# Statistical Models for Nucleon Structure Function

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# CONTENTS

# PREFACE

During the eighties and nineties, many statistical/thermodynamical models emerged to describe the nucleon structure functions and the energy distribution of quarks. Most of these models describe the compound quarks and gluons inside the nucleon as a Fermi-Dirac or Bose-Einstein gas, confined in an MIT bag with continuous energy levels. These models obtained some relevant features of the nucleons, like the asymmetries between $\bar{d}$ and $\bar{u}$, the spin-dependent structure functions, and the ratio $F_2^n/F_2^p$, for instance.

In this work, we reviewed the hadronic models. They use statistical/thermodynamic features to describe the hadrons' structure functions, polarized and unpolarized. The revised works were described, as far as possible, in chronological order. We believe this book is convenient for researchers and students because it put together several studies about the thermodynamic features of nucleons and their structure function. Because this subject is yet to undergo research, there is no unique approach or model that is in complete accord.

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# CHAPTER 1

# INTRODUCTION

QCD (Quantum Chromodynamics) is the theory of strong interaction in the standard model. This theory describes the short-range interactions among the subnuclear components, the quarks. The gluons mediate the interactions and the main difference with the QED (Quantum Electrodynamics) is the use of a Lagrangian, where gluons may interact among them (in opposition to the photons from QED, that don't interact). Although this theory describes many relevant aspects, some open questions have been studied with effective alternative models, respecting the basic principles of QCD. One of the questions that concerns the structure-function is the distribution of energy of quarks inside the nucleon. We show some reasons to use effective models in the following:

The perturbative theories (short distances) describe the interactions inside the nucleon (strong interactions between quarks and gluons). However, including more diagrams and details makes this method too complex. We need to include many loopings and renormalizations, which is almost impracticable. In principle, this is already a 3-body problem, without an analytical solution, even in classical cases. On the other hand, the lattice quantum field theory also demands great computational efforts.

Some effective models do not include Fermi-Dirac and Bose-Einstein statistics as relevant physical effect, such as, the valon model and perturbative chiral. The parameters fit according to the experimental data available. Field and Feynman[6] pointed out that including the Fermi-Dirac statistics is relevant to describing the sea asymmetry in the nucleon $\bar{d} > \bar{u}$, which explains the violation of the Gottfried sum rule[7]. Another very important characteristic is the confining and the asymptotic freedom, predicted by QCD, considered in the models through the effective potential. The effective temperature for each model is also relevant, and it is interesting to compare the obtained temperature of the different models.

The Deep Inelastic scattering (DIS) process between leptons and nucleons has been an indispensable tool in describing the hadron structure. Interest in describing the partonic distribution and the different phenomena involved, such as the sea asymmetry, strangeness in the nucleon, EMC (Euro Muon Collaboration) effect, and the ratio $F_2^n/F_2^p$ has generated many theoretical models in recent decades.

These models have the appeal of simplicity and are physically well-founded. The phenomenology to explain such models is in the following way: even though valence quarks must lie in discrete energy levels, they can emit gluons that may split in a quark-antiquark pair, with continuous energy.

In the framework of the MIT bag model[5], an estimate for the structure-function was presented by Jaffe[8]. As it can be speculated, with partons bound in the wee volume of the nucleon, we have not only the dynamic but also the statistical properties; for example, the Pauli exclusion principle should have a relevant effect on PDF (Particle Distribution Function). Most statistical/thermodynamic models proposed in the eighties and nineties consider the confinement given by the MIT bag model, and treat the quark/gluons inside de nucleon as a Fermi-Dirac and Bose-Einstein gas of free particles with a continuous spectrum).

