, \overline{\xi}$. Since all hypotheses of Theorem 4 are fulfilled, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p_+ \times R^m_+ \times R^s$ is a vector saddle point of the Lagrange function L_E defined for the E-vector optimization problem (VP$_E$). This means, by Definition 14, that the following conditions

$$L_{p_E}\left(\overline{x}, \overline{\lambda}, \mu, \xi\right) \leqq L_{p_E}\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \;\; \forall \mu \in R^m_+, \; \forall \xi \in R^s, \tag{41}$$

$$L_{p_E}\left(x, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \nleq L_{p_E}\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \;\; \forall x \in \Omega_E \tag{42}$$

hold. By the definition of the Lagrange function L_{p_E}, (41) and (42) yield, respectively,

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \mu_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \xi_t h_t(E(\overline{x})) \leqq$$

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(\overline{x})) \; \forall \mu \in R^m_+, \; \forall \xi \in R^s \tag{43}$$

and

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(x)) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(x)) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(x)) \nleq$$

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t h_t(E(\overline{x})) \quad \forall x \in \Omega_E. \tag{44}$$

Since $E : R^n \to R^n$ is an one-to-one and onto operator, this means that, for any $x \in \Omega_E$, there exists $z \in \Omega$, such that $z = E\left(x\right)$. Hence, (22) and (23) yield, respectively,

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m} \mu_j g_j(E(\overline{x})) + \sum_{t=1}^{s} \xi_t h_t(E(\overline{x})) \leqq$$

$$\sum_{i=1}^{p}\overline{\lambda}_i f_i(E(\overline{x}))+\sum_{j=1}^{m}\overline{\mu}_j g_j(E(\overline{x}))+\sum_{t=1}^{s}\overline{\xi}_t h_t(E(\overline{x}))\ \forall\mu\in R_+^m,\ \forall\xi\in R^s \tag{45}$$

and

$$\sum_{i=1}^{p}\overline{\lambda}_i f_i\left(z\right)+\sum_{j=1}^{m}\overline{\mu}_j g_j\left(z\right)+\sum_{t=1}^{s}\overline{\xi}_t h_t\left(z\right)\nleq$$
$$\sum_{i=1}^{p}\overline{\lambda}_i f_i(E(\overline{x}))+\sum_{j=1}^{m}\overline{\mu}_j g_j(E(\overline{x}))+\sum_{t=1}^{s}\overline{\xi}_t h_t(E(\overline{x}))\quad\forall z\in\Omega. \tag{46}$$

By Definition 15, (45) and (46) imply that $\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\in\Omega\times R_+^p\times R_+^m\times R^s$ is a vector E-saddle point of the vector-valued Lagrange function L_p defined for the considered multiobjective programming problem (MOP).

From Theorem 5, it follows directly the following result:

Corollary 3. *Let $\overline{x}\in\Omega$ be a (weak) E-Pareto solution of the E-vector optimization problem (VP$_E$) and, thus, $E\left(\overline{x}\right)$ be a (weak) E-Pareto solution of the considered multiobjective programming (MOP). Further, assume that the E-Karush–Kuhn–Tucker necessary optimality conditions (4)-(6) are satisfied at $E\left(\overline{x}\right)$ with Lagrange multiplier $\overline{\lambda}\in R^p$, $\overline{\mu}\in R^m$, and $\overline{\xi}\in R^s$. If all hypotheses of Theorem 4 are satisfied, then $\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a vector E-saddle point of the vector-valued Lagrange function L_p defined for the multiobjective programming problem (MOP).*

As it follows from the above proofs, the definition of a vector E-saddle point of the vector-valued Lagrange function L_p can be formulated as follows:

Definition 16. *Let $\overline{x}$ be a feasible solution of the E-vector optimization problem (VP$_E$). A point $\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\in\Omega\times R_+^p\times R_+^m\times R^s$ is said to be an E-saddle point for the considered multiobjective programming problem (MOP) if*

i) $L_p\left(E\left(\overline{x}\right),\overline{\lambda},\mu,\xi\right)\leqq L_p\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\quad\forall\mu\in R_+^m,\ \forall\xi\in R^s,$
ii) $L_p\left(E\left(x\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\nleq L_p\left(E\left(\overline{x}\right),\overline{\lambda},\overline{\mu},\overline{\xi}\right)\quad\forall E\left(x\right)\in\Omega.$

