# Selected Topics on Antenna Synthesis

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## *The Unity of Mathematical Formalisms and Technical Sense*

By

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To the memory of my supervisor and colleague Vadim Nikolaevich Dymsky, who encouraged me to believe in the harmony of electromagnetic laws and strive for the unity of mathematical formalism and its physical sense.

“Students should not learn thoughts—they should learn to think”
Immanuel Kant (1724–1804)

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# PREFACE

I firmly believe that the core quality of any technician that makes them successful in the profession is a special style of mind, an engineering style of thinking. The features of this book stemmed from the fact that the author set himself the goal of not only providing knowledge on the designated issue, but also, perhaps mainly, honing the engineering style of thinking. Antenna synthesis provides an excellent opportunity to achieve this goal, since it differs from many other technical disciplines in the organic unity of complex mathematical abstractions and physical manifestations, which is a vital aspect of the engineering style of thinking. The photo in Fig. 3-3 can serve as a visual confirmation of this fact. It shows a setup that implements such a mathematical abstraction as an adjoint operator in the form of an electromagnetic field. Amazing! Isn't it?

**The mentioned features are as follows.**

- First, the emphasis is not on the results of the synthesis, but on methodological aspects: how to formulate and solve the corresponding problems. At the same time, physical meaning and compliance with practical needs are the focus of attention. For example, in the synthesis of a given amplitude antenna radiation pattern (ARP), the deviation of ARPs in the form, taking into account random errors of reproduction of the nominal solution, need to be considered.
- Second, it is best to teach creativity with your own ideas and developments. That is why the book is not an overview of the works in the field of antenna synthesis, so its bibliography contains only 49 references. The subject matter of the book is what the author did.
- Third, Chapter 1 is devoted to the basics of functional analysis, without which it is hardly possible to understand the essence of antenna synthesis. This material is of broader interest, as is the Introduction, which discusses what an engineering style of thinking is.

**Who is this book for?** Naturally, it is primarily for those who are busy with or interested in the development of antennas, since antenna synthesis is the theoretical basis of such activities.

In addition, due to its focus on the formation of technical thinking, it is useful to students, graduate students and young engineers of many specialties and first radio communications.

**As for the style of the book**, at the end of the chapters there are training tasks and Mathcad programs in the Appendices, which allow the interested reader to think over or conduct their own research on the issues posed.

In some places, the author's comments interrupt the text. Smaller fonts and an enlarged left margin highlight them so you can skip them.

Graphic visualization of everything is one of the facets of technical thinking. It is therefore not surprising that the book has so many figures.

## Acknowledgments

The author considers it his pleasant duty to express his gratitude to his colleagues for many years of communication and cooperation, thanks to which his professional philosophy has become what it is.

# INTRODUCTION: ENGINEERING STYLE OF THINKING

The field of activity associated with techniques and technologies is extremely diverse and extensive. Nevertheless, some universal psychological qualities largely predetermine the ability to assimilate the relevant knowledge, to develop necessary skills, and success in the profession. A special mindset, which could be called an engineering way of thinking, is the most important, I think.

The list that indexes different facets of thinking (thinking styles, methods and ways of thinking) consists of more than a hundred items. Not surprisingly, there are several scales used by psychologists to classify styles of thinking in accordance with their penchant for philosophy, sociology and management [1, 2], or psychology [3].

Along with Kirton's adaptive vs. innovative cognitive styles [3], V. Gulenko defined four styles of thinking from another point of view [4]: causal, dialectic-algorithmic, holographic and vortex. Note that psychologists understand style of thinking merely as universal cognitive techniques without its relation to the subject of a thought. The engineering style of thinking, as I understand it, is much narrower and refers to the techniques of comprehending the world of technique and technologies. It relates closely to the criteria of competence, which is a very important measure of purpose and quality of training in the high school [5, 6].

## Of what the engineering style of thinking consists

The following mental qualities and skills form the engineering style of thinking:

Firstly, curiosity and an overwhelming desire to understand the essence of any natural phenomena and the laws of the technical world. Of course, human curiosity does not restrict itself by the field of technique. Once during the spring I thought, "The spring sap will start moving. I wonder how the sap reaches the foliage at an altitude of 15 or 20 meters. What makes it rise? What pushes it or pulls it?" My reader, try and answer this. If you think of pressure, it should be of unbelievable values of 1.5 atm. Think of something else.