In this book, we may check the main features of the statistical model. In the next chapters, we describe the statistical models, clarifying focus, motivation, and results. The following papers are studied in chronological order, as follows:

1. Angelini and Pazzi's works (1982-1983)[9] used a statistical model with a Boltzmann distribution and scaling violation.
2. The Cleymans-Thews' model (1988)[10] started with the transition rate of scattering in the framework of the temperature-dependent field theory and explored a statistical way to generate compatible pdfs.
3. The Mac and Ugaz's work (1989)[11] incorporated first-order QCD corrections, introduced by Altarelli and Parisi[12].
4. Bickerstaff and Londergan (1990)[15] interpreted the finite-temperature property to mimic some volume-dependent effect due to confining. They also discussed the theoretical validity of the ideal gas assumption in detail.
5. The Ganesamurthy, Devanathan, Rajasekaran and Karthiyaini (1994,1996)[16,17,18] proposed a thermodynamical bag model, which evolves as a function of $x$. The structure-function they got is practicable for $x > 0.2$ and has the correct asymptotic behavior for $x \to 1$; in addition, they parametrized on $T$ and exhibited the scaling behavior.

6. Soffer-Bourrely-Bucella have been working with parametrization based on the Fermi-Dirac and Bose-Einstein distributions (1995-today)[19,20,22,23,21]. We reviewed the polarized case (1995-today).
7. Bhalerao and Bhalerao et al. (1996, 2000)[25,26] introduced finite size corrections to the statistical model and got more accurate results for unpolarized and polarized structure functions.
8. Trevisan, Tomio, Mirez, Frederico (1999 and 2008)[27,28] presented a statistical model based on the Dirac equation with a linear confining potential. They also obtained the strangeness in the nucleon.
9. Trevisan and Mirez presented "A very simple statistical model to quarks asymmetry"[267] that considers the meson-hadron fluctuations as energy states with some probability varying according to the temperature.
10. Zhang, Shao and Ma (2009)[29] intended to present a statistical model using few parameters to fit the data. They also studied the EMC[79] effect with the statistical model.
11. Deppman[33], and Deppman et al. [34], presented a relation between the fractal structure of the nucleons and Tsallis statistic (nonextensivity).
12. Trevisan and Mirez[35] presented a statistical model that considered the nonextensivity introduced by Tsallis[112,113].
13. Trevisan, Mirez, and Silva presented a model with different sizes for quarks and gluons to fix the low moment carried by gluons in the previous statistical model.[37]
14. Trevisan and Mirez gave a version of the nonextensive statistical model applied to the EMC effect.
15. On the framework of the valon model[38,30,39], Mirjalili and collaborators studied the statistical approach[40,41].

Interestingly, despite the fact that the models basically start from the same physical description, there are some remarkable variations such as scale variance, the dependence of the temperature with the Bjorken variable $x$, and different ways of taking into account the polarization.

# PART I

# REVIEW ON PARTONS MODEL

# CHAPTER 2

# PARTONS MODEL AND THE NUCLEON'S STRUCTURE FUNCTION

## 2.1 Introduction

This chapter is a brief introduction to the main concepts of QCD quark models[31,32], which we will use in the following chapters.

For the process $e^- \mu^- \to e^- \mu^-$, we initially describe the kinematics for the proton model. After, we do the elastic and inelastic electron-proton scattering process. Finally, we study the momentum distribution of the Bjorken scale.

## 2.2 Process $e^- \mu^- \to e^- \mu^-$

The Feynman's diagram for scattering $e^- \mu^- \to e^- \mu^-$ is illustrated in Fig. (2.1). Applying Feynman's rules, we calculate the invariant amplitude

$$A = -\, e^2 \bar{u}(k')\gamma^\mu u(k)\, \frac{1}{q^2}\, \bar{u}(p')\gamma_\mu u(p) \ . \tag{2.1}$$

From Fig. (2.1), we have the quad-moment $q = k - k'$. We can calculate the non-polarized shock section by simply squaring the amplitude and summing the spins (we sum the spins separately for each electron and muon), in the form

$$\overline{|A|}^{\,2} = \frac{e^4}{q^4}\, L_e^{\mu\nu}\, L_{\mu\nu}^{muon} \ , \tag{2.2}$$

where the tensor for the electron vertex is

$$L_e^{\mu\nu} = \frac{1}{2} \sum_{\text{spins-electron}} \left[\bar{u}(k')\gamma^{\mu}u(k)\right]\left[\bar{u}(k')\gamma^{\nu}u(k)\right]^{\star} \tag{2.3}$$

The same is true for the tensor $L_{\mu\nu}^{muon}$.