Example 2. Consider the following nondifferentiable vector optimization problem

$$f\left(x_1,x_2\right)=\left(\sqrt[3]{x_1^2}-\sqrt[3]{x_1}\ ,\ e^{\sqrt[3]{x_2}}-\sqrt[3]{x_1}\right)\to\min$$
$$g\left(x_1,x_2\right)=1-e^{\sqrt[3]{x_2}}\leqq 0, \qquad \text{(MOP2)}$$
$$h\left(x_1,x_2\right)=e^{\sqrt[3]{x_1}}-e^{\sqrt[3]{x_2}}=0.$$

Note that $\Omega=\left\{\left(x_1,x_2\right)\in R^2:1-e^{\sqrt[3]{x_2}}\leqq 0\wedge e^{\sqrt[3]{x_1}}-e^{\sqrt[3]{x_2}}=0\right\}$. Let $E:R^2\to R^2$ be an operator defined by $E\left(x_1,x_2\right)=\left(x_1^3,x_2^3\right)$. For the considered vector optimization problem (MOP2), we define its associated E-vector optimization problem (VP$_E$) as follows

$$f\left(E\left(x_1,x_2\right)\right)=\left(x_1^2-x_1\ ,\ e^{x_2}-x_1\right)\to\min$$
$$g\left(E\left(x_1,x_2\right)\right)=1-e^{x_2}\leqq 0, \qquad \text{(VP2}_E\text{)}$$
$$h\left(E\left(x_1,x_2\right)\right)=e^{x_1}-e^{x_2}=0.$$

Then, $\Omega_E = \{(x_1, x_2) \in R^2 : 1 - e^{x_2} \leqq 0 \wedge e^{x_1} - e^{x_2} = 0\}$ and $\overline{x} = (0, 0)$ is a feasible solution. Let η be defined by $\eta(E(x), E(\overline{x})) = (x_1 + 1, e^{x_2})$. Further, note that all functions constituting the considered vector optimization problem (MOP2) are E-differentiable at $\overline{x} = (0, 0)$. Then, it can also be shown that the E-Karush–Kuhn–Tucker necessary optimality conditions (4)-(6) are fulfilled at $\overline{x} = (0, 0)$ with Lagrange multipliers $\overline{\lambda}_1 = 0$, $\overline{\lambda}_2 = \frac{1}{2}$, $\overline{\mu} = 0$, and $\overline{\xi} = \frac{1}{2}$. Since all hypotheses of Theorem 4 are fulfilled, $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a saddle point of the vector-valued Lagrange function L_{p_E} defined for the E-vector optimization problem (VP2$_E$). Furthermore, by Theorem 5, it follows that $(E(\overline{x}), \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is an E-saddle point of the Lagrange function L_p of the considered multiobjective programming problem (MOP2). Moreover, it is not possible to use the saddle point criteria from [4] since that the functions constituting (VP2$_E$) are not E-convex at $\overline{x}$ on Ω_E. However, it is possible to use E-saddle point criteria established in the paper since that the functions constituting (VP2$_E$) are E-invex at $\overline{x}$ on Ω_E with respect to η.

5 Conclusion

In this work, the saddle point criteria have been presented for the class of (not necessarily) differentiable multiobjective programming problems with both inequality and equality constraints in which the involved functions are E-differentia-ble. Both scalar and vector E-saddle point criteria have been established for the considered E-differentiable multiobjective programming problem by using the classical scalar and vector saddle point criteria established for its associated E-vector optimization problem under E-invexity hypotheses. In other words, the equivalence between scalar and vector E-saddle points and (weak) E-Pareto solutions of the considered E-differentiable vector optimization problems with E-invex functions has been established. Hence, the mentioned results have been proved for a new class of nonconvex (not necessarily) differentiable vector optimization problems with both inequality and equality constraints by the help of similarly results established for differentiable vector optimization problems.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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On a class of interval-valued variational control problems with first-order PDE constraints