Secondly, a strong predilection for the quantitative analysis of physical processes and things. Three skills are of importance here: 1) to select measurable characteristics and parameters of the essential manifestations; 2) to ignore irrelevant factors; 3) to identify quantitative relationships and laws that govern the world of technology.

Thirdly, a proper arsenal of basic mathematical methods and techniques needed to solve a wide range of tasks in the relevant technical area.

Fourthly, the ability to perceive the physical nature of analyzed processes, hidden in mathematical formalism. This not only gives a meaningful interpretation of the obtained results, but also can often speed up calculations through the use of heuristic assumptions and reasonable simplifications. Besides, very often it happens that among strict mathematical solutions, just a few comply with the terms of technical feasibility or other practical limitations, and we have to separate them from the formal solutions. To do this we must understand the physical nature of the processes and quantitative laws reflected by mathematical expressions.

In Fig. 1, the diagram illustrates the role of thinking in an engineer's activities. Just like any chart of a socio-psychological nature, it does not require detailed comments. We note only that there are two main functions of engineering thinking, as a summand of engineer competencies. It is the capability to plunge easily from the real physical world into the virtual world of mathematical descriptions (mathematical formalism), and then just as easily make the reverse transition to the physical reality, to the interpretation and understanding of the formal results.

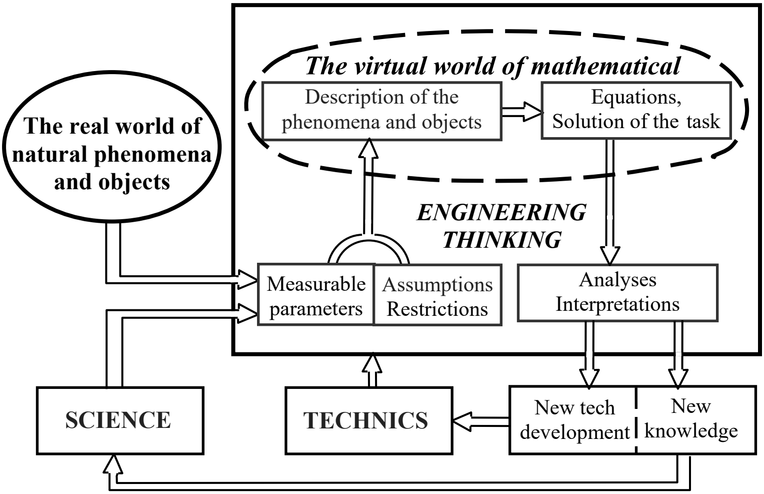



Figure 1: Activities of an engineer

It is impossible and hardly necessary to devise several (say, ten) specialized lessons, the result of which the engineering way of thinking would have been acquired by all the students. This is the aim of the whole process of technical training in its entirety.

Education, in general, is a product of the interaction of three subjects: the Authority's manager, the teacher and the student (Fig. 2).

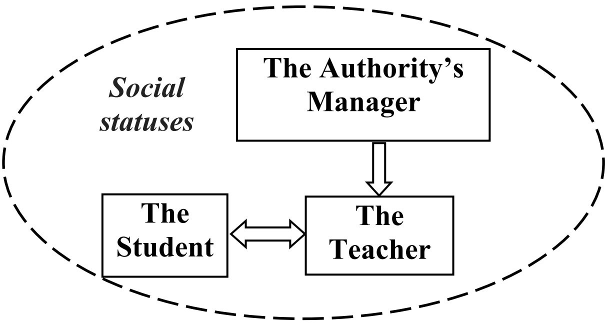



Figure 2: Three subjects of the educational process

Certainly, educational standards and curriculums (partly the product of officials' efforts) are necessary and important. However, they have a very distant relation towards the formation of an engineering style of thinking.

How clever and ambitious (or not) school graduates are, those who go to universities is largely dependent on social prestige of the profession they have chosen. While prestige is a moral issue, in relation to this topic it has a very clear quantitative measure: the income that an average employee receives in the profession.