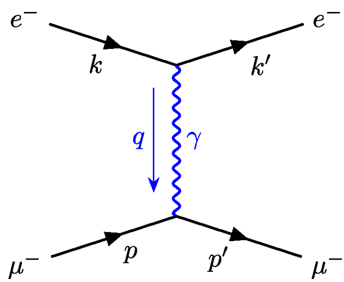


Figure 2.1: Feynman diagram for electron-muon scattering.

Applying properties of the traces, we have from Fig. (2.1)

$$L_e^{\mu\nu} = \frac{1}{2}\mathrm{Tr}\left(k'\,\gamma^{\mu}k\,\gamma^{\nu}\right) + \frac{1}{2}m_e^2\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\right) = 2\left[k'^{\mu}.k^{\nu} + k'^{\nu}.k^{\mu} - \left(k'k - m_e^2\right).g^{\mu\nu}\right], \tag{2.4}$$

where $m_e$ is the electron mass.

For muon, we have the same procedure to do

$$L_{\mu\nu}^{muon} = 2\left[p'_{\mu}.p_{\nu} + p'_{\nu}.p_{\mu} - \left(p'.p - M^2\right)g_{\mu\nu}\right], \tag{2.5}$$

where $M$ is the muon mass.

Now by multiplying both terms $L_e^{\mu\nu}L_{\mu\nu}^{muon}$, we get

$$\overline{|A|}^2 = \frac{8e^4}{q^4}\left[\left(k'.p'\right)\left(k.p\right) + \left(k'.p\right)\left(k.p'\right) - m_e^2\,p'.p - M^2k'.k + 2m_e^2\,M^2\right], \tag{2.6}$$

At the "relativistic limit" we assume that $m_e^2, M^2 \to 0$, so squared amplitude will be reduced to the following expression

$$\overline{|A|}^2 = \frac{8e^4}{(k-k')^4}\left[(k'.p')(k.p) + (k'.p)(k.p')\right] \ . \tag{2.7}$$

The Mandelstam variables at the relativistic limit are

$$s = (k+p)^2 = m_e^2 + 2k.p + M^2 \approx 2k.p \approx 2k'.p' \tag{2.8}$$

$$t = (k-k')^2 \approx -2k.k' \approx -2p.p' \tag{2.9}$$

$$u = (k-p')^2 \approx -2k.p' \approx -2k'.p' \ , \tag{2.10}$$

therefore the spreading amplitude takes the final form

$$\overline{|A|}^2 = 2e^4\left(\frac{s^2+u^2}{t^2}\right) \ . \tag{2.11}$$

### 2.2.1 Process e$_\mu^-$ → e$_\mu^-$ in the lab frame

For the scattering form

$$e^-(k) + \mu^-(p) \to \mu^-(p') + e^-(k')$$

considering the electron mass $m_e^2 = 0$ and the muon mass $M^2$, we have the amplitude

$$\overline{|A|}^2 = \frac{8e^4}{q^4}\left[(k'.p')(k.p) + (k'.p)(k.p) - M^2 k'.k\right] \ . \tag{2.12}$$

The scattering process in the laboratory reference frame is illustrated in Fig. (2.2)

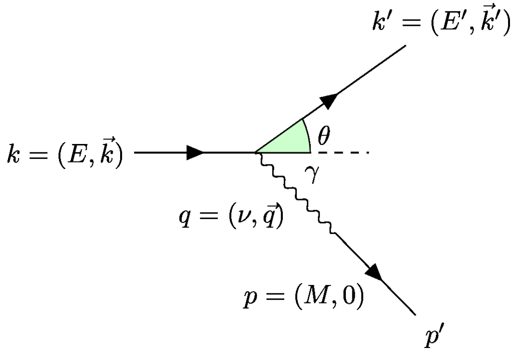


Figure 2.2: Process $e^- \mu^- \to e^- \mu^-$, in the lab frame.

