Measurement and Data Science

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PREFACE

Nowadays, all of us enjoy the benefits of the global rebirth of measurement and data science due to a revolution in sensory devices and the amazing data transmission, storage and processing capabilities that have become available and are now embedded everywhere. Thanks to the unbelievable amount of recorded information and the theoretical results of measurement and data science, many newly developed products invade our surroundings and enable previously inconceivable smart services and support.

This volume consists of selected chapters covering the scientific results of researchers working at the Department of Measurement and Information Systems at Budapest University of Technology and Economics, Hungary. The authors decided to reconsider the achievements of their previous research—summarized typically in their dissertations—and place these results in a larger contextual framework.

The intention of this volume is to provide a comprehensive picture of the knowledge, ways of thinking, and working methods of researchers in measurement science, and arouse interest in the study, acquisition, and application of these achievements. This book is recommended for MSc and PhD students, as well as partners in research and development. The topics covered display the experience and references the authors have, i.e. the fields in which they can contribute and cooperate.

Measurement science is a cognitive science dealing with the observation and evaluation of phenomena and interaction in the surrounding world. We consider a measurement process to be the systematic activity of broadening our previous knowledge based on more and more, possibly repeated, observations. Within this process, first the intensity relations of different quantities are explored and recorded as data. This is followed by the processing of those data that convey information about the features being investigated. Finally, the process is completed with the interpretation of this newly developed and expanded knowledge.

While every human activity has its measurement science, and the information delivered by the data is rather far-reaching, it is a valuable feature that the methodology of these measurement processes can be placed—in several respects—into a common framework. In this volume we attempt to contribute to this process.

In utilizing measurement science in any given domain, we carry out four types of interrelated activity:

- *Modelling:* We rearrange our prior knowledge in order to achieve improved understanding. On the one hand, we create/extend the model of the investigated phenomena and their interactions and, simultaneously, we make it clear which feature is to be the focus of observation. This possibly iterative process is called modelling and results in an approximate description of reality, of limited extent, considered to be sufficient to find out further information. During this process we cannot neglect that our resources are limited both in time and space. We must also strive to keep expenses at an acceptable level.
- *Measurement design:* In order to get new information, we try to separate useful interactions from effects caused by distortion/disturbance, and, if needed, we apply appropriate test, excitation, or training signals/samples. This is the process of measurement design. It cannot be separated from modelling and also requires prior information concerning test/excitation signals and distortion/disturbance effects. During measurement design, we seek to perform such interactions for the observation of which we have appropriate devices and experience.
- *Data acquisition:* We perform the targeted observations using sensors and measuring channels. The results are then converted into a format that can be processed. This activity is the most domain-specific, as domain-specific features are characterized by appropriate data.
- *Evaluation:* Using the model of the investigated features and their interactions, we evaluate our observations. We develop conclusions and also formulate and characterize new information. The computational demand of data, coming partly from sensors and partly from databases, may be remarkably large. This is an exciting issue if observation-based conclusions and evaluations specify the necessity of intervention and/or determine their parameters. The foundation of such real-time, typically autonomous measurement systems requires the sound coordination of modelling, measurement design, data acquisition, and evaluation.

All the studies in this volume provide overviews of specific fields and research results that enable the coordination of the above noted procedures. Furthermore, to some extent, the conceptual foundation of measurement science is also extended, and new measurement methods, opening up completely new vistas, are also introduced.

All the chapters of this volume are available to readers as independent studies and cross-references are occasional. The authors were free to apply the wording and notation they are accustomed to.

The first chapter, entitled *Structure and Interpretation of Model-based Signal Processing*, argues that the measurement of directly non-measurable quantities can be efficiently solved by the application of the so-called observer-based approach. An observer is a mechanism that is capable of following, as a simulator, an observed object. Its operation is based on the model of the observed object and thus the simulator tracks not only the observations, but also the internal (unknown) quantities. The operation of the observer is controlled, in a negative feedback loop, by the difference between the observed values and the simulated output. A measurement process that follows this strategy can estimate an unknown quantity based on the corresponding internal variable of the observer. The simulation proceeds in parallel with data acquisition until the required accuracy is achieved, or the results of simulation are needed. The resulting evaluation is based on recursive expressions that fit the requirements of real-time operation well. The recursive expressions of the observer-based approach also provide an inclusive framework for signal representation and corresponding signal analysis. The resulting recursive signal processing structure can serve as a universal tool, as it decomposes the signals into components that can be used in signal synthesis and can implement arbitrary linear filters and transformations. Thanks to the passivity and orthogonality of the structure, it displays excellent properties concerning stability, limit-cycle avoidance, round-off errors, and transient behaviour.