Fig. 2 presents the subjects of educational processes along with social conditions that, catalyst-like, "do not participate in the reaction, but can greatly speed up or slow down it." Always and in all circumstances, the professionalism of teachers is the most essential thing concerning the quality of education. Over many years of teaching at the Electronics and Telecommunications Institute (courses "Antennas and microwave devices," "Electrodynamics" and others) pedagogical methods were worked out, that can be hopefully useful at teaching any technical discipline.

## Psychological issues

1. It is very helpful to keep in mind that terms, definitions, physical and functional characteristics are as vital for any technical discipline, as the structural elements (bricks, beams, logs, etc.) for the building. That is why the question "What is it?" is pronounced so often in exams. Draw the students' attention to a pleasant and inspiring fact that each term and each definition contains a small (easily exhaustible) amount of information. No matter how paradoxical it may sound, any student can and should know as clearly as professors or academicians what the spectrum of a signal, a band pass filter, the electric field vector, the antenna gain etc. are.

2. A good student or a qualified engineer can make the right judgment whether he knew what was discussed, or not. So, sometimes he answers, "I do not know." An unsuccessful student or engineer, as a rule, has no certainty on this matter. Typically, this is due to unsystematic learning—attempts to grasp the current topic without mastering previous often-simpler themes.

A good piece of advice to such people is the following. Develop the habit, when faced with difficulties, of asking yourself two questions: a) whether the terms of the studied text are clear, b) whether what it is about is clear (not in detail, but in general). If necessary, leaf through the textbook again. In critical situations, seek someone's help. In addition, of course, train your inner voice: "I know it," or "I don't understand it." Such a strategy will not only result in successful learning but also will reduce the time consumption.

3. For so many students and engineers, a formula is a fetish: it is impossible to derive it and to find it in a textbook is the only way. There is a need to convince them that not even all the books together contain the answers to all situations. A strategy that is much more practical is to understand the general principles and laws and remember a small number of the corresponding formulas and relations. This will make it possible to get a formula for a particular case, without wasting time on an unsuccessful search.

4. Some do not know how to control the calculations, find errors and assess the acceptability of results. There are standard techniques that help to cope with these limitations:

First, dimensional analysis is elementary and surprisingly effective. If the obtained formula results in calculating the sine of 3 kg or the logarithm of 1 mm or something like that, do not do any calculations and look for errors. If this formula is from a textbook, it is likely you do not correctly perceive the notation of the variables.

Second, it is useful to check formulas at special values of their variables, for which you know what it should be: in case of a short circuit or discontinuity, at zero or infinitely large value, etc.

5. A very useful habit is to not rush into solving a new task, but try to understand it as a whole, find out its general structure or its features in order to pave the best way to a solution. I call this tactic a "shell approach."

## Nature and its description

A feature of the engineering profession as the brainchild of Applied Physics and Applied Mathematics is that it deals with something not completely known. For example, electrodynamics, based on paradigm of the macroscopic, explores notion of electromagnetic field (EMF). It elucidates the relevant lows that create the solid foundations for developing antennas, high-frequency devices, radio systems, and many other applications. Although a comprehensive answer to the question "What is EMF itself?" still awaited. Perception of the real complex and multifaceted world of nature through its manifestations is the fundamental principle of engineering thinking. Regularities of these manifestations established as strict quantitative relationships pave the way for the use of physical phenomena in technology.

The transition from the world of reality to the world of quantitative descriptions of its manifestation begins with an introduction of significant measurable characteristics of the considered phenomenon, structure or device. To the engineer, it is useful to bear in mind that the choice of these characteristics is not strictly predetermined. They are largely the result of professionals' conventions.

For example, apart from in our perception, the electric field vector, the "famous" vector **E**, does not exist. What does exist is an ability of EMF to exert a force **F** on a particle of matter, which is arranged in the given point in space and has a charge q. Coulomb's law has proven useful in characterizing this property of EMF by vector **E**=**F**/q. Similarly, Ampere's law regarding the force interaction of electric currents had led to an agreement to characterize the magnetic field associated with the currents by vector **H**. The pair of vectors **E** and **H** so fully characterize the properties of EMF, that specialists perceive these vectors as EMF itself.