The scattering process in the laboratory reference frame is illustrated in Fig. (2.2). So we have $q = k - k'$, $p' = k - k' + p$, $k^2 = k'^2 = 0$ and $q^2 \approx -2k.k'$, if we replace in Eq. (2.12), we get

$$\overline{|A|}^2 = \frac{8e^4}{q^4} \left[ \frac{-q^2}{2} (k.p - k'.p) + 2 (k'.p) . (k.p) + \frac{1}{2} M^2 q^2 \right] . \quad (2.13)$$

If we consider that the muon is initially at rest, that is $p = (M, 0)$, then we have

$$\overline{|A|}^2 = \frac{8e^4}{q^4} \left\{ \frac{-q^2}{2} \left[ (\underbrace{k - k'}_{E-E'}).M \right] + 2 (E'.M) (E.M) + \frac{1}{2} M^2 q^2 \right\} , \quad (2.14)$$

and since the kinematic relations

$$q^2 \approx -2k.k' \approx -2EE' (1 - \cos\theta) = -4EE' \sin^2 \left( \frac{\theta}{2} \right) , \quad (2.15)$$

then, for the amplitude we have

$$\overline{|A|}^2 = \frac{8e^4}{q^4} 2M^2 E' E \left[ \cos^2 \left( \frac{\theta}{2} \right) - \left( \frac{q^2}{2M^2} \right) \sin^2 \left( \frac{\theta}{2} \right) \right] , \quad (2.16)$$

and we squared $q + p = p'$, where $q = (\nu, \vec{q})$, $p = (M, 0)$ so

$$\begin{pmatrix} \nu & \vec{q} \end{pmatrix}\begin{pmatrix} \nu \\ \vec{q} \end{pmatrix} + 2\begin{pmatrix} \nu & \vec{q} \end{pmatrix}\begin{pmatrix} M \\ 0 \end{pmatrix} + \begin{pmatrix} M & 0 \end{pmatrix}\begin{pmatrix} M \\ 0 \end{pmatrix} =$$
$$= \begin{pmatrix} M & \vec{q} \end{pmatrix}\begin{pmatrix} M \\ \vec{q} \end{pmatrix}$$

that results

$$\nu = E - E' = \frac{q^2}{2M} \quad .$$

Having these relations, and using the formula that relates amplitude and shock sections (see Eq. (4.27) from Ref. 31), we obtain

$$d\sigma = \frac{1}{4ME}\frac{\overline{|A|}^2}{4\pi}\frac{1}{2}E'dE'd\Omega\frac{d^3p'}{2p'_0}\delta^4\left(p+q-p'\right) \quad . \tag{2.17}$$

By performing the integration into the Dirac delta functions, we have

$$\int \frac{d^3p}{2p_0}\delta^4\left(p+q-p'\right) = \frac{1}{2M}\delta\left(\nu + \frac{q^2}{2M}\right) \quad . \tag{2.18}$$

Inserting Eq. (2.16) into Eq. (2.17) and using the delta integration of Eq. (2.18), we get

$$\frac{d\sigma}{dE'd\Omega} = \frac{(2\alpha E')^2}{q^4}\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2M^2}\sin^2\left(\frac{\theta}{2}\right)\right]\delta\left(\nu + \frac{q^2}{2M}\right) \quad . \tag{2.19}$$

Performing integration over $dE'$ and replacing $q^2$ with $-4EE'\sin^2(\theta/2)$, we get the differential shock section in the lab frame

$$\frac{d\sigma}{d\Omega}\Big|_{\mathrm{lab}} = \left(\frac{\alpha^2}{4E^2\sin^4(\theta/2)}\right)\frac{E'}{E}\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2M^2}\sin^2\left(\frac{\theta}{2}\right)\right] \quad , \tag{2.20}$$

where $\alpha = e^2/4\pi$.