The first part of the second chapter, which is entitled *Adaptive Spectral Estimation and Active Noise Control*, is devoted to the high-precision measurement and tracking of periodic signal components where the basic harmonic varies in time or is not exactly known. The proposed *adaptive Fourier analyser* (AFA) follows the model-based approach introduced in the first chapter and, correspondingly, is composed of tunable resonators that are capable of generating harmonic signal components. Several strategies are offered for tuning the resonators, each of which concerns various aspects of convergence. The second part of this chapter deals with the consequences of data loss on spectral estimation and proposes modifications of the original methods to avoid distorting effects. As a first step, the author introduces data loss models, which are followed by solutions, based on both recursive and fast Fourier transformations. The third part of the chapter covers the problem of active noise control. First, using the model-based approach, a possible method for the active control

of periodic disturbances in the acoustic environment is treated. To achieve spatial noise reduction, loudspeakers receiving input from tunable resonators are used to counteract periodic acoustic noise components. The elaborated method has proven successful in linear system identification, automatic offset compensation, and in reducing nonlinear distortion. As a next step, the author presents an improved version of the least mean square (LMS) algorithm that is capable of reducing wide-band and stochastic disturbances. The last part of this chapter introduces, as a test application, an active noise control system based on a wireless sensor network.

The third chapter, entitled *Inverse Problems and Algorithms of Measurement Science*, deals with the problem of how the accuracy of the devices used to observe our environment can be improved by digital signal processing. Its starting point is that our observations are affected by distortions and disturbances; therefore, compensating these effects, i.e. reconstructing the real value of the quantity to be measured, is of serious interest. In the first part of the chapter, the author presents an overview of methods that can minimize distortions due to limited bandwidth, assuming that the signal path can be characterized by a linear model, and an appropriate criterion of minimization is also given. These methods are called inverse algorithms and they try to compensate known distortions while simultaneously suppressing disturbances. As an important extension, the author provides an automatic process to optimize inverse filter parameters and a method to compensate sampling jitter. The efficiency of the proposed methods is proven through practical examples. As a next step, the possible compensation of nonlinearities is discussed, followed by the introduction of redundant observation setups where the abundance and/or diversity of sensors enables simultaneous and/or redundant observations. In this case, sensor fusion offers further solutions for successfully compensating distortions and disturbances. The next part of the chapter concerns the case of directly non-measurable quantities, for which data path compensation and measured data reconstruction cannot be separated. In such situations the estimation process should involve the partial or complete identification of the observed system, and the derivation of the quantity to be measured can be achieved via the identified system. In the last part of the chapter, the author introduces some of his contributions to the solution of practical inverse problems.

The author of the fourth chapter, entitled *Optimized Random Multisines in Nonlinear System Characterization*, deals with a rather large family of measurements, namely with the nonparametric identification of dynamic systems. His investigations primarily concern measurement methods of the frequency response function. The first part sums up the

features of the knowledge, attainable by measurement that is needed to identify such systems, which are modelled assuming linearity and time invariance. Furthermore, it presents the consequences of the finite time duration of measurement records. As a next step, he introduces methods to measure the frequency response function, assuming both periodic and random signals, and systems with multiple-input and multiple-output, or even feedback. Based on his investigations, the author concludes that it is more beneficial to perform the measurement of the frequency response function if it is designed assuming multisine excitation and frequency domain interpretation. The second subchapter is devoted to the nonparametric identification of such systems, which cannot be identified properly under the assumption of linearity and time invariance, i.e. the consequences of deviation from linearity should also be considered. The investigations of the author concern systems excited by random multisines that have nonlinear behaviour, which can be efficiently characterized by Volterra systems. As a first step, he interprets the best linear approximation frequency characteristic and attaches a model of nonlinear distortions. As a next step, the author shows how, based on a systematic design of the multisine excitation, i.e. by the proper composition of the frequency grid, nonlinear distortions can be separated and/or suppressed. In the following, two significant methods for measuring the best linear approximation are provided, which are extended to measurements in closed loops and for systems having multiple inputs and multiple outputs. The last subchapter provides practical considerations concerning the proposed methods to help application designers and other users.

The authors of the fifth chapter, entitled *Methods for Processing Measured Sinusoidal Signals and their Application in Analogue-to-digital Converter Classification*, deal with the reduction of problems arising in the implementation of the IEEE 1241 standard, elaborated to regulate analogue-to-digital converter (ADC) classification, and, in certain respects, to suggest new, more efficient solutions that exceed the specifications of the standard. ADCs assign digital codes to analogue signal levels. The classification of a converter is based on knowledge of the actual threshold levels at which code transition occurs. The determination of these threshold levels requires an appropriate excitation signal, and, practically speaking, this is possible only via an indirect evaluation using statistical methods. To test ADCs, as excitation signals, sine waves consisting of an integer number of periods are preferred, together with the additional condition that the total number of digitized samples and the number of periods are relative primes. The histogram test can contribute to the appropriate evaluation of the digitized samples,

which, if the excitation signal parameters are known, gives the occurrence statistics of the samples within a given code bin. In this chapter, the authors first review some of the properties of ADCs, the method of least squares (LS) sine-fitting, and that of the histogram test. This is followed by the presentation of methods that allow the verification of the correctness of the excitation signal settings and the unbiasedness of the ADC measurement. As a next step, it is pointed out that the numerical errors of least squares sine-fitting algorithms, unless properly handled, can be several orders of magnitude larger than the round-off errors of the numerical representation. Following a proper analysis, new methods are proposed that can significantly increase the numerical stability of the investigated algorithms. As a possible further improvement, the maximum likelihood (ML) estimation of ADC and excitation signal parameters is introduced and developed in two directions. The first one is a method, which can, by an appropriate approximation of the ADC static transfer characteristic, significantly decrease the size of the parameter space, while the second is a proposition to estimate the aperture jitter in the sense of ML estimation.