Such identification we use very often. For instance, we say, "transmit energy," although the energy does not exist as a material something. After all, it is a property of a matter, or a body, or a field, and apart from its carrier does not exist. The same situation concerns information: it is customary to speak of it, as if it is something tangible and exists in itself. We must understand the conventionality of such expressions used for brevity.

## Creativity and ingenuity

To exercise the creativity and ingenuity inherent to engineering thinking, unpretentious crafty tasks are useful. For the sake of concreteness, here are two examples of such tasks, which do not require the knowledge beyond the high school physics. I hope that they will please inquisitive readers, who, I hope, will try to answer the tasks before reading the paragraphs "***If you still have not coped with the task****, read the answer*."

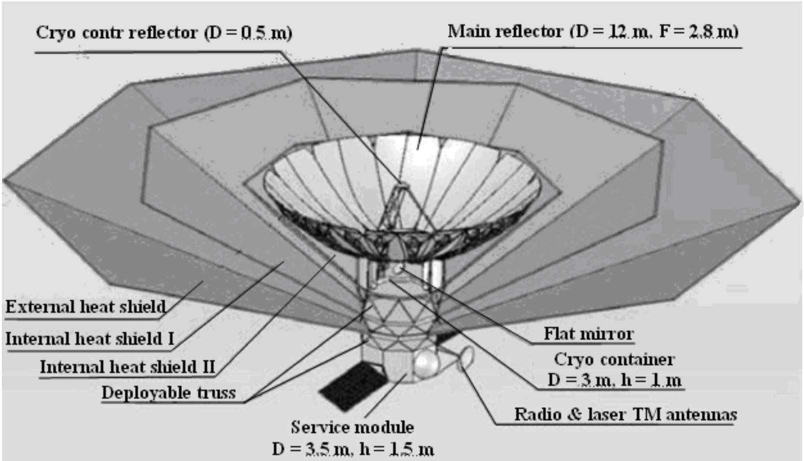



Figure 3: Layout of the 12 m space telescope (a picture from [7])

The first of them I thought up under the impression of the Russia-European project named "Millimetron." The plan is to launch a large space telescope for exploring the universe [7]. Fig. 3 depicts its overall view, which is so grandiose that I cannot resist showing it. Among others, there are two amazing moments. 1) To achieve the desired angular resolution and needed mirror efficiency the overall surface accuracy should be 10 microns for the whole 12 meter dish surface! 2) To detect the relict radiation (verifying the theory of primary blast is one of the many aims of the project), the required sensitivity is so high that no super receiver can achieve it. The only option is photon counters [8]!

The last fact provoked the idea of the following **task A**.

**Task A**

*Conditions:*

Let us assume that a signal of a feeble cell phone's transmitter of the power $P$ = 5 mW, is radiated uniformly in all directions of a full solid angle. The carrier frequency of the signal is ν = 1 GHz.

*What to do:*

Estimate the distance R from the cellular phone, which can serve as a boundary between two areas: one where the phone field exists in the form of a continuous electromagnetic wave (EMW) and the other where it become a discrete flux of photons (Fig. 4).

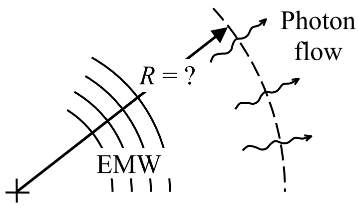



Figure 4: The subject of Task A

*Some tips:*

1) Note that EMF can be something continuous in space (for example, EMW) only under the condition that the density of photons—elementary particles of the field—is extremely high.
2) Photon energy $E$ is determined by Planck's relation $E = h\,\nu$, where $h = 6.626\ 10^{-34}$ [J s] is the Planck constant and ν is the frequency.
3) EMF propagates in free space in the form of a spherical wave, therefore the density of flaw of electromagnetic energy subsidizes along the distance $r$ in the proportion of $1/r^2$.

*Discussion:*

Of course, this task has no exact single-valued answer, because the condition that determines the boundary between the electromagnetic wave and the "photon discrete flux" is uncertain—there is no distinct determination. In fact, in this lies the usefulness of the task—a reader must make his own independent decision based on common sense. It is very typical for engineers to need to make a decision in the situation of essential uncertainty.