## 2.3 Electron-Proton scattering

### 2.3.1 Elastic Scattering

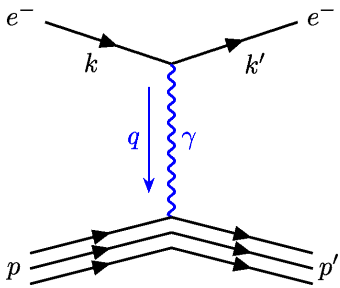


Figure 2.3: Elastic scattering process electron-proton: $e^- p \to e^- p$

The scattering amplitude for the Fig. (2.3) is given by the following expression:

$$T_{fi} = -i \int j_\mu \left( - \frac{1}{q^2} \right) J^\mu \, dx^4 \,, \tag{2.21}$$

where the transition currents of the electron (the electron mates with the photon as a Dirac particle) and the proton (not a Dirac particle because it has an internal structure) are respectively

$$j^\mu = -e \, \bar{u} \, (k') \, \gamma^\mu u \, (k) \exp \left[ i(k' - k).x \right] \tag{2.22}$$

$$J^\mu = e \, \bar{u} \, (p') \, [[ \;\; ]] \, u \, (p) \exp \left[ i(p' - p).x \right] \quad . \tag{2.23}$$

As we do not know the structure of the proton, we will use the most general combination of Dirac arrays between square brackets. Matrix terms like $\gamma^5$ are discarded due to parity conservation because of the $\gamma^5$ anti-commutation matrix. The $[[ \;\; ]]$ object (parameterized the coupling of the proton with the photon) has the general form expressed as

$$[[ \;\; ]] = \left[ F_1 \left( q^2 \right) \gamma^\mu + \frac{\kappa}{2M} F_2 \left( q^2 \right) i \sigma^{\mu\nu} q_\nu \right] \quad , \tag{2.24}$$

where $F_1$ and $F_2$ are two independent form factors and $\kappa$ is the anomalous magnetic moment. Using Eq. (2.24) to calculate the differential shock section of the electron-proton elastic scattering we obtain the Rosembluth[31] formula.

$$\frac{d\sigma}{d\Omega}\Big|_{\mathrm{lab}} = \left(\frac{\alpha^2}{4E^2\sin^4(\theta/2)}\right)\frac{E'}{E}\left[\left(F_1^2 - \frac{\kappa^2 q^2}{4M^2}F_2^2\right)\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2M^2}\left(F_1+\kappa F_2\right)^2\sin^2\left(\frac{\theta}{2}\right)\right] . \tag{2.25}$$

The form factors $F_1$ and $F_2$ (anomalous magnetic moment coupling) represent the fact that the proton is not an elementary particle and these factors are determined experimentally by measuring $d\sigma/d\Omega$ as a function of $\theta$ and $q^2$. These form factors are energy-dependent and $F_1(q^2=0)=F_2(0)=1$. In Eq. (2.25) the value of "$\kappa = 1.9$" is the anomalous magnetic moment of the proton. If we increase the energy and the proton is broken, then we can study its internal structure.

If the proton were a point-like particle (without structure) like the electron (or muon), having a charge "$e$", and magnetic moment of Dirac "$e/2M$", the result for the scattering $e^-\mu^-$ would also be valid for the proton case, exchanging the mass of muon for the proton. So in Eq. (2.25), we have $\kappa = 0$ and $F_1(q^2)=1$ for every $q^2$. So we get Eq. (2.20)

$$\frac{d\sigma}{d\Omega}\Big|_{\mathrm{lab}} = \left(\frac{\alpha^2}{4E^2\sin^4(\theta/2)}\right)\frac{E'}{E}\left[\cos^2\left(\frac{\theta}{2}\right) - \left(\frac{q^2}{2M^2}\right)\sin^2\left(\frac{\theta}{2}\right)\right] , \tag{2.26}$$

where the factor

$$\frac{E'}{E} = \frac{1}{1+2\dfrac{E}{M}\sin^2(\theta/2)} , \tag{2.27}$$

originates from the retreat of the target.