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CHAPTER ONE

STRUCTURE AND INTERPRETATION OF MODEL-BASED SIGNAL PROCESSING

GÁBOR PÉCELI

1.1 Introduction

The objective of this chapter is to interpret and implement measurement processes and related signal processing algorithms as *observer mechanisms* (Luenberger 1971).

The basic concept of this *observer-based* approach is that the measurement of directly non-measurable quantities is enabled by *simulation* of the investigated real-world phenomena. This *simulation* is based partly on prior knowledge, including the executable *relevant model* of the system to be measured, and partly on observed data. The *simulator* device, which is typically a digital computer, tries to *copy* the events and processes of the environment, and thus the quantity to be measured will have an *available estimate* from among the quantities computed by the simulator.

In a domain of interest, the *relevant model* of the system to be measured is an ordered set of indispensable prior knowledge required for the success of the measurement. Here, success means that new and useful information becomes available, i.e. our knowledge will become deeper and/or wider.

The *simulator* operates the *relevant model* of the system, while the measurement process itself is governed as an *observer*. The differences of the observed values and their estimates, provided by the simulator, force the simulator to behave as a model-copy of the investigated real-world phenomena. Any process that follows this type of operation is an *observer*.

The application of this *observer-based approach* to measurements shows that evaluation of the observed data can be performed in parallel with data acquisition. This process can be continued until the required accuracy is achieved, or the simulation results are useful.

In the following, we will concern ourselves with such observations, for which, using an independent variable (e.g. time or space coordinates), the fact and measure of interrelation can be characterized and is interpretable. Without providing a detailed specification of the concept of a signal, in the following we will refer to the evaluation of such observations as *digital signal processing*, and the entire process as signal processing.

This chapter addresses *cognitive processes* that can be considered as observers. Since such processes typically work on interrelated, multiple observations and apply the relevant model of the system to be discovered, we will refer to this as *model-based signal processing*. At the same time, in order to enable real-time evaluation, only signal processing algorithms that do not require prior knowledge of measured data sequences or blocks will be considered.

In engineering practice, the result of the *measurement process* is typically *observation-based* inference. Essentially this inference, at an appropriate level of abstraction, is the solution of an equation, even if the interrelation of observations and unknowns may be rather uncertain and changeable in time and space.

In this context, model-based signal processing is the *solving* of different *equations* in such a way that the new information available in parallel with the solution, to improve its quality, is utilized within the process itself. As a result, we get *iterative* or even *recursive* solutions.

This iterative nature is inherent to the measurement process. In the following, firstly, the attributes of the measurement processes are reviewed and then the recursive evaluation of equations related to signal processing problems is considered. Such an approach is applied both to the measurements and the signal processing problems, where we estimate the unknown value, according to prior observations, then make a new observation, before finally improving/refining our estimate in accordance with the following scheme:

new estimate = prediction based on previous estimate + correction based on new observation

It is a speciality of the applied approach that the quantities are represented by *discrete time* (and/or *space*) samples, hence the available new information is also related to discrete time and/or space coordinates. The capability of determining the value at other time or space coordinates is ensured by meeting the conditions of the *sampling theorems*. Accordingly, the *model-based simulation* applied is also discrete.

1.2 Attributes of measurement processes

This section presents the main topic of this chapter: the *observer-based* interpretation of measurement processes together with the wide-ranging relationships of related signal processing algorithms.

The goal of the measurement process (Pavese and Forbes 2009) is to *improve and refine* available prior knowledge and information. More precisely, the measurements are targeted to characterize various real-world phenomena. Preferably, this characterization relies on quantities/features that show *stability* in some respect. We can discover such quantities/ features by *abstraction*. The role of the following quantities/features is of basic importance:

- ***State variables*** (***x***), which change with time and/or space and are related to *energy processes* due to *interactions/counteractions* of objects in the real world (e.g. voltage, pressure, temperature, speed, etc.);
- ***Parameters*** (***a***), which characterize the *intensity* of *interactions or counteractions*;
- ***Structures*** (***S***), which describe the *relationships* between the objects (system components).

The goal of a measurement process is the determination of such quantities or features; these are the unknowns of the equation set to be solved as part of the measurement process. The measurement process can be divided into two main parts (See Fig. 1-1).