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Abstract. In this chapter, a modified interval-valued variational control problem involving first-order partial differential equations (PDEs) and inequality constraints is investigated. Specifically, under some generalized convexity assumptions, we formulate and prove LU-optimality conditions for the considered interval-valued variational control problem. Also, to illustrate the main results and their effectiveness, an application is provided. Moreover, we study the connections between the LU-optimal solutions of the interval-valued variational control problem and the saddle-points associated with the interval-valued Lagrange functional corresponding to the modified interval-valued variational control problem.

Keywords: interval-valued variational control problem; invexity; pseudoinvexity; LU-optimality conditions; saddle-point optimality criteria.

1 Introduction

In recent years, saddle-point optimality criteria and the modified objective function method in optimization problems have been investigated increasingly more. In this regard, we mention the works of Sposito and David [11], Smith and Vandelinde [10], Duc et al. [3], Li [5] and dos Santos et al. [8]. Also, it is well known that considerable interest has been given in obtaining new easier methods for solving the initial optimization problem and its duals by considering some associated optimization problems (see, for instance, Antczak [1], Bhatia [2], Jayswal et al. [4], Singh et al. [9] and Mishra et al. [6]).

In this chapter, taking into account the applications of interval analysis in various fields and motivated and inspired by the above-mentioned works, we extend the previous studies for a new class of interval-valued variational control problems with mixed constraints involving first-order PDEs. Specifically, based on a class of interval-valued variational control problems recently introduced by Treanţă [15], we formulate and prove LU-optimality conditions in the considered first-order PDE-constrained modified interval-valued variational control problem. It can be easily observed that the modified interval-valued variational control problem is simpler to study than the initial interval-valued variational control problem. Also, we establish some connections between the LU-optimal solutions of the interval-valued variational control problem and the saddle-points associated with the interval-valued Lagrange functional corresponding to the modified interval-valued variational control problem. Consequently, the present study provides important mathematical tools and ideas for further research in various fields.

The chapter is structured as follows. Section 2 contains notations, definitions and the preliminary results to be used in the sequel. Section 3 introduces a modified interval-valued variational control problem governed by first-order partial differential equations and inequality constraints. Also, under invexity and pseudoinvexity hypotheses, some connections between the original interval-valued variational control problem and the modified interval-valued variational control problem are established. In order to illustrate the mathematical development and its effectiveness, we present an application. In Section 4, we formulate saddle-point criteria via an interval-valued Lagrange functional. Finally, Section 5 gives the conclusions of the present study.

2 Notations and preliminaries

In this section, we introduce the notations, working hypotheses and the preliminary results to be used throughout the present chapter. Thus, we consider:

∗ $\Omega \subset \mathbb{R}^m$ is a compact domain and $t = (t^\alpha)$, $\alpha = \overline{1,m}$, is a point in Ω;

∗ let $\mathcal{X}$ be the space of piecewise smooth *state functions* $x : \Omega \to \mathbb{R}^n$ with the norm

$$\| x \| = \| x \|_\infty + \sum_{\alpha=1}^{m} \| x_\alpha \|_\infty, \quad \forall x \in \mathcal{X},$$

where x_α denotes $\dfrac{\partial x}{\partial t^\alpha}$;

∗ also, denote by $\mathcal{U}$ the space of piecewise continuous *control functions* $u : \Omega \to \mathbb{R}^k$ with the uniform norm $\| \cdot \|_\infty$;

∗ for $\mathcal{P} := \Omega \times \mathbb{R}^n \times \mathbb{R}^k$, we define the following continuously differentiable functions

$$X = \left(X^i_\alpha\right) : \mathcal{P} \to \mathbb{R}^{nm}, \quad i = \overline{1,n},\ \alpha = \overline{1,m},$$

$$Y = (Y_\beta) : \mathcal{P} \to \mathbb{R}^q, \quad \beta = \overline{1,q};$$

∗ $dv := dt^1 dt^2 \cdots dt^m$ represents the volume element on $\mathbb{R}^m \supset \Omega$;