***If you still have not coped with the task,*** *read the answer:*

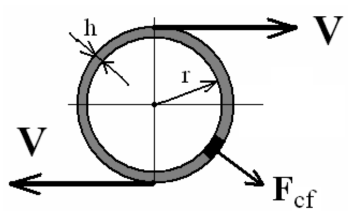
As to the task, assume that we have to consider EMF as a discrete flux of photons if less then $N_{ph} = 10^5$ photons fly through a surface area $dS$ of 1 mm$^2$ during a time span $dt$ of 1c. Then the obvious equation $P\ dt\ dS\ /\ (4\pi R^2) = N_{ph}\ h\ \nu$ takes place, which results in the estimation $R \approx 77.5$ km.

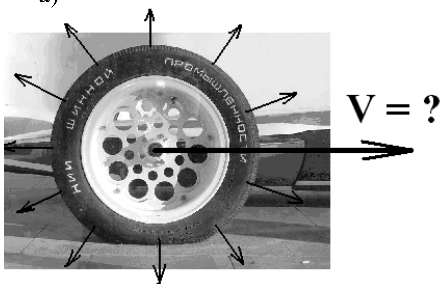
**Task B**

*Conditions:*

A tire of a rear wheel of a sports car became flat (Fig. 5a). There was no spare wheel in the car. A car garage was 5 km away along a highway. The driver became very distressed, since the tire would inevitably come into disrepair while driving or towing to the garage. Fortunately, he was a good engineer before becoming a professional driver. Therefore, clever thoughts came to his mind (Fig. 5b). He closed his eyes, moved his lips, shouted “Eureka!” and pressed the accelerator pedal down.


a)




b) c)

Figure 5: A flat tire in three situations: a) a standing car; b) the driver’s thinking; c) a moving car

*What to do:*

Estimate the speed *V* at which the driver had to drive in order to save the tire (Fig. 5c).

*Some tips:*

1) Note that the centrifugal force can essentially substitute the force of air pressure expanding the tire in the normal situation.
2) Neglecting the sidewalls, we can consider a tire as a section of cylindrical tube (Fig. 5b).

*Discussion:*

This task, like the previous one, has no exact quantitative answer. However, we deal with assessing. Therefore, physical dependences and common sense fight uncertainty and pave the way to a meaningful estimation. Indeed, the pressure of a pumped tire $P_0$ [Pa] gives rise to the force $F_0 = P_0\ \Delta S$ [N] on area ΔS of the tread surface. On the other hand, the centrifugal force $F_{cf}$ introduced by the rotation of a tire due to moving the car with a speed $V$ is $F_{cf} = \rho\ (h\ \Delta S)\ V^2 / r$ where ρ, $h$ and $r$ are the density, the depth and the radius of the tire, respectively. The desired estimation results from equation $F_{cf} = F_0$.

***If you still have not coped with the task,*** *read the answer:*

Let us assume that the pressure $P_0$ is about 2 atm, the wheel radius $r$ is 25 cm, tire thickness $h$ is 25mm, and the tire density ρ exceeds the density of water due to steel cords at $1.1\ 10^3$ kg/m$^3$. After transforming all the values to the uniform units, the estimation V ≈ 154.5 km/h comes in force.

# ABBREVIATIONS

| | |
|---|---|
| AA | antenna array |
| AAA | adaptive antenna array |
| APhD | amplitude and phase distribution |
| ARP | antenna radiation pattern |
| CCPW | converging cluster of plane wavs |
| EMF | electromagnetic field |
| EMW | electromagnetic wave |
| PhF | phase front |
| LPhC | local phase center |
| LPF | low-pass filter |
| MMRA | modified minimal residual algorithm |
| MRA | minimal residual algorithm |
| PhC | phase center |
| PEMW | plane electromagnetic wave |
| RMSD | root-mean-square deviation |
| WLM | Woodward-Lawson method |
| WV | weighting vector |

| | |
|---|---|
| $k = 2\pi/\lambda$ | wave number in free space |
| […] | dimension of physical value |

$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$ is the so-called Kronecker delta

$\mathrm{sinc}(x) = \sin(x) / x$ is the so-called sinc-function

# CHAPTER ONE

# SOME FUNDAMENTALS OF FUNCTIONAL ANALYSIS

## 1.1. Introduction

To master the topics of the following chapters with understanding, the knowledge of some fundamentals of functional analysis is necessary. The corresponding issues, as a rule, lie on the periphery of such university courses as “Special sections of mathematics” or “Methods of computational mathematics.” My teaching experience shows that many students do not know that. However, if you are an advanced reader, you may skip this chapter. Nevertheless, in my opinion, Section 1.3 is worth studying or, at least, reading. It outlines the ideology and effective algorithms used to analyze, synthesize and design not only antennas but also many other radio and electronics devices.

