Dimensional Regularization and Non-Renormalizable Quantum Field Theories

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By

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Chapter 1

Introduction

1.1 Preface

This book is a quantum field theory treatise that aims to simplify the subject by including some mathematical techniques devised in the 50's and 60's which have not yet percolated on the physicists' family. With them what one might call a Schwartz' distributions approach to quantum field theory can be devised. Our purpose is twofold: i) to make accessible the concomitant design, on the one hand, and ii) to spread it by making the construct more well known, on the other. The conjunction of these mathematics with the dimensional regularization approach of Giambiagi and Bollini vastly expands the outreach of quantum field theory (QFT), as we will show here.

The main difficulty of conventional QFT concerns infinities. QFT is plagued by puzzling infinities, that emerge when dealing with basic entities called quantum propagators. One is not exaggerating by asserting that quantum propagators (QP) are the very quantities around which quantum field theory revolves. QP constitute its main predictors. The central idea to be advanced here is that CP are Schwartz' distributions (SD). This book is mainly concerned with this last statement. We will in it step-by-step develop an SD approach to quantum field theory, which is absent from all extant QFT textbooks.

1.2 Why is this book needed?

As stated in the Preface, the main difficulty of quantum field theory (QFT) concerns infinities. QFT is plagued by puzzling infinities. For instance, a familiar example is posed by the energy density of a static electric field (EE) [1]. It is proportional to the square of its EE intensity E. The EE intensity at a distance r due to a charge Q uniformly distributed over a spherical surface of radius R is, in turn, proportional to Q/r^2 for $r > R$ and 0 for $r < R$ [1]. Therefore, the total energy V of the field is given by [1],

$$V = \frac{Q^2}{6\pi c^2}(1/R), \tag{1.2.0.1}$$

with c light's speed. This quantity is called the self-energy of the particle. It obviously diverges as $R \to 0$. Thus, an EE emanating from a point charge displays infinite energy. Theorizing with point particles, as in classical analysis, may produce valid results, but when it involves manipulations of the particles' fields, these results may be contaminated by the infinite energy of the field of a point charge [1]. The QFT infinities appear in dealing with quantum propagators. Quantum propagators are in a sense the quantities around which quantum field theory (QFT) revolves. They are its main predictors and can be regarded as Green functions and, crucially, also as *Schwartz' distributions (SD)*. This book is mainly concerned with this last statement. We will in it develop an SD approach to quantum field theory, which is absent from all extant QFT textbooks.

The infinities of QFT appear in the guise of products (or, more precisely, of convolutions) of SDs, We need these products to develop the QFT theory.

There are other ways of discussing QFT, but this book concentrates only on the SD approach to QFT. Why? Because it is the only one that permits an easy handling of non-renormalizable problems, the main QFT blockage, as we shall see in the next chapters. Thus, SDs will be one of our main protagonists here. Immediately below, we denote the propagators as ρ^{-1}. Ours is the only book, as far as we know, that discusses in detail the SD approach to QFT..

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Dimensional regularization (DR) is another main QFT issue, and our second leitmo in this effort. DR is a method for dealing with infinities that works by taking the system's dimension ν as a continuous variable. Calculations use this continuous variable ν, and at the end of the computations, the limit is taken $\nu \to d$, with $d = 1, 2, 3, 4$, as the case may be. Thus, this book is the first to detailedly discuss the two central themes of SD and DR, together with the intimate connection between them. If you read it, you will get easy access to the deepest secrets of QFT, that are not accessible elsewhere in all their gory details.

1.3 Prerequisites

What prior knowledge is needed to embark into this book? It treats very technical issues in theoretical physics, so that a kind of warning is the honest way to proceed. To make plain sailing of what follows the reader necessitates acquaintance with a variety of themes. We might say that one requires the experience equivalent to have followed two semesters of quantum mechanics (including quantum electrodynamics), one of quantum field theory, and one of functional analysis. We emphasize here familiarity with Feynman diagrams A very useful source of mathematical information (or physical one) is provided by the two mathematics/physics sections of Wikipedia [1], to which we will refer when some knowledge is required that we do not have the space here to discuss at length.

1.4 Book's organization

Chapters 2 and 3 are of a preparatory character. The first introduces the absolutely central concept of Schwartz distribution (SD) while the second provides a foretaste of the kind of infinities this book tries to deal with. Chapter 4 studies in detail the mathematical concept of Ultradistributions, a special kind of SD that will become our main weapon to face infinities. Chapter 5 discusses a more involved and historically important approach to avoiding infinities: Bollini-Giamgiaggi's dimensional regularization. In Chapter 6 we introduce our two most important physical quantities: the Feynman and Wheeler propagators, that can be regarded as special SDs. All infinities in quantum field theory can be shown to emerge as we face the convolution of ultradistibution (UD)s, that we analyze in Chapter 7. We specialize this subject to even tempered UDs in Chapter 8, to Lorentz invariant UDs in chapter 9, and to UDs of exponential type in chapter 10. Final words regarding all these UDs are found in chapter 11. The formidable mathematical apparatus developed in the preceding chapters is applied to two important physical problems in chapters 12 an 13. An epilogue closes the book in Chapter 14.