∗ we assume that the continuously differentiable functions

$$X_\alpha = \left(X^i_\alpha\right) : \mathcal{P} \to \mathbb{R}^n, \quad i = \overline{1,n},\ \alpha = \overline{1,m},$$

fulfill the complete integrability conditions, that is,

$$D_\zeta X^i_\alpha = D_\alpha X^i_\zeta, \quad \alpha, \zeta = \overline{1,m},\ \alpha \neq \zeta,\ i = \overline{1,n},$$

where D_ζ is the total derivative operator;

∗ for any two vectors $w = (w_1, ..., w_p)\,, l = (l_1, ..., l_p)$ in $\mathbb{R}^p$, the following convention will be used throughout the chapter:

$$w = l \Leftrightarrow w_i = l_i, \quad w \leq l \Leftrightarrow w_i \leq l_i,$$

$$w < l \Leftrightarrow w_i < l_i, \quad w \preceq l \Leftrightarrow w \leq l,\ w \neq l, \quad i = \overline{1,p}.$$

In the following, in order to formulate and prove the main results included in this chapter, we present the invexity and pseudoinvexity associated with a multiple integral functional.

Consider a continuously differentiable function

$$h : J^1(\mathbb{R}^m, \mathbb{R}^n) \times \mathbb{R}^k \to \mathbb{R},\ h = h\left(t, x(t), x_\alpha(t), u(t)\right),$$

where $J^1(\mathbb{R}^m, \mathbb{R}^n)$ is the first-order jet bundle associated to $\mathbb{R}^m$ and $\mathbb{R}^n$. For $x \in \mathcal{X}$ and $u \in \mathcal{U}$, we introduce the following scalar functional

$$H : \mathcal{X} \times \mathcal{U} \to \mathbb{R}, \quad H(x, u) = \int_{\Omega} h(t, x(t), x_\alpha(t), u(t))\, dv.$$

Taking into account Treanţă [14], according to Treanţă and Arana-Jiménez [12, 13] and following Mititelu and Treanţă [7], we formulate the next definitions. Further, we use the notation $(\Lambda_{xu}) = \big(t, x(t), u(t), x^0(t), u^0(t)\big)$.

Definition 1. *If there exist*

$$\eta : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,$$

$$\eta = \eta(\Lambda_{xu}) = (\eta_i(\Lambda_{xu})), \quad i = \overline{1, n},$$

of C^1-class with $\eta(\Lambda_{x^0u^0}) = 0$, $\forall t \in \Omega$, $\eta|_{\partial\Omega} = 0$, and

$$\xi : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k,$$

$$\xi = \xi(\Lambda_{xu}) = (\xi_j(\Lambda_{xu})), \quad j = \overline{1, k},$$

of C^0-class with $\xi(\Lambda_{x^0u^0}) = 0$, $\forall t \in \Omega$, $\xi|_{\partial\Omega} = 0$, such that for every $(x, u) \in \mathcal{X} \times \mathcal{U}$:

$$H(x, u) - H\big(x^0, u^0\big)$$

$$\geq \int_{\Omega} \left[h_x\big(t, x^0(t), x^0_\alpha(t), u^0(t)\big)\eta + h_{x_\alpha}\big(t, x^0(t), x^0_\alpha(t), u^0(t)\big) D_\alpha \eta\right] dv$$

$$+ \int_{\Omega} \left[h_u\big(t, x^0(t), x^0_\alpha(t), u^0(t)\big)\xi\right] dv,$$

then H is said to be invex at $\big(x^0, u^0\big) \in \mathcal{X} \times \mathcal{U}$ with respect to η and ξ.