The first part is the *observation process*, which is devoted to getting information—this is called *observation*. These observations are inaccurate due to distortion and/or noise in the observation channel. The effect of distortion and/or noise in the channel can be reduced if we gather more information, i.e. if we increase the number of observations. The determination of *unknown quantities* or *features* is based on observation. This is the second major part of the measurement process, which can also be interpreted as the *inverse of the observation process*. The observation process corresponds to the setting up of the equation set to be solved, while the inverse operation corresponds to solving the equation set. The result of the measurement process, depending on its nature, is interpreted as a *decision* or *estimation* and also has a *qualification*. *This qualification* is typically a measure of *uncertainty*, generally characterized by *variance* and *bias.*

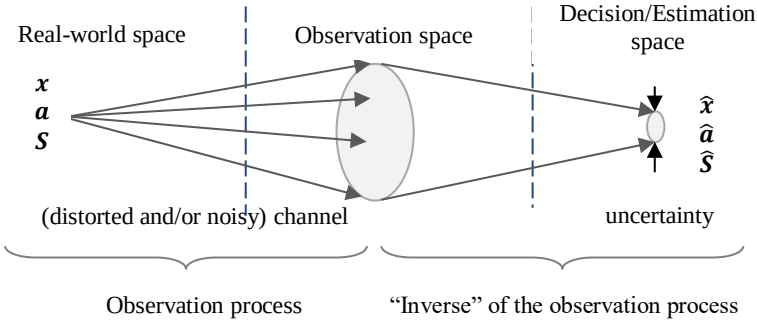

Fig. 1-1. The measurement process.

The *real-world space* is an *abstraction* where the values of the investigated quantities or features correspond to one point in this space. Before measurement these coordinates are unknown. Through the measurement process we intend to determine (measure) the coordinates of such a point by mapping it. This mapping is called observation. The path between the value to be measured and the observed value is called the measurement or transmission channel.

The *observation space* is an *abstraction* where the values of the observed quantities or features correspond to one point in this space.

Finally, the *decision/estimation space* is an *abstraction* where values of the measurement results correspond to one point in this space.

In the following subsections, this general framework is fleshed out. We will acquaint the reader with the observer-based way of thinking, which provides a receptive environment for many measurement processes. Most of these techniques have been presented separately in the literature, many times using particular notations and wording, which makes it difficult to discover similarities and analogies. In the following sections, the unification of these notations is a major concern. For certain processes, this leads us to neglect traditions or usual conventions; however, this all contributes to the identification of a common framework.

1.2.1 Observation in the case of noiseless system and observation models

As a first step, let us suppose that the real world and the observation can be described at discrete time instants with the help of the following linear (state and observation) equations:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) \tag{1.1}$$

$$\boldsymbol{y}(n) = \boldsymbol{C}\boldsymbol{x}(n) \tag{1.2}$$

In the following, the state equation describes the system model, while the observation equation describes the observation model. Let us suppose that the state transition matrix $\boldsymbol{A}$ and the observation matrix $\boldsymbol{C}$ are known. The "real-world" is supposed to be an *autonomous system*, therefore in (1.1) no external *excitation* or *noise/disturbance* is added. Similarly, the observation is also free of noise/disturbance. The object of the observation is the unknown state vector $\boldsymbol{x}(n)$, which, at first glance, can be derived as the solution of equation (1.2). However, this is possible only if the number of the observed data, i.e. the dimension of vector $\boldsymbol{y}(n)$, is equal to or higher than the dimension of the state vector $\boldsymbol{x}(n)$.

With respect to equations (1.1) and (1.2), we investigate cases where the state vector $\boldsymbol{x}(n)$ is of dimension N; the state transition matrix $\boldsymbol{A}$ is of dimension $N * N$; the observation vector $\boldsymbol{y}(n)$ is generally of dimension $M < N$; and, correspondingly, the observation matrix $\boldsymbol{C}$ is of dimension $M * N$. Our aim is to estimate the state vector $\boldsymbol{x}(n)$. Within the given framework, this is not possible in a single step, because the required minimum number of observations (measured data) is only attainable in more than one step. It is the particularity of the "real-world" that the unknown state vector changes with time and, according to (1.1), in the next step it will have a different value. In this case, the *observer* can be a proper tool of estimation, which, thanks to a dedicated *correction/ learning/adaptation mechanism*, tends to behave as a *copy* of the "real-world", providing an iterative solution of the equation to be solved.

Fig. 1-2 shows the block-diagram of a discrete-time *observer*. The input of the observer is the observation vector $\boldsymbol{y}(n)$, which is compared to its estimate $\hat{\boldsymbol{y}}(n)$, generated by simulating the observed system. The difference $\boldsymbol{y}(n) - \hat{\boldsymbol{y}}(n)$ controls the simulator in such a way that its output values will follow the output of the observed system.

After convergence, the "result" of the measurement $\hat{\boldsymbol{x}}(n)$ can be read from the observer (see Fig. 1-3). In the figure, $\boldsymbol{z}^{-1}$ stands for a one-step time difference/delay for all the components of $\hat{\boldsymbol{x}}(n+1)$. The state and the observation equations of the observer are:

$$\hat{\boldsymbol{x}}(n+1) = \boldsymbol{A}\hat{\boldsymbol{x}}(n) + \boldsymbol{G}\boldsymbol{e}(n) = \boldsymbol{A}\hat{\boldsymbol{x}}(n) + \boldsymbol{G}\big(\boldsymbol{y}(n) - \hat{\boldsymbol{y}}(n)\big) \tag{1.3}$$

$$\hat{\boldsymbol{y}}(n) = \boldsymbol{C}\hat{\boldsymbol{x}}(n) \tag{1.4}$$

where the correction matrix $\boldsymbol{G}$ is of dimension $N * M$.