Chapter 2

Distribution Theory

2.1 Introduction to Schwartz' distributions

SDs are special linear functionals. They are continuous and one defines them over a space of infinitely differentiable functions. The pertinent derivatives are themselves generalized functions as well [1]. The most commonly encountered generalized function is the delta function. These mathematical objects are the main mathematical tools of this book. Schwartz' distributions (generalized functions) (SD) are thus mathematical objects devised with the intent of generalizing the concept of function that we learned at an early age [1].

Distribution theory (DT) constitutes a powerful tool for physics endeavors. In particular, for physicists the paradigmatic example is the Dirac's delta that we learned at college.

DT regards distributions as *linear functionals* acting on a space of so-called test functions [1]. Now, which set of test functions is appealed to constitutes the essential question for this book. A good choice is important in order to tackle difficult problems.

As stated, SD act by integration over a test function. Thus, the particular choice we make for the space of test functions, when we are given several options, is of crucial importance, because each choice leads to a different space of distributions. Selecting as test functions smooth functions with compact support leads to the conventional, standard Schwartz distributions one finds in textbooks.

One can instead appeal to a very nice space, that of smooth and rapidly diminishing (faster than any polynomial growth) test functions (also called Schwartz' test functions). This choice yields the so-called tempered distributions, that will play a protagonist's role below. They possess a well defined (distributional) Fourier transform.

We have above defined distributions as a special class of linear functionals: those that map a set of test functions into the set of complex numbers C. In the simplest situation, the set of test functions to have in mind is the set of functions $\mathcal{D} = \{\phi\}$ displaying two properties i) ϕ is *infinitely differentiable* (smooth) and 2) ϕ has *compact support.* A Schwartz' distribution (SD) T is the linear mapping $T : \mathcal{D} \to C$. For the delta function one writes $< \delta, \phi >= \phi(0)$, which entails that δ computes a test function at the origin [1]. A SD can be multiplied by complex numbers and added together, yielding another SD. They may also be multiplied by smooth functions and we keep looking at a SD [1]. We usually denote by T_f the distribution

$$T_f =< T, \phi >= \int_R f(x)\phi(x)dx, \tag{2.1.0.1}$$

for $\phi \in D$ and $x \in R$, with R the set of reals. One would like to be able to select a definition for the *derivative* of a distribution which displays the property that $T'_f = T_{f'}$ [1]. A distribution is called regular if f is an ordinary function [1]. In this case, integrating by parts one has

$$< f', \phi >= \int_R f'(x)\phi(x)dx =$$

$$[f(x)\phi(x)]_{-\infty}^{\infty} - \int_R f(x)\phi'(x)dx = - < f, \phi' >, \tag{2.1.0.2}$$

which leads us to define, for a SD in general,

$$< T', \phi >= - < T, \phi' > . \tag{2.1.0.3}$$

2.2 Explicitly Lorentz invariant distributions

In special relativity, physics equations and important quantities should have the same form in all inertial frames. This invariance of form is called Lorentz invariance and is usually expressed in Minkowski's space [1]. In this Section you will find important definitions. We consider first the case of the ν-dimensional *Minkowskian space* $\mathbf{M}_\nu$ of special relativity. Let $\mathbf{S}^{'}$ be the space of the above mentioned Schwartz *test tempered distributions*, belonging to the space of smooth and rapidly diminishing (faster than any polynomial growth) test functions [5, 6] and consider an element $g \in \mathbf{S}^{'}$. Focus attention now upon a new set $\mathbf{S}_L^{'}$ defined below. In such a context we say that $g \in \mathbf{S}_L^{'}$ if and only if

$$g(\rho) = \frac{d^l}{d\rho^l} f(\rho), \tag{2.2.0.1}$$

where the derivative is to be regarded in the sense of distributions discussed above, l *is a natural number*, $\rho = k^2 = k_0^2 - k_1^2 - k_2^2 - \cdots - k_{\nu-1}^2$, and our as yet unknown $f \in \mathbf{M}_\nu$ satisfies

$$\int_{-\infty}^{\infty} \frac{|f(\rho)|}{(1+\rho^2)^n} d\rho < \infty, \tag{2.2.0.2}$$

and is also continuous in $\mathbf{M}_\nu$. *The exponent* n *is a natural number.* We assert then that f belongs to a new set $\mathbf{T}_{1L}$. This new set is also called in a different way ($\mathbf{S}_{LA}^{'}$) via the equality $\mathbf{T}_{1L} = \mathbf{S}_{LA}^{'}$.