Definition 2. *If there exist*

$$\eta : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,$$

$$\eta = \eta(\Lambda_{xu}) = (\eta_i(\Lambda_{xu})), \quad i = \overline{1, n},$$

of C^1-class with $\eta(\Lambda_{x^0u^0}) = 0$, $\forall t \in \Omega$, $\eta|_{\partial\Omega} = 0$, and

$$\xi : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k,$$

$$\xi = \xi(\Lambda_{xu}) = (\xi_j(\Lambda_{xu})), \quad j = \overline{1, k},$$

of C^0-class with $\xi(\Lambda_{x^0u^0}) = 0$, $\forall t \in \Omega$, $\xi|_{\partial\Omega} = 0$, such that for every $(x, u) \in \mathcal{X} \times \mathcal{U}$:

$$H(x, u) - H\big(x^0, u^0\big) < 0$$

$$\Rightarrow \int_{\Omega} \left[h_x\big(t, x^0(t), x^0_\alpha(t), u^0(t)\big)\eta + h_{x_\alpha}\big(t, x^0(t), x^0_\alpha(t), u^0(t)\big) D_\alpha \eta\right] dv$$

$$+ \int_{\Omega} \left[h_u\big(t, x^0(t), x^0_\alpha(t), u^0(t)\big)\xi\right] dv < 0,$$

or, equivalently,

$$\int_{\Omega}\left[h_x\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\eta+h_{x_\alpha}\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)D_\alpha\eta\right]dv$$

$$+\int_{\Omega}\left[h_u\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\xi\right]dv\geq 0\Rightarrow H\left(x,u\right)-H\left(x^0,u^0\right)\geq 0,$$

then H is said to be pseudoinvex at $\left(x^0,u^0\right)\in\mathcal{X}\times\mathcal{U}$ with respect to η and ξ.

Further, let K be the set of all closed and bounded real intervals. Denote by $A=[a^L,a^U]$ a closed and bounded real interval, where a^L and a^U indicate the lower and upper bounds of A, respectively. Throughout this chapter, the interval operations can be performed as follows:

1) $A=B\Longrightarrow a^L=b^L$ and $a^U=b^U$;
2) if $a^L=a^U=a$ then $A=[a,a]=a$;
3) $A+B=[a^L+b^L,a^U+b^U]$;
4) $-A=-[a^L,a^U]=[-a^U,-a^L]$;
5) $A-B=[a^L-b^U,a^U-b^L]$;
6) $k+A=[k+a^L,k+a^U],\ k\in\mathbb{R}$;
7) $kA=[ka^L,ka^U],\ k\in\mathbb{R},\ k\geq 0$;
7′) $kA=[ka^U,ka^L],\ k\in\mathbb{R},\ k<0$.

Definition 3. *([15]) Let $A,B\in K$ be two closed and bounded real intervals. We write $A\preceq_{LU}B$ if and only if $a^L\leq b^L$ and $a^U\leq b^U$.*

Definition 4. *([15]) Let $A,B\in K$ be two closed and bounded real intervals. We write $A\prec_{LU}B$ if and only if $A\preceq_{LU}B$ and $A\neq B$.*

Definition 5. *([15]) A function $f:\mathcal{P}\to K$, defined by*

$$f\chi_{xu}(t)=[f^L\chi_{xu}(t),f^U\chi_{xu}(t)],\quad t\in\Omega,$$

where $\chi_{xu}(t):=(t,x(t),u(t))$, $f^L\chi_{xu}(t)$ and $f^U\chi_{xu}(t)$ are real-valued functions and satisfy the condition $f^L\chi_{xu}(t)\leq f^U\chi_{xu}(t)$, $t\in\Omega$, is said to be interval-valued function.

Now, we introduce the following class of interval-valued variational control problems, where the objective functional $F\left(x,u\right)=\int_{\Omega}f\chi_{xu}(t)dv,\ (x,u)\in\mathcal{X}\times\mathcal{U}$, is considered as interval-valued (for more details, see Treanţă [15]):

$$(CP)\quad\min_{(x,u)}\left\{\int_{\Omega}f\chi_{xu}(t)dv=\left[\int_{\Omega}f^L\chi_{xu}(t)dv,\int_{\Omega}f^U\chi_{xu}(t)dv\right]\right\}$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t)=X^i_\alpha\chi_{xu}(t),\quad i=\overline{1,n},\ \alpha=\overline{1,m},\ t\in\Omega,\tag{1}$$

$$Y\chi_{xu}(t)\leq 0,\quad t\in\Omega,\tag{2}$$

$$x(t)|_{\partial\Omega}=\varphi(t)=\text{given},\tag{3}$$

where $f : \mathcal{P} \to K$ is an interval-valued function and $f^L, f^U : \mathcal{P} \to \mathbb{R}$ are continuously differentiable real-valued functions.