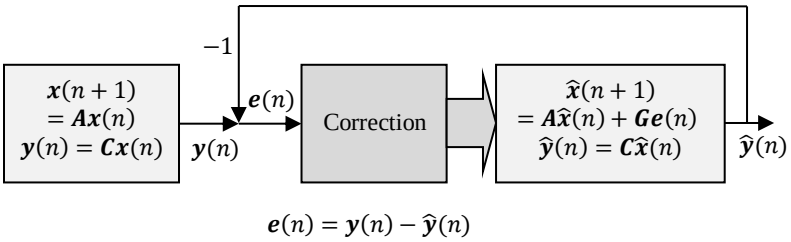

Fig. 1-2. Measurement using discrete-time observer.

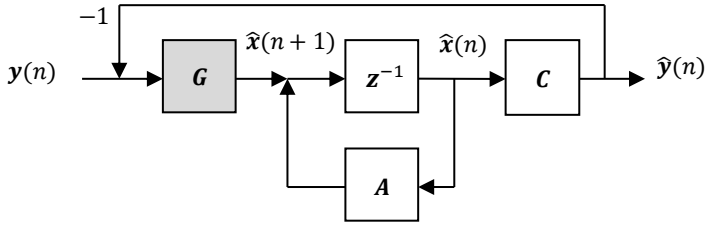

Fig. 1-3. The discrete-time observer.

The matrix $\boldsymbol{G}$ is designed to produce $\hat{\boldsymbol{x}}(n) \to \boldsymbol{x}(n)$. Taking the difference of (1.1) and (1.3):

$$\begin{aligned} \boldsymbol{x}(n+1) - \hat{\boldsymbol{x}}(n+1) &= \boldsymbol{A}\boldsymbol{x}(n) - \boldsymbol{A}\hat{\boldsymbol{x}}(n) - \boldsymbol{G}\boldsymbol{e}(n) \\ &= (\boldsymbol{A} - \boldsymbol{G}\boldsymbol{C})[\boldsymbol{x}(n) - \hat{\boldsymbol{x}}(n)]. \end{aligned} \tag{1.5}$$

Let us introduce notations $\boldsymbol{\varepsilon}(n) = \boldsymbol{x}(n) - \hat{\boldsymbol{x}}(n)$ and $\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{G}\boldsymbol{C}$. The state transition matrix of the so-called *error system* will be:

$$\boldsymbol{\varepsilon}(n+1) = \boldsymbol{F}\boldsymbol{\varepsilon}(n). \tag{1.6}$$

Using this interpretation, we find the solution to the equation, i.e. the value of the unknown state vector, if the *state error* $\boldsymbol{\varepsilon}(n)$ (the state variable of the error system) achieves a value of zero in finite steps. If, for some reason, this is not be possible, it can be reduced to below the level of required accuracy. The correction matrix $\boldsymbol{G}$ should be designed in such a way that matrix $\boldsymbol{F}$ reduces the size/norm of state error $\boldsymbol{\varepsilon}(n)$.

Remarks: To resolve the error, a monotonic decrease is not necessary. What is required is the stability of the error system, i.e. its convergence to zero in the case of zero input. This property can be interpreted as follows: in order to reach a stable state, the error system dissipates its internal energy. If it dissipates its internal energy at every step, then the size reduction of the error vector will be a monotonic process.

Cases:

1. $\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{GC} = \boldsymbol{0}$. In this case $\boldsymbol{G} = \boldsymbol{AC}^{-1}$. This is possible if matrix $\boldsymbol{C}$ is quadratic, i.e. the observation vector has as many components as the state vector. In this case the value of the unknown state vector can be found without iteration in one step. This means that the observer, and within the observer the "copy" of the state, follows the observed (physical) system.

2. $\boldsymbol{F}^N = (\boldsymbol{A} - \boldsymbol{GC})^N = \boldsymbol{0}$. In this case the error system converges in N steps:

$$\boldsymbol{x}(N) - \hat{\boldsymbol{x}}(N) = (\boldsymbol{A} - \boldsymbol{GC})^N [\boldsymbol{x}(0) - \hat{\boldsymbol{x}}(0)] = \boldsymbol{0} \tag{1.7}$$

   Matrices having the property $\boldsymbol{F}^N = \boldsymbol{0}$ are called *non-derogatory nilpotent* matrices. An important property of these matrices is that all their eigenvalues are zero (Halmos 1995). Systems having state transition matrices with this property have a finite impulse response (FIR systems), because the initial error disappears in a finite number of steps. (Remark: if $\boldsymbol{F}^M = \boldsymbol{0}$, with $M < N$, then matrix $\boldsymbol{F}$ is a *derogatory nilpotent* matrix, for the convergence of which fewer than N steps are needed.)