In the case of the *Euclidean space* $\mathbf{R}_\nu$, let $g \in \mathbf{S}^{'}$. We say that $g \in \mathbf{S}_R^{'}$ if and only if

$$g(k) = \frac{d^l}{dk^l} f(k), \tag{2.2.0.3}$$

where $k^2 = k_0^2 + k_1^2 + k_2^2 + \cdots + k_{\nu-1}^2$, with $f(k)$ satisfying

$$\int_{0}^{\infty} \frac{|f(k)|}{(1+k^2)^n} dk < \infty, \tag{2.2.0.4}$$

and $f(k)$ is continuous in $\mathbf{R}_\nu$. We assert then that $f \in \mathbf{T}_{1R} = \mathbf{S}_{RA}^{'}$, a new subset.

We will grandly, but also aptly call $\mathbf{S}^{'}_{\mathsf{LA}}$ and $\mathbf{S}^{'}_{\mathsf{RA}}$ the Fourier Anti-transformed spaces of $\mathbf{S}^{'}_{\mathsf{L}}$ and $\mathbf{S}^{'}_{\mathsf{R}}$, respectively.

Chapter 3

Analytical regularization

Let us have a taste of the problems to be tackled in this book. We will now speak of propagators, quantities that specify at time t the probability amplitudes for traveling from one site to another [1]. Precisely, one of the main problems in quantum field theory (QFT) is the *convolution* of propagators [2]. The purpose of this Section is to describe how the convolution of *two* propagators, our leit motif in this book, is to be calculated using *analytical regularization* (AR) [2, 3]. This is a sophisticated procedure employed to convert some kind of mathematical problems into other simpler ones [1]. Historically, one wished to transform those boundary value problems that can be written as Fredholm integral equations (of the first kind) involving singular operators into equivalent Fredholm integral equations of the second kind [1]. Why? Because the latter may be easier to analytically treat and can be studied with discretization schemes (like the finite differences method) because they are point-wise convergent [1]. A modified AR constituted the prerequisite step, developed by Bollini and Giambiagi (BG), that led a posteriori to the discovery of dimensional regularization [3]. All that the reader needs in this respect is explained below. BG confronted the convolution of propagators ρ^{-1} corresponding to a scalar field without mass, the simplest scenario, working in a Euclidean space, that is able to deal with spin zero particles [3]. One must consider the quantity $K(x,t;x',t') =< x|\hat{U}(t,t')|x' >$, where $\hat{U}(t,t')$ is the unitary time-evolution operator for the system taking states at time t' to states at time t. Then, the propagators' convolution

$$\rho^{-1} * \rho^{-1} = \int \frac{d^4p}{\vec{p}^2(\vec{p}-\vec{k})^2} \tag{3.1.0.1}$$

where $\rho^{-1} = K^{-2}$ is the *field propagator.* In these circumstances we define the analytic extension of the convolution by introducing a complex arbitrary number α and writing the α-generalized expression [3]

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \int \frac{d^4p}{\vec{p}^{2\alpha}(\vec{p}-\vec{k})^{2\alpha}}. \tag{3.1.0.2}$$

Introduce now two auxiliary quantities A and B, called generalized *Feynman's parameters* [1]

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[Ax+B(1-x)]^{\alpha+\beta}} dx, \tag{3.1.0.3}$$

and cast the above convolution in the fashion

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int d^4p \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[(\vec{p}-\vec{k})^2x+\vec{p}^2(1-x)]^{2\alpha}} dx =$$

$$\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int \frac{d^4p}{[(\vec{p}-\vec{k})^2x+\vec{p}^2(1-x)]^{2\alpha}}, \tag{3.1.0.4}$$

or, in simpler manner

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha =$$

$$\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int \frac{d^4p}{[(\vec{p}-\vec{k}x)^2+\vec{k}^2x(1-x)]^{2\alpha}}. \tag{3.1.0.5}$$

Making now the change of variables $\vec{s} = \vec{p} - \vec{k}x$ and calling $a = \vec{k}^2x(1-x)$ we obtain

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int \frac{d^4s}{,} (\vec{s}^2+a)^{2\alpha}, \tag{3.1.0.6}$$

or

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = 2\pi^2 \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int_0^\infty \frac{s^3}{(s^2+a)^{2\alpha}} ds. \tag{3.1.0.7}$$

Making another change of variables $y = s^2$ we have

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \pi^2 \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int_0^\infty \frac{y}{(y+a)^{2\alpha}} dy. \tag{3.1.0.8}$$

Using now the essential reference for any theorist [4], we can indeed calculate the previous integral, obtaining

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \pi^2 \frac{\Gamma(2\alpha-2)}{[\Gamma(\alpha)]^2} \int_0^1 \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{[k^2 x(1-x)]^{2\alpha-2}} dx, \tag{3.1.0.9}$$

or

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \pi^2 \frac{\Gamma(2\alpha-2)}{[\Gamma(\alpha)]^2} k^{4-4\alpha} \int_0^1 [x(1-x)]^{1-\alpha} dx. \tag{3.1.0.10}$$