Define the *set $\mathcal{D}$ of all feasible solutions (domain)* in (CP) as

$$\mathcal{D} := \{(x, u) \,|\, x \in \mathcal{X},\ u \in \mathcal{U} \text{ satisfying } (1),\ (2),\ (3)\}.$$

Definition 6. *([15]) A feasible solution $(x^0, u^0) \in \mathcal{D}$ in interval-valued variational control problem (CP) is called LU-optimal solution if there exists no other feasible solution $(x, u) \in \mathcal{D}$ such that $F(x, u) \prec_{LU} F(x^0, u^0)$.*

The next result formulates necessary LU-optimality conditions for a feasible point in (CP). In the following, summation over the repeated indices is assumed.

Theorem 1. *([15]) Under constraint qualification assumptions, if (x^0, u^0) is LU-optimal solution in (CP), then there exist the piecewise smooth functions $\theta : \Omega \to \mathbb{R}^2$, $\theta(t) = (\theta^L(t), \theta^U(t))$, $\mu : \Omega \to \mathbb{R}^q$ and $\lambda : \Omega \to \mathbb{R}^{nm}$, with $\mu(t) = (\mu^\beta(t)) \in \mathbb{R}^q$, $\lambda(t) = (\lambda_i^\alpha(t)) \in \mathbb{R}^{nm}$, such that:*

$$\theta^L(t)\frac{\partial f^L}{\partial x^i}\chi_{x^0u^0}(t) + \theta^U(t)\frac{\partial f^U}{\partial x^i}\chi_{x^0u^0}(t) + \lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial x^i}\chi_{x^0u^0}(t) \tag{4}$$

$$+\mu^\beta(t)\frac{\partial Y_\beta}{\partial x^i}\chi_{x^0u^0}(t) + \frac{\partial \lambda_i^\alpha}{\partial t^\alpha}(t) = 0, \quad i = \overline{1, n},$$

$$\theta^L(t)\frac{\partial f^L}{\partial u^j}\chi_{x^0u^0}(t) + \theta^U(t)\frac{\partial f^U}{\partial u^j}\chi_{x^0u^0}(t) + \lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial u^j}\chi_{x^0u^0}(t) \tag{5}$$

$$+\mu^\beta(t)\frac{\partial Y_\beta}{\partial u^j}\chi_{x^0u^0}(t) = 0, \quad j = \overline{1, k},$$

$$\mu^\beta(t)Y_\beta\chi_{x^0u^0}(t) = 0 \quad \textit{(no summation)}, \quad (\theta(t), \mu(t)) \succeq 0, \tag{6}$$

for all $t \in \Omega$, except at discontinuities.

Definition 7. *([15]) The LU-optimal solution $(x^0, u^0) \in \mathcal{D}$ in (CP) is said to be normal LU-optimal solution if $\theta(t) > 0$.*

3 Modified interval-valued variational control problem

In this section, we define a modified interval-valued variational control problem associated with (CP) and, under some invexity and pseudoinvexity assumptions, we establish LU-optimality conditions for the considered interval-valued optimization problems.

For an arbitrary given feasible solution $(x^0, u^0) \in \mathcal{D}$ in (CP) and for η, ξ defined as in Definitions 1 and 2, we introduce a *modified interval-valued variational control problem* associated with (CP), as follows:

$$(CP_{\eta,\xi}(x^0, u^0)) \qquad \min_{(x,u)} \int_\Omega \left(f_x\chi_{x^0u^0}(t)\eta + f_u\chi_{x^0u^0}(t)\xi\right) dv$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha \chi_{xu}(t), \quad i = \overline{1,n},\ \alpha = \overline{1,m},\ t \in \Omega,$$
$$Y\chi_{xu}(t) \leq 0, \quad t \in \Omega,$$
$$x(t)|_{\partial\Omega} = \varphi(t) = \text{ given},$$