I got acquainted with functional analysis [9, 10], a fascinating branch of mathematics, during my postgraduate studies, when I was working on a dissertation devoted to the problems of antenna synthesis. Since then, I have never ceased to admire how exquisitely and elegantly mathematicians are able to comprehend the fundamental structural concepts of the real world and to convey them in a generalized form, naturally, to the world of mathematical abstractions. It is impossible not to admire the effectiveness of methods and algorithmic tools generated from these abstractions to solve various tasks. In this there is something very much in common with the engineering style of thinking, more precisely its facet, connected with the transition from reality to quantitative descriptions reflecting the main manifestations of physical phenomena or a technical object.

Therefore, I hope that this chapter will provide the reader with ideas about universal means of obtaining the best approximation to the desired functional characteristics of the devices designed. At the same time, it will contribute to the formation of a certain philosophical framework that serves as reliable support and a compass in dealing with complex technical problems.

Of course, only the basic issues of functional analysis will be discussed here without in-depth nuances, but with an emphasis on aspects that are important for the synthesis of antennas.

## 1.2. Metric and space of functions

Let us think about "What is space?" Try to define it not from the position of its physical essence (whether it is emptiness, whether a special kind of matter called ether), but in a purely ordinary sense as a universal container (without boundaries) for all that exists. By the way, any material object, existing in space, has spatial properties itself. Back to the question, what sense is behind our intuitive feel of space, the space in which we live? The question sounds very simple, but it is not easy to find a meaningful answer for it.

Mathematics, with exceptional elegance, answers this question: "the space X is the set of points $\{x\}$, to each pair of which there is associated a number, called the distance $\rho(x_i, x_j)$ or the metric[1] of space." The distance/metric, more precisely the rule of its definition, must satisfy the following axioms: for arbitrary points $x_i$, $x_j$, $x_k$ in X

- the distance is a positive real value, i.e. $\rho(x_i, x_j) \geq 0$;
- if $x_i = x_j$, then $\rho(x_i, x_j) = 0$, and vice versa (!) (the axiom of identity);
- $\rho(x_i, x_j) = \rho(x_j, x_i)$ (the axiom of symmetry);
- $\rho(x_i, x_j) \leq \rho(x_i, x_k) + \rho(x_k, x_j)$ (the axiom of triangle inequality).

The first axiom (the distance cannot be negative) follows from the remaining axioms. Therefore, strictly speaking, it can be omitted, and mathematicians do this [10]. However, for us engineers, it is worth mentioning this property explicitly because it expresses an important fragment of intuitive human notions of distance in space. In some mathematical books, it is also present.

It is obvious that these axioms are not the invention of mathematicians. It is the formalization of our intuitive perception of space and distance in it. In particular, the triangle inequality corresponds to the fact that the path from $x_i$ to $x_j$ is shorter or at least not longer than the path first traversed from $x_i$ to $x_k$, and then from $x_k$ to $x_j$. After intuition transformed into clear mathematical relations, huge possibilities arise for the use of the concept

[1] If the elements {x} of the set X are functions or objects of nature other than ordinary (Euclidean) points, then the term "distance" is not linguistically right. Therefore, the term "metric" is preferable, although both have the same meaning.

of distance as a quantitative measure of the closeness of elements of any nature, in particular functions.

Consider the set $\mathcal{F}$ of functions $f(x)$, referring to a particular function $f_i(x)$ and treating it as an abstract point of $\mathcal{F}$. If we define a rule according to which the quantity $\rho(f_i, f_j)$ satisfying the above axioms is determined (computed), then the concept of the distance between points in the set $\mathcal{F}$ (the distance between functions!) is introduced, and the set turns into a functional space—a space of functions. Since linguistically the phrase "distance between two functions" sounds strange, it is customary to use the more abstract term "metric" as applied to the functional space.