Employing again results given in [4] we have now

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \pi^2 \frac{\Gamma(2\alpha-2)}{[\Gamma(\alpha)]^2} k^{4-4\alpha} \frac{[\Gamma(2-\alpha)]^2}{\Gamma(4-2\alpha)}. \tag{3.1.0.11}$$

At this point we tell you, dear reader, that our four-dimensional convolution is to be obtained as the *residue* in the *pole* when α tends to one (see [3]), a fact that we ask you to graciously accept. Thus,

$$\rho^{-1} * \rho^{-1} = \lim_{\alpha\to 1} \frac{\partial}{\partial\alpha} \left\{ (\alpha-1)\pi^2 \frac{\Gamma(2\alpha-2)}{[\Gamma(\alpha)]^2} k^{4-4\alpha} \frac{[\Gamma(2-\alpha)]^2}{\Gamma(4-2\alpha)} \right\}. \tag{3.1.0.12}$$

Using once again [4] we have

$$\Gamma(4-2\alpha) = 2^{3-2\alpha} \pi^{-\frac{1}{2}} \Gamma(2-\alpha) \Gamma\left(\frac{5}{2}-\alpha\right). \tag{3.1.0.13}$$

Evaluating the limit, we get for the four-dimensional *convolution*, without much pain, the desired final result

$$\rho^{-1} * \rho^{-1} = -\pi^2 [\ln \rho - 1]. \tag{3.1.0.14}$$

We have seen that, in order to face Eq. (3.1.0.1), we made an α-detour that allowed for a simple solution. This detour was grandly called an *analytical regularization*, which sounds wise and sophisticated enough, but is essentially simple.

Chapter 4

Ultradistributions

4.1 Distributions of exponential type

For the benefit of the reader, we present here a brief description of the main properties of the so called tempered ultradistributions and of ultradistributions of *exponential type* (UET). We need for this purpose to recapitulate some well known ideas regarding Hilbert spaces [1].

Remember first from elementary quantum mechanics that a countable *Hilbert* $\mathcal{H}$ space is one possessing a countable basis [1]. One defines at this junction the related notion of nuclear spaces. These are spaces that retain some convenient features of finite-dimensional vector spaces, particularly with respect to their topology [1]. In this regard, we think of those special semi-norms whose unit balls' sizes diminish in rapid fashion. All finite-dimensional vector spaces are nuclear, of course.

Let us pass now to the important notion of rigged Hilbert space (RHS) [1]. A RHS consists of 1) a Hilbert space H plus 2) a subspace $\mathbf{\Phi}$ which carries a finer topology that of H [1]. Thus, $\mathbf{\Phi} \subset \mathsf{H}$. Then, we define the RHS in terms of the inequalities $\mathbf{\Phi} \subset \mathsf{H} \subset \mathbf{\Phi}'$, with $\mathbf{\Phi}'$ the dual space to $\mathbf{\Phi}$ [1], a triplet of symbols [1].

Notations. Our notation is almost textually taken from Ref. [9]. Let $\mathbb{R}^n$ (respectively $\mathbb{C}^n$) be the real (respectively complex) n-dimensional space whose points are denoted by $x = (x_1, x_2, ..., x_n)$ (resp. $z = (z_1, z_2, ..., z_n)$). We shall use the following notations

(i) $x+y = (x_1+y_1, x_2+y_2, ..., x_n+y_n)$; $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$
(ii)$x \geqq 0$ means $x_1 \geqq 0, x_2 \geqq 0, ..., x_n \geqq 0$
(iii)$x \cdot y = \sum_{j=1}^{n} x_j y_j$
(iV)$| x |= \sum_{j=1}^{n} | x_j |$

Consider the set of n-tuples of natural numbers $\mathbb{N}^n$. If $p \in \mathbb{N}^n$, then $p = (p_1, p_2, ..., p_n)$, where p_j is a natural number, $1 \leqq j \leqq n$. $p + q$ denote $(p_1 + q_1, p_2 + q_2, ..., p_n + q_n)$ and $p \geqq q$ means $p_1 \geqq q_1, p_2 \geqq q_2, ..., p_n \geqq q_n$. x^p means $x_1^{p_1} x_2^{p_2} ... x_n^{p_n}$. We denote by $| p |= \sum_{j=1}^{n} p_j$ and by D^p we understand the differential operator $\partial^{p_1+p_2+...+p_n}/\partial {x_1}^{p_1} \partial {x_2}^{p_2} ... \partial {x_n}^{p_n}$.
For any natural number k we define $x^k = x_1^k x_2^k ... x_n^k$ and $\partial^k/\partial x^k = \partial^{nk}/\partial x_1^k \partial x_2^k ... \partial x_n^k$.