3. If $\boldsymbol{F}^N = (\boldsymbol{A} - \boldsymbol{GC})^N \neq \boldsymbol{0}$, then, if the error system is designed for stability, the size of the state vector of the error system will decay exponentially. Such a system will be stable if all the eigenvalues are within the unit circle. Systems having this property have an infinite impulse response (IIR systems), because the initial error will disappear after an infinite number of steps.

Remarks:

1. Both models (the system model and its copy within the observer) of Fig. 1-2 can be excited by a common input. Since the models

themselves are linear, due to the superposition theorem the convergence of the observer remains valid.

2. The observer in Fig. 1-2 is called a Luenberger observer ("Almost any System is an Observer", Luenberger 1971). The condition of the capacity to behave as an observer is that the observer should be "faster" than the observed system; otherwise it will not be able to follow the changes.

3. In the case of a resistance or impedance measuring bridge, the branch containing the unknown element implements the physical model of the real world, while the branch containing the component for bridge balancing corresponds to the adjustable model within the observer. Both branches divide the voltage of the common driving source and the difference in voltage controls the correction mechanism. If the difference is zero, the set value of the compensating component is used to determine the unknown parameter. Such a circuit, together with the feedback performed by the operator, implements an observer.

1.2.2 Observation in the case of noisy system and observation models

In measuring processes, it is usual that some details of the observation are modelled as random observation noise, because they cannot be modelled or are difficult to model using deterministic tools. Based on similar considerations, we try to bridge the problem of being unable to give the accurate next value of the unknown state variable by applying random plant noise. With the introduction of these changes to the models, if we apply the observer concept, the expectation $\boldsymbol{\varepsilon}(n) \underset{n\to\infty}{\longrightarrow} \mathbf{0}$ is no longer realistic; instead, the requirement of achieving the smallest error of an approximate solution is set. The smallest error is defined according to an appropriate error criterion.

With the appearance of noise processes, errors can be characterized only by statistical methods; as such, the best approximate solutions are based on minimizing the values of appropriate statistical error criteria. Minimization here means finding the optimum value of the free parameters; in our case, this is the value of the correction matrix $\boldsymbol{G}$. In the case of equalities such as (1.5), which is linear in its free parameters, mean square error criteria are preferred. This is because under this condition the optimum parameters are given as the solution of a set of linear equations.

If we extend the equations (1.3) and (1.4) with random processes, then both the observation and the quantity to be measured become stochastic processes. Therefore, instead of minimizing the state variable of the error system, a squared function of this error is minimized. In the following, the covariance matrix $\boldsymbol{P}(n) = E\{\boldsymbol{\varepsilon}(n)\boldsymbol{\varepsilon}^T(n)\}$ plays an important role, the trace of which $tr\boldsymbol{P}(n) = E\{\boldsymbol{\varepsilon}^T(n)\boldsymbol{\varepsilon}(n)\}$ will be a suitable error criterion and its minimization can serve our purpose. With the introduction of the covariance matrix $\boldsymbol{P}(n)$ the state equation of the error system (1.6) can be replaced by:

$$\boldsymbol{P}(n+1) = E\{\boldsymbol{\varepsilon}(n+1)\boldsymbol{\varepsilon}^T(n+1)\} = \boldsymbol{F}\boldsymbol{P}(n)\boldsymbol{F}^T + \text{perturbations}^1. \quad (1.8)$$

This type of error matrix has a significant role in the case of the famous Kalman predictor and filter (Anderson and Moore 1979). In the following we will consider the predictor, because it better fits the point of this chapter.

Remarks:

In the following, we will use the sign of transposition $(\ )^T$ for vectors and matrices, the effect of which is the transformation of rows into columns, and vice versa. In the case of complex-valued matrices/vectors, together with transposition, conjugation is also applied, which is not indicated separately. If we transpose possibly complex-valued vectors/ matrices, this is indicated by applying: $(\ )'$.

Optimum recursive minimum mean square error estimator (Kalman predictor):

In accordance with the above considerations, the linear system and observation models with which we attempt to describe the behaviour of the real world can be described by:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{w}(n) \quad (1.9)$$

$$\boldsymbol{y}(n) = \boldsymbol{C}\boldsymbol{x}(n) + \boldsymbol{n}(n), \quad (1.10)$$

where the state transition matrix $\boldsymbol{A}$ and the observation matrix $\boldsymbol{C}$ are supposed to be known; $\boldsymbol{w}(n)$ is the plant (or system) noise; and $\boldsymbol{n}(n)$

[1]Due to the plant and observation noises, $\boldsymbol{P}(n+1)$ is not only a function of $\boldsymbol{P}(n)$.

stands for observation noise. Concerning the noise processes, we suppose that they are zero-mean white Gaussian noise processes that are independent of each other and the state of the system. Their covariance matrices are:

$$\boldsymbol{Q}(n) = E\{\boldsymbol{w}(n)\boldsymbol{w}^T(n)\},\ \boldsymbol{R}(n) = E\{\boldsymbol{n}(n)\boldsymbol{n}^T(n)\}. \tag{1.11}$$

In the following, we repeat Fig. 1-2 and indicate the differences in the model (see Fig. 1-4). Formally, the only difference is that the observed system has been excited by the plant (system) noise and the observed values are perturbed by the addition of observation noise.