Among the vast variety of possible metrics and the functional spaces generated by them, two metrics are most often used. The first is a metric defined by the equality:

$$\rho_{C_2}(f_1(x), f_2(x)) = \sqrt{\int_a^b \left| f_1(x) - f_2(x) \right|^2} \tag{1-1}$$

which means the root-mean-square deviation (RMSD) of the functions. The relevant space is commonly denoted as $C_2$.

The second is a metric defined as follows:

$$\rho_L(f_1(x), f_2(x)) = \max_{a \le x \le b} \left| f_1(x) - f_2(x) \right| \tag{1-2}$$

that measures the value of the maximum deviation of the functions in the interval $(a, b)$ where they exist. This metric name is "linear," and L usually denotes the functional space that it generates.

Figure 1-1 illustrates the difference in how the functions approach the same desired function $f_0(x)$ if the $C_2$ or L metric is used. Taking into account that the metric Ex. 1-1 stands for RMSD, we can see that the value $\rho_{C_2}(f_1(x), f_0(x))$ is remarkably bigger than $\rho_{C_2}(f_2(x), f_0(x))$, therefore, the function $f_2(x)$ approaches $f_0(x)$ better than $f_1(x)$. The metric Ex. 1-2 controls local deviations and therefore, contrary to the previous situation, the value $\rho_L(f_1(x), f_0(x))$ is smaller than $\rho_L(f_2(x), f_0(x))$, and the function $f_1(x)$ approaches to $f_0(x)$ nearer than the function $f_2(x)$.

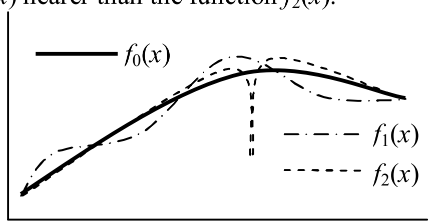



Figure 1-1: Functions $f_1(x)$ and $f_2(x)$ approaching the function $f_0(x)$

## 1.3. Hilbert space of functions

### 1.3.1. Introductory note

Another, perhaps even more impressive, example of the extremely sophisticated generalization of the notion of our Euclidean space to abstract mathematical spaces is the notion of orthogonality, in particular, the orthogonality of functions.

In common Euclidean space, the space in which we live, each point $x$ can be determined by the radius vector **x**—a line segment directed from the coordinate system's origin to the point $x$. The angle between the vectors and, in particular, their perpendicularity represents an important property of Euclidean space. To represent an arbitrary vector **x** as the sum of the basis vectors $\{\xi_n\}$: $\mathbf{x} = \sum_n a_n \xi_n$ it is extremely convenient to use a system of mutually perpendicular unit vectors $\{\xi_n\}$ forming the so-called orthonormal basis. The fact is that in this case each of the coefficients $a_n$ of the expansion is determined independently of the others, as the projection of the vector **x** onto the basis vector $\xi_n$: $a_n = (\mathbf{x}, \xi_n)$. If the basis vectors are not perpendicular to each other, then the procedure for finding the coefficients $a_n$ becomes much more complicated and, as a result, requires solving a system of algebraic equations.

Naturally, for function spaces the expansion of an arbitrary function $f(x)$ over the system of basis functions $\{g_n(x)\}$ of $f(x) = \sum_n a_n\, g_n(x)$ plays the same important role as for the vector space. It is tempting that the basis functions have the same property of mutual "perpendicularity." However, how do you define this property for functions? Can we introduce the notion of "angle" between functions and the property of their "perpendicularity"? Is it possible to introduce something analogous to the notion of "angle" between functions or the property of their "perpendicularity"? This problem is definitely more complicated than the notion of the distance between functions. Indeed, perceiving distance as a quantitative measure of the deviation of functions from each other, we can easily guess we should use the standard deviation as the distance between them. To introduce the concept of "perpendicularity" of functions, a much more sophisticated idea is required. To solve this problem, mathematicians generalize the notion of an inner product.