4.1.1 An important set of test functions

This is the space $\mathcal{H}$ of test functions such that $e^{p|x|}|D^q\phi(x)|$ is bounded for any natural numbers p and q. This test-space is of great importance for our present purposes. It is defined (Ref. [7]) by means of the countable set of norms

$$\|\hat{\phi}\|_p = \sup_{0 \leq q \leq p,\, x} e^{p|x|} \left|D^q \hat{\phi}(x)\right| \quad , \quad p = 0, 1, 2, ... \qquad (4.1.1.1)$$

According to reference [8], $\mathcal{H}$ is a *countable* and nuclear Hilbert space $\mathcal{K}\{\mathbf{M_p}\}$ with

$$M_p(x) = e^{(p-1)|x|} \quad , \quad p = 1, 2, ... \qquad (4.1.1.2)$$

$\mathcal{K}\{\mathbf{e}^{(\mathbf{p}-\mathbf{1})|\mathbf{x}|}\}$ complies with a special mathematical demand called $(\mathcal{N})$ by *Guelfand* (Ref. [8]), whose details we do not really need to enter into. Let us insist on the fact that $\mathcal{K}\{\mathbf{e}^{(\mathbf{p}-\mathbf{1})|\mathbf{x}|}\}$ is a countable Hilbert and nuclear space

$$\mathcal{K}\{\mathbf{e}^{(\mathbf{p}-\mathbf{1})|\mathbf{x}|}\} = \mathcal{H} = \bigcap_{p=1}^{\infty} \mathcal{H}_\mathbf{p} \qquad (4.1.1.3)$$

where $\mathcal{H}_p$ is obtained by completing $\mathcal{H}$ with the norm induced by the scalar product

$$<\hat{\phi},\hat{\psi}>_p = \int\limits_{-\infty}^{\infty} e^{2(p-1)|x|}\sum_{q=0}^{p} D^q\overline{\hat{\phi}}(x)D^q\hat{\psi}(x)\ dx \quad ; \quad p=1,2,... \tag{4.1.1.4}$$

where $dx = dx_1\ dx_2 ... dx_n$.

If we take the conventional *scalar product*

$$<\hat{\phi},\hat{\psi}>= \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x)\ dx, \tag{4.1.1.5}$$

then $\mathcal{H}_o$, completed with (4.1.1.5), is the familiar old Hilbert space H of *square integrable* functions.

4.1.2 The associated distributions

Consider the space generated by exponentials $e^{(p)|x|}$, with p real. Distributions of *exponential type* (Ref. [9]) are those belonging to the space of continuous linear functionals defined on $\mathcal{H} = \mathcal{K}\{e^{(p-1)|x|}\}$. The new space is itself a Hilbert space, the dual of $\mathcal{H}$, with the same dimension. We are speaking of the space Λ_∞. To repeat, this space can be identified with the set of continuous linear functionals. We will badly need such space, that we call the one of distributions of *exponential type* (Ref. [9]).

Let H_y stand for the Heaviside function. The Fourier transform of a distribution of exponential type $\hat{F}$ is called a tempered ultradistribution, being given by (see [9, 10])

$$F(k) = \int\limits_{-\infty}^{\infty} H_y[\Im(k)]H_y[\Re(x)] - H_y[-\Im(k)]H_y[-\Re(x)]\hat{F}(x)e^{ikx}\ dx =$$

$$H_y[\Im(k)]\int\limits_{0}^{\infty}\hat{F}(x)e^{ikx} - H_y[-\Im(k)]\int\limits_{-\infty}^{0}\hat{F}(x)e^{ikx} \tag{4.1.2.6}$$

where F is, as stated above, the corresponding *tempered ultradistribution* (see also the next section).

The triplet

$$\mathfrak{H} = (\mathcal{H}, \mathsf{H}, \Lambda_\infty) \qquad (4.1.2.7)$$

is a rigged Hilbert Space (or a Guelfand's triplet [8]). Moreover, we have $\mathcal{H} \subset \mathcal{S} \subset \mathsf{H} \subset \mathcal{S}' \subset \Lambda_\infty$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing test functions (Refs. [5, 6]).

Any rigged Hilbert space $\mathfrak{G} = (\Phi, \mathsf{H}, \Phi')$ displays the fundamental property that a linear and symmetric operator on Φ, which admits an extension to a self-adjoint operator in H, has a complete set of generalized eigenfunctions in Φ', with real eigenvalues [1].