### 2.3.2 Inelastic scattering

For high energy scattering where $q^2 \to \infty$, the proton becomes a complicated multi-particle system, illustrated by Fig. (2.4)

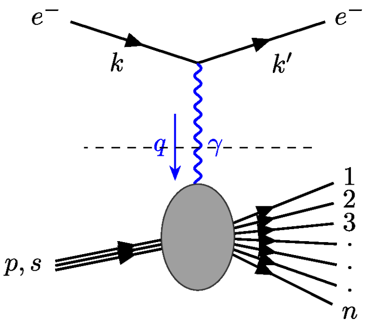


Figure 2.4: Diagram for the scattering $e^-p \to e^-X$

The differential shock section is of the form

$$d\sigma = L^e_{\mu\nu} (L^p)^{\mu\nu} \quad , \tag{2.28}$$

is described more generally

$$d\sigma = L^e_{\mu\nu} W^{\mu\nu} \quad , \tag{2.29}$$

where $L^e_{\mu\nu}$ represents the leptonic tensor. The most general form of the hadronic tensor $W^{\mu\nu}$ should be constructed with $g^{\mu\nu}$ and the independent moments $p$ and $q$ $(p' = p + q)$.

So we have for the hadronic tensor

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M^2} p^\mu p^\nu + \frac{W_4}{M^2} q^\mu q^\nu + \frac{W_5}{M^2} (p^\mu q^\nu + q^\mu p^\nu) \tag{2.30}$$

we disregard the antisymmetric contributions to $W^{\mu\nu}$ because they disappear after we insert them into Eq. (2.29) because the $L^e_{\mu\nu}$ tensor is symmetrical. $W_5$ is reserved for a parity-violating structure when a beam of neutrinos interacts rather than electrons so that the virtual photon is replaced by a weak boson.

The conservation of current $\partial_\mu \tilde{J}^\mu = 0$ (ou $q_\mu \tilde{J}^\mu = 0$ in the space of the moments) implies that $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$

$$-W_1 q^\mu + \frac{W_2}{M^2} (p.q)\, p^\mu + \frac{W_4}{M^2} q^2 q^\mu + \frac{W_5}{M^2} \left[q^2 p^\mu + (p.q) q^\mu\right] = 0 \tag{2.31}$$

by grouping terms, we observe

$$W_5 = -W_2 \frac{p.q}{q^2}$$

$$W_4 = W_1 . \frac{M^2}{q^2} - W_5 . (p.q)$$

$$W_4 = W_1 \frac{M^2}{q^2} + W_2 \frac{(p.q)^2}{q^2} ,$$

So only two of the four inelastic structure functions are independent, so

$$W^{\mu\nu} = W_1 \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + W_2 \frac{1}{M^2} \left( p^\mu - \frac{(p.q)}{q^2} q^\mu \right) \cdot \left( p^\nu - \frac{p.q}{q^2} q^\nu \right) , \tag{2.32}$$

so in inelastic scattering we have 02 (two) important variables

$$q^2 \quad \text{e} \quad \nu \equiv \frac{p.q}{M} . \tag{2.33}$$

The invariant mass $W$ of the final hadronic system is related to $\nu$ and $q^2$ per

$$W^2 = (p+q)^2 = M^2 + 2M\nu + q^2 ,$$

and we have the dimensionless variables

$$x = -\frac{q^2}{2p.q} = -\frac{q^2}{2M\nu} , \quad y = \frac{pq}{pk} , \tag{2.34}$$

so the kinematic region is $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Also consider that in the resting frame of the target proton, we have

$$\nu = E - E' , \quad y = \frac{E - E'}{E} ,$$

where $E$ and $E'$ are the start and end energy of the electron, respectively. The shock section for $e^- p \to e^- X$ is similar to the $e^- \mu^- \to e^- \mu^-$, replacing $L^{muon}_{\mu\nu}$ with $W_{\mu\nu}$, so using the expression

$$L_e^{\mu\nu} = \frac{1}{2}\text{Tr}\left(K^{\nu}\gamma^{\mu}k\gamma^{\nu}\right) + \frac{1}{2}m^2\text{Tr}\left(\gamma^{\mu}\gamma^{\nu}\right) = 2\left[k^{\mu}k^{\nu} + k^{\nu}k^{\mu} - \left(k'.k - m^2\right)g^{\mu\nu}\right] , \tag{2.35}$$

and considering that $q^{\mu}L^{e}_{\mu\nu} = q^{\nu}L^{e}_{\mu\nu} = 0$ (current conservation), we have

$$\left(L^e\right)^{\mu\nu}W_{\mu\nu} = 4W_1\left(k.k'\right) + 2\frac{W_2}{M^2}\left[2\left(p.q\right)\left(p.k'\right) - M^2\left(k.k'\right)\right] . \tag{2.36}$$