where

$$\int_\Omega (f_x \chi_{x^0u^0}(t)\eta + f_u \chi_{x^0u^0}(t)\xi)\, dv$$
$$:= \left[\int_\Omega \left(f_x^L \chi_{x^0u^0}(t)\eta + f_u^L \chi_{x^0u^0}(t)\xi\right) dv, \int_\Omega \left(f_x^U \chi_{x^0u^0}(t)\eta + f_u^U \chi_{x^0u^0}(t)\xi\right) dv\right].$$

Remark 1. We observe that the set of all feasible solutions for the considered modified interval-valued control problem $(CP_{\eta,\xi}(x^0, u^0))$ is $\mathcal{D}$, as well.

Definition 8. *A point $(\hat{x}, \hat{u}) \in \mathcal{D}$ is said to be an LU-optimal solution for $(CP_{\eta,\xi}(x^0, u^0))$ if*

$$\int_\Omega (f_x \chi_{x^0u^0}(t)\eta\,(\Lambda_{xu}))\, dv + \int_\Omega (f_u \chi_{x^0u^0}(t)\xi\,(\Lambda_{xu}))\, dv$$
$$\succeq_{LU} \int_\Omega (f_x \chi_{x^0u^0}(t)\eta\,(\Lambda_{\hat{x}\hat{u}}))\, dv + \int_\Omega (f_u \chi_{x^0u^0}(t)\xi\,(\Lambda_{\hat{x}\hat{u}}))\, dv,$$

for every $(x, u) \in \mathcal{D}$.

Further, under some invexity assumptions, we establish the equivalence between LU-optimal solutions associated with (CP) and $(CP_{\eta,\xi}(x^0, u^0))$.

Theorem 2. *If $(x^0, u^0) \in \mathcal{D}$ is a normal LU-optimal solution in (CP) and $\int_\Omega \mu^\beta(t) Y_\beta \chi_{xu}(t) dv$, $\int_\Omega \lambda_i^\alpha(t) \left(X^i_\alpha \chi_{xu}(t) - \frac{\partial x^i}{\partial t^\alpha}(t)\right) dv$ are invex at $(x^0, u^0) \in \mathcal{D}$ with respect to η and ξ, then $(x^0, u^0) \in \mathcal{D}$ is an LU-optimal solution in $(CP_{\eta,\xi}(x^0, u^0))$.*

Proof. By hypothesis, the relations (4) − (6), with $\theta(t) = (1, 1)$ (for instance), are satisfied for all $t \in \Omega$, except at discontinuities. By reductio ad absurdum, consider that $(x^0, u^0) \in \mathcal{D}$ is not an LU-optimal solution in $(CP_{\eta,\xi}(x^0, u^0))$. Thus, there exists $(\overline{x}, \overline{u}) \in \mathcal{D}$ such that

$$\int_\Omega (f_x \chi_{x^0u^0}(t)\eta\,(\Lambda_{\overline{xu}}))\, dv + \int_\Omega (f_u \chi_{x^0u^0}(t)\xi\,(\Lambda_{\overline{xu}}))\, dv$$
$$\prec_{LU} \int_\Omega (f_x \chi_{x^0u^0}(t)\eta\,(\Lambda_{x^0u^0}))\, dv + \int_\Omega (f_u \chi_{x^0u^0}(t)\xi\,(\Lambda_{x^0u^0}))\, dv.$$

Taking into account that

$$\eta\,(\Lambda_{x^0u^0}) = 0, \quad \xi\,(\Lambda_{x^0u^0}) = 0, \quad \forall t \in \Omega,$$

we get

$$\int_\Omega (f_x \chi_{x^0u^0}(t)\eta\,(\Lambda_{\overline{xu}}))\, dv + \int_\Omega (f_u \chi_{x^0u^0}(t)\xi\,(\Lambda_{\overline{xu}}))\, dv \prec_{LU} [0, 0]. \tag{7}$$