4.2 Tempered ultradistributions

They are the Fourier transforms of distributions of exponential type. The Fourier transform of a function $\hat{\phi} \in \mathcal{H}$ is

$$\phi(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\, e^{iz\cdot x}\, dx \qquad (4.2.0.1)$$

Here $\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We call $\mathfrak{H}$ the set of all such functions.

$$\mathfrak{H} = \mathcal{F}\{\mathcal{H}\} \qquad (4.2.0.2)$$

It is called a $\mathcal{Z}\{\mathbf{M_p}\}$ countably normed and complete space (Ref. [7]), with

$$M_p(z) = (1 + |z|)^p \qquad (4.2.0.3)$$

$\mathfrak{H}$ is a *nuclear space* defined with the norms

$$\|\phi\|_{pn} = \sup_{z \in V_n} (1 + |z|)^p |\phi(z)| \qquad (4.2.0.4)$$

where $V_k = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n \mid Im\, z_j \,| \leqq k, 1 \leqq j \leqq n\}$,

We can define the usual *scalar product*

$$< \phi(z), \psi(z) >= \int_{-\infty}^{\infty} \phi(z)\psi_1(z)\, dz = \int_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x)\, dx \qquad (4.2.0.5)$$

where

$$\psi_1(z) = \int_{-\infty}^{\infty} \hat{\psi}(x)\, e^{-iz\cdot x}\, dx$$

and $dz = dz_1\, dz_2 ... dz_n$. By completing $\mathfrak{H}$ with the norm induced by (4.2.0.5) we once again obtain the Hilbert space of square integrable functions. The dual of $\mathfrak{H}$ is the space $\boldsymbol{\mathcal{U}}$ of tempered ultradistributions (Refs. [9, 10]). Namely, a *tempered ultradistribution* is a continuous linear functional defined on the space $\mathfrak{H}$ of entire functions rapidly decreasing on straight lines parallel to the real axis. The set $\mathfrak{U} = (\mathfrak{H}, \mathsf{H}, \boldsymbol{\mathcal{U}})$ is also a rigged Hilbert space.

Moreover, we have $\mathfrak{H} \subset \boldsymbol{\mathcal{S}} \subset \mathsf{H} \subset \boldsymbol{\mathcal{S}}' \subset \boldsymbol{\mathcal{U}}$. Now, $\boldsymbol{\mathcal{U}}$ can also be characterized in the following way (Ref. [9]): let $\boldsymbol{\mathcal{A}}_\omega$ be the space of all special functions $F(z)$, that will become very important to us, such that

A) $F(z)$ is analytic on the set $\{z \in \mathbb{C}^n | Im(z_1)| > p, |Im(z_2)| > p, ..., |Im(z_n)| > p\}$.

B) $F(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n | Im(z_1)| \geqq p, |Im(z_2)| \geqq p, ..., |Im(z_n)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $F(z)$.

Let $\boldsymbol{\Pi}$ be the set of all z-dependent pseudo-polynomials (defined below) $z \in \mathbb{C}^n$. Then $\boldsymbol{\mathcal{U}}$ is the *quotient space*

C) $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{A}}_\omega / \boldsymbol{\Pi}$. Let us clarify that, by a *pseudo-polynomial*, we refer to a function of z of the form
$\sum_s z_j^s G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n)$ with $G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \boldsymbol{\mathcal{A}}_\omega$.

Due to these properties it is possible to represent any ultradistribution as (Ref. [9])

$$F(\phi) =< F(z), \phi(z) >= \oint_\Gamma F(z)\phi(z)\, dz, \qquad (4.2.0.6)$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup ... \Gamma_n$. The path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z_j) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $F(z)$). Recall now that a branch cut [1] is a special curve (with ends possibly open, closed, or half-open) in the complex plane [1]. Across it, an analytic multivalued function is discontinuous. For convenience, branch cuts are often taken as lines or line segments [1]. Branch cuts (even those consisting of curves) are also known as cut lines [1].

Formula (4.2.0.6) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of "*Dirac's formula*" for ultradistributions (Ref. [10])

$$F(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)}\, dt \tag{4.2.0.7}$$

where the "density" $f(t)$ is the cut of $F(z)$ along the real axis and satisfies

$$\oint_{\Gamma} F(z)\phi(z)\, dz = \int_{-\infty}^{\infty} f(t)\phi(t)\, dt. \tag{4.2.0.8}$$

While $F(z)$ is analytic on Γ, the density $f(t)$ is in general singular, so that the r.h.s. of (4.2.0.7) should be interpreted in the sense of Schwartz' distribution's theory. Another important property of this *analytic representation* is the fact that on Γ, $F(z)$ is bounded by a power of z (Ref. [9])

$$|F(z)| \leq C|z|^p, \tag{4.2.0.9}$$

where C and p depend on F. The representation (4.2.0.6) implies that the sum of a *pseudo-polynomial* $P(z)$ to $F(z)$ does not alter the ultradistribution

$$\oint_{\Gamma} \{F(z) + P(z)\}\phi(z)\, dz = \oint_{\Gamma} F(z)\phi(z)\, dz + \oint_{\Gamma} P(z)\phi(z)\, dz$$

. However,

$$\oint_{\Gamma} P(z)\phi(z)\, dz = 0$$

as $P(z)\phi(z)$ is entire analytic in some of the variables z_j (and rapidly decreasing),

$$\therefore \quad \oint_{\Gamma} \{F(z) + P(z)\}\phi(z)\, dz = \oint_{\Gamma} F(z)\phi(z)\, dz. \tag{4.2.0.10}$$

The *inverse Fourier transform* of (4.1.2.6) is given by

$$\hat{F}(x) = \frac{1}{2\pi} \oint_{\Gamma} F(k) e^{-ikx}\, dk = \int_{-\infty}^{\infty} f(k) e^{-ikx}\, dx. \tag{4.2.0.11}$$

4.3 Ultradistributions of exponential type

Any vector space has associated to it what is called a dual space. It consists of linear functionals of the elements of the original space [1].