The measurement process assigned to the extended model is also an observer:

$$\begin{aligned}\hat{\boldsymbol{x}}(n+1) &= \boldsymbol{A}\hat{\mathrm{x}}(n) + \boldsymbol{G}(n)\boldsymbol{e}(n)\\ &= \boldsymbol{A}\hat{\mathrm{x}}(n) + \boldsymbol{G}(n)[\boldsymbol{y}(n) - \boldsymbol{C}\hat{\boldsymbol{x}}(n)],\end{aligned} \tag{1.12}$$

which differs from (1.3) only in that matrix $\boldsymbol{G}$ is a function of discrete time (see Fig. 1-5).

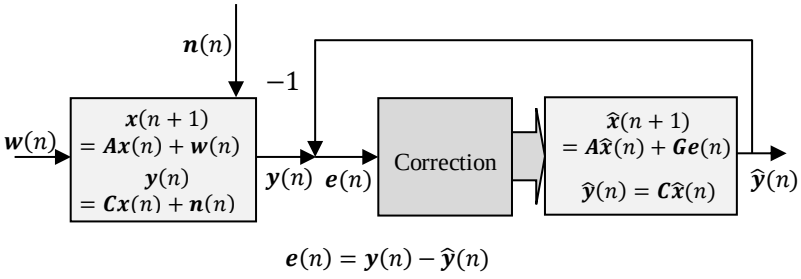

Fig. 1-4. Measurement using the Kalman predictor.

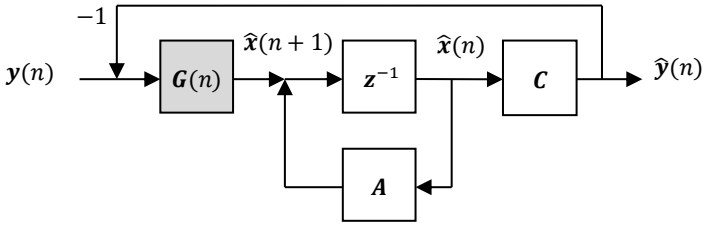

Fig. 1-5. The Kalman predictor as an observer.

We are looking for matrix $\boldsymbol{G}(n)$ (the so-called predictor gain), which minimizes the trace of the error covariance matrix:

$$\begin{aligned}&\boldsymbol{P}(n+1)\\&= E\{[\boldsymbol{x}(n+1)-\hat{\boldsymbol{x}}(n+1)][\boldsymbol{x}(n+1)-\hat{\boldsymbol{x}}(n+1)]^T\}\\&= E\{\boldsymbol{\varepsilon}(n+1)\boldsymbol{\varepsilon}^T(n+1)\}.\end{aligned} \tag{1.13}$$

To find this minimum, we compute the derivative by $\boldsymbol{G}(n)$ of the trace of (1.13), i.e. that of $tr\boldsymbol{P}(n+1) = E\{\boldsymbol{\varepsilon}^T(n+1)\boldsymbol{\varepsilon}(n+1)\}$. The optimum value of $\boldsymbol{G}(n)$ is given as the derivative equal to zero.

If we substitute the system model (1.9), the observation model (1.10) and the recursive predictor model (1.12) into (1.13), then the optimum $\boldsymbol{G}(n)$ vector can be derived from the following equation:

$$\begin{aligned}&\frac{\partial tr\boldsymbol{P}(n+1)}{\partial \boldsymbol{G}(n)}\\&= \frac{\partial tr\{[\boldsymbol{A}-\boldsymbol{G}(n)\boldsymbol{C}]\boldsymbol{P}(n)[\boldsymbol{A}-\boldsymbol{G}(n)\boldsymbol{C}]^T+\boldsymbol{Q}(n)+\boldsymbol{G}(n)\boldsymbol{R}(n)\boldsymbol{G}^T(n)\}}{\partial \boldsymbol{G}(n)} = \boldsymbol{0}.\end{aligned} \tag{1.14}$$

In (1.14), we have utilized our prior knowledge concerning independence:

$$E\{\boldsymbol{\varepsilon}(n)\boldsymbol{n}^T(n)\} = \boldsymbol{0}, E\{\boldsymbol{w}(n)\boldsymbol{\varepsilon}^T(n)\} = \boldsymbol{0}, E\{\boldsymbol{w}(n)\boldsymbol{n}^T(n)\} = \boldsymbol{0}, \tag{1.15}$$

i.e. the fact that the nth value of the noise processes is independent of the nth value of $\boldsymbol{x}(n)$ and $\hat{\boldsymbol{x}}(n)$. (The noise-effects only influence the samples indexed by $n+1$.)

After completing the derivations in (1.14), the optimum predictor gain can be expressed as:

$$\begin{aligned}&\frac{\partial tr\boldsymbol{P}(n+1)}{\partial \boldsymbol{G}(n)} = -2[\boldsymbol{A}-\boldsymbol{G}(n)\boldsymbol{C}]\boldsymbol{P}(n)\boldsymbol{C}^T+2\boldsymbol{G}(n)\boldsymbol{R}(n) = \boldsymbol{0}.\\&\boldsymbol{G}(n) = \boldsymbol{A}\boldsymbol{P}(n)\boldsymbol{C}^T[\boldsymbol{C}\boldsymbol{P}(n)\boldsymbol{C}^T+\boldsymbol{R}(n)]^{-1}\end{aligned} \tag{1.16}$$

The following derivation rules are applied:

$$\frac{\partial tr[\boldsymbol{X}\boldsymbol{W}\boldsymbol{X}^T]}{\partial \boldsymbol{X}} = \boldsymbol{X}\boldsymbol{W}^T+\boldsymbol{X}\boldsymbol{W}; \frac{\partial tr[\boldsymbol{X}\boldsymbol{W}]}{\partial \boldsymbol{X}} = \boldsymbol{W}^T; \frac{\partial tr[\boldsymbol{W}\boldsymbol{X}^T]}{\partial \boldsymbol{X}} = \boldsymbol{W}, \tag{1.17}$$

where matrix $\boldsymbol{W}$ is independent of matrix $\boldsymbol{X}$.

Using the notation $\boldsymbol{F}(n) = \boldsymbol{A}-\boldsymbol{G}(n)\boldsymbol{C}$, the covariance matrix of the

estimation error expressed in (1.13) can be expressed in the following way:

$$\begin{aligned}&\boldsymbol{P}(n+1)\\&= E\left\{\begin{matrix}[\boldsymbol{F}(n)\boldsymbol{\varepsilon}(n)+\boldsymbol{w}(n)-\boldsymbol{G}(n)\boldsymbol{n}(n)]\\ \times[\boldsymbol{F}(n)\boldsymbol{\varepsilon}(n)+\boldsymbol{w}(n)-\boldsymbol{G}(n)\boldsymbol{n}(n)]^T\end{matrix}\right\}\\&= \boldsymbol{F}(n)\boldsymbol{P}(n)\boldsymbol{F}^T(n)+\boldsymbol{Q}(n)+\boldsymbol{G}(n)\boldsymbol{R}(n)\boldsymbol{G}^T(n)\end{aligned} \tag{1.18}$$

Remarks: Based on (1.18) it can be seen that the covariance matrix of the estimation error changes due to three effects:

- The reducing effect, thanks to the contractivity property of the matrix $\boldsymbol{F}(n)=\boldsymbol{A}-\boldsymbol{G}(n)\boldsymbol{C}$, as has already been discussed with respect to the observer, and which, due to the quadratic criterion, appears here in squared form.
- The statistical error-increasing effect, represented by the covariance matrix of the plant noise, which is present here since the value $\boldsymbol{w}(n)$ influences $\boldsymbol{x}(n+1)$, the predicted value based on the previous state $\boldsymbol{x}(n)$.
- Another statistical error-increasing effect, represented by the covariance matrix of the observation noise and weighted by the predictor gain, which is presented here as the value $\boldsymbol{n}(n)$, influences $\widehat{\boldsymbol{x}}(n+1)$, the predicted value based on the previous estimate $\widehat{\boldsymbol{x}}(n)$.

Equation (1.18) can be written in a more compact form. Let us replace $\boldsymbol{F}(n)=\boldsymbol{A}-\boldsymbol{G}(n)\boldsymbol{C}$ and by expressing the first term in more detail:

$$\begin{aligned}\boldsymbol{P}(n+1)=&\boldsymbol{A}\boldsymbol{P}(n)\boldsymbol{A}^T-\boldsymbol{A}\boldsymbol{P}(n)\boldsymbol{C}^T\boldsymbol{G}^T(n)-\boldsymbol{G}(n)\boldsymbol{C}\boldsymbol{P}(n)\boldsymbol{A}^T\\&+\boldsymbol{G}(n)\boldsymbol{C}\boldsymbol{P}(n)\boldsymbol{C}^T\boldsymbol{G}^T(n)+\boldsymbol{G}(n)\boldsymbol{R}(n)\boldsymbol{G}^T(n)+\boldsymbol{Q}(n)\end{aligned} \tag{1.19}$$

If we combine the fourth and fifth term with the second expression of (1.16), we have $\boldsymbol{G}(n)[\boldsymbol{C}\boldsymbol{P}(n)\boldsymbol{C}^T+\boldsymbol{R}(n)]\boldsymbol{G}^T(n)=\boldsymbol{A}\boldsymbol{P}(n)\boldsymbol{C}^T\boldsymbol{G}^T(n)$, which cancels the second term of (1.19). What remains is the first, third and sixth terms:

$$\boldsymbol{P}(n+1)=[\boldsymbol{A}-\boldsymbol{G}(n)\boldsymbol{C}]\boldsymbol{P}(n)\boldsymbol{A}^T+\boldsymbol{Q}(n) \tag{1.20}$$

In summary:

If the system model has the form $\boldsymbol{x}(n+1)=\boldsymbol{A}\boldsymbol{x}(n)+\boldsymbol{w}(n)$, and the observation model is given by $\boldsymbol{y}(n)=\boldsymbol{C}\boldsymbol{x}(n)+\boldsymbol{n}(n)$, then the equations