Consider the Schwartz space of rapidly decreasing test functions $\mathcal{S}$. Let Λ_j be the region of the complex plane defined as

$$\Lambda_j = \{z \in \mathbb{C} | \Im(z)| < j;\ j \in \mathbb{N}\} \tag{4.3.0.1}$$

According to Ref. [10, 11] the space of test functions $\hat{\phi} \in \mathfrak{V}_j$ is constituted by the set of all entire analytic functions of $\mathcal{S}$ for which

$$||\hat{\phi}||_j = \max_{k \leq j} \left\{ \sup_{z \in \Lambda_j} \left[e^{(j|\Re(z)|)} |\hat{\phi}^{(k)}(z)| \right] \right\} \tag{4.3.0.2}$$

is finite.

The new space $\mathfrak{Z}$ is then defined as

$$\mathfrak{Z} = \bigcap_{j=0}^{\infty} \mathfrak{V}_j . \tag{4.3.0.3}$$

It is a complete countably normed space with the topology generated by the set of *semi-norms* $\{||\cdot||_j\}_{j\in\mathbb{N}}$. The *topological dual* of $\mathfrak{Z}$, denoted by $\mathfrak{B}$, is by definition the space of Ultradistributions of *exponential type* (Ref.[9, 10, 11]). Let $\mathfrak{S}$ be the space of rapidly decreasing sequences. According to Ref.[8] $\mathfrak{S}$ is a nuclear space. We consider now the space of sequences $\mathfrak{P}$ generated by the Taylor expansion of $\hat{\phi} \in \mathfrak{Z}$

$$\mathfrak{P} = \left\{ \mathfrak{Q} \left(\hat{\phi}(0), \hat{\phi}'(0), \frac{\hat{\phi}''(0)}{2}, ..., \frac{\hat{\phi}^{(n)}(0)}{n!}, ... \right); \hat{\phi} \in \mathfrak{Z} \right\}. \tag{4.3.0.4}$$

The norms that define the topology of $\mathfrak{P}$ are given by

$$||\hat{\phi}||_p^{'} = \sup_n \frac{n^p}{n!} |\hat{\phi}^n(0)|. \tag{4.3.0.5}$$

$\mathfrak{P}$ is a subspace of $\mathfrak{S}$ and as consequence is a nuclear space. The norms $||\cdot||_j$ and $||\cdot||_p^{'}$ are equivalent, the correspondence

$$\mathfrak{Z} \Longleftrightarrow \mathfrak{P} \tag{4.3.0.6}$$

being an isomorphism and, therefore, $\mathfrak{Z}$ is a countably normed nuclear space. We define now the set of scalar products

$$< \hat{\phi}(z), \hat{\psi}(z) >_n = \sum_{q=0}^{n} \int_{-\infty}^{\infty} e^{2n|z|} \overline{\hat{\phi}^{(q)}}(z) \hat{\psi}^{(q)}(z) \, dz =$$

$$\sum_{q=0}^{n} \int_{-\infty}^{\infty} e^{2n|x|} \overline{\hat{\phi}^{(q)}}(x) \hat{\psi}^{(q)}(x) \, dx. \qquad (4.3.0.7)$$

They induce the norm

$$||\hat{\phi}||_n^{''} = [< \hat{\phi}(x), \hat{\phi}(x) >_n]^{\frac{1}{2}}. \qquad (4.3.0.8)$$

The norms $||\cdot||_j$ and $||\cdot||_n^{''}$ are equivalent, and therefore $\mathfrak{Z}$ is a *countably hilbert* nuclear space. Thus, if we call now $\mathfrak{Z}_p$ the completion of $\mathfrak{Z}$ by the norm p given in (4.3.0.5), we have

$$\mathfrak{Z} = \bigcap_{p=0}^{\infty} \mathfrak{Z}_p, \qquad (4.3.0.9)$$

where

$$\mathfrak{Z}_0 = H, \qquad (4.3.0.10)$$

is the Hilbert space of square integrable functions.

As a consequence the triplet

$$\mathfrak{U} = (\mathfrak{Z}, H, \mathfrak{B}) \qquad (4.3.0.11)$$

is also a Guelfand's triplet.

$\mathfrak{B}$ can also be characterized in the following way (Refs. [9],[10]): let $\mathfrak{C}_\omega$ be the space of all functions $\hat{F}(z)$ such that **A**) $\hat{F}(z)$ is an analytic function for $\{z \in \mathbb{C} | Im(z)| > p\}$. **B**)- $\hat{F}(z)e^{-p|\Re(z)|}/z^p$ is a bounded continuous function in $\{z \in \mathbb{C} | Im(z)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $\hat{F}(z)$.
Let further $\mathfrak{N}$ be $\mathfrak{N} = \{\hat{F}(z) \in \mathfrak{C}_\omega \hat{F}(z)$ is entire analytic$\}$. Then

$\mathfrak{B}$ is the *quotient space* **C**) $\mathfrak{B} = \mathfrak{C}_\omega / \mathfrak{N}$

Due to these properties it is possible to represent any ultradistribution of *exponential type* as [9, 10]

$$\hat{F}(\hat{\phi}) =< \hat{F}(z), \hat{\phi}(z) >= \oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz, \tag{4.3.0.12}$$

where the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $\Im(z) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $\Im(z) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $\hat{F}(z)$).

Eq. (4.3.0.12) will be our fundamental representation for a ultradistribution of exponential type. The " *Dirac's formula*" for ultradistributions of exponential type is (Refs. [9, 10])

$$\hat{F}(z) \equiv \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\hat{f}(t)}{t-z}\ dt \equiv \frac{\cosh(\lambda z)}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t-z)\cosh(\lambda t)}\ dt, \tag{4.3.0.13}$$

where the "density" $\hat{f}(t)$ is such that

$$\oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz = \int\limits_{-\infty}^{\infty} \hat{f}(t)\hat{\phi}(t)\ dt. \tag{4.3.0.14}$$

Eq. (4.3.0.13) should be used carefully. While $\hat{F}(z)$ is an analytic function on Γ, the density $\hat{f}(t)$ is in general singular, so that the right hand side of (4.3.0.14) should be interpreted again in the sense of distribution's theory.

Another important property of the analytic representation is the fact that, on Γ, $\hat{F}(z)$ is bounded by a exponential and a power of z (Ref. [9, 10])

$$|\hat{F}(z)| \le C|z|^p e^{p|\Re(z)|}, \tag{4.3.0.15}$$

where C and p depend on $\hat{F}$.

The representation (4.3.0.12) implies that the addition of any entire function $\hat{G}(z) \in \mathfrak{H}$ to $\hat{F}(z)$ does not alter the ultradistribution

$$\oint\limits_{\Gamma} \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\ dz = \oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz + \oint\limits_{\Gamma} \hat{G}(z)\hat{\phi}(z)\ dz$$

. However,

$$\oint_{\Gamma} \hat{G}(z)\hat{\phi}(z)\,dz = 0$$

as $\hat{G}(z)\hat{\phi}(z)$ is an entire analytic function,

$$\therefore \quad \oint_{\Gamma} \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\,dz = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z)\,dz. \tag{4.3.0.16}$$

Another very important property of $\mathfrak{B}$ is that $\mathfrak{B}$ is *reflexive* under the Fourier transform

$$\mathfrak{B} = \mathcal{F}_c\left\{\mathfrak{B}\right\} = \mathcal{F}\left\{\mathfrak{B}\right\}, \tag{4.3.0.17}$$

where the *complex Fourier transform* $F(k)$ of $\hat{F}(z) \in \mathfrak{B}$ is given by

$$F(k) = H_y[\Im(k)]\int_{\Gamma_+} \hat{F}(z)e^{ikz}\,dz - H_y[-\Im(k)]\int_{\Gamma_-} \hat{F}(z)e^{ikz}\,dz =$$

$$\oint_{\Gamma} \{H_y[\Im(k)H_y[\Re(z)] - H_y[-\Im(k)H_y[-\Re(z)]\}\hat{F}(z)e^{ikz}\,dz =$$

$$H[\Im(k)]\int_0^{\infty} \hat{f}(x)e^{ikx}\,dx - H[-\Im(k)]\int_{-\infty}^{0} \hat{f}(x)e^{ikx}\,dx \tag{4.3.0.18}$$

Here, Γ_+ is the part of Γ with $\Re(z) \geq 0$ and Γ_- is the part of Γ with $\Re(z) \leq 0$ Using (4.3.0.18) we can interpret Dirac's formula as

$$F(k) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(s)}{s-k}\,ds \equiv \mathcal{F}_c\left\{\mathcal{F}^{-1}\left\{f(s)\right\}\right\}. \tag{4.3.0.19}$$

The *inverse Fourier transform* corresponding to (4.3.0.18) is given by

$$\hat{F}(z) = \frac{1}{2\pi}\oint_{\Gamma} \{H_y[\Im(z)]H_y[-\Re(k)] - H_y[-\Im(z)]H_y[\Re(k)]\}F(k)e^{-ikz}\,dk. \tag{4.3.0.20}$$

The treatment of ultradistributions of exponential type defined on $\mathbb{C}^n$ is similar to that for the case of just one variable. Thus let Λ_j be given as

$$\Lambda_j = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n | \Im(z_k)| \leq j \quad 1 \leq k \leq n\}, \tag{4.3.0.21}$$