Using the kinematic relations Eq. (2.15), we obtain in the laboratory reference

$$\left(L^e\right)^{\mu\nu}W_{\mu\nu} = 4EE'\left[\cos^2\left(\frac{\theta}{2}\right)W_2\left(\nu, q^2\right) + 2\,\sin^2\left(\frac{\theta}{2}\right)W_1\left(\nu, q^2\right)\right] , \tag{2.37}$$

including the flow factor and the phase space factor, we have the differential shock section even for the electron-proton inelastic scattering $e^- p \to e^- X$

$$d\sigma = \frac{1}{4\left[\left(k.p\right)^2 - m^2M^2\right]^{\frac{1}{2}}}\left[4\pi M\,\frac{e^4}{q^4}\left(L^e\right)^{\mu\nu}W_{\mu\nu}\right]\frac{d^3k'}{2E'\left(2\pi\right)^3} . \tag{2.38}$$

The extra factor $4\pi M$ arises by normalization of $W^{\mu\nu}$. Inserting Eq. (2.37) into Eq. (2.38), we finally get

$$\frac{d\sigma}{dE'd\Omega}_{lab} = \frac{\alpha^2}{4E^2\sin^4\left(\theta/2\right)}\left[W_2\left(\nu, q^2\right)\cos^2\left(\frac{\theta}{2}\right) + 2\,W_1\left(\nu, q^2\right)\sin^2\left(\frac{\theta}{2}\right)\right] , \tag{2.39}$$

where the mass of the electron $m_e$ is neglected.

The test for the proton to be composed of point particles is the behavior of the differential shock section, within the process $e^- p \to e^- X$ the differential shock section is given as follows

$$\frac{d\sigma}{dE'd\Omega} = \frac{4\alpha^2E'^2}{q^4}\left[W_2\left(\nu, q^2\right)\cos^2\left(\frac{\theta}{2}\right) + 2\,W_1\left(\nu, q^2\right)\sin^2\left(\frac{\theta}{2}\right)\right] . \tag{2.40}$$

If we compare the result with the elastic shock section with a point proton, in the scattering of high-energy virtual photons $-q^2 \to \infty$, we can write

$$\frac{d\sigma}{dE'd\Omega} = \frac{4\alpha^2 E'^2}{q^4}\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2m_q^2}\sin^2\left(\frac{\theta}{2}\right)\right]\delta\left(\nu + \frac{q^2}{2m_q}\right) . \tag{2.41}$$

So the proton structure function is

$$2W_1^{\text{pontual}} = \frac{Q^2}{2m_q^2}\delta\left(\nu - \frac{Q^2}{2m_q}\right) \tag{2.42}$$

$$W_2^{\text{pontual}} = \delta\left(\nu - \frac{Q^2}{2m_q}\right) . \tag{2.43}$$

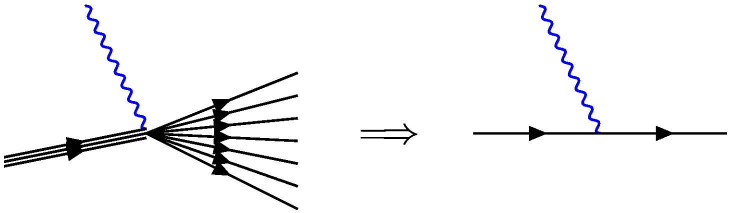

Figure 2.5: Illustration of the Eq. (2.43)

## 2.4 Bjorken scale and the Partons model

When entering a positive variable $Q^2 = -q^2$, where $m_q$ is the mass of the quark, then at $Q^2 \to \infty$ in DIS, electron-proton scattering is an elastic scattering of an electron by a free quark within the proton.

Using the delta identity of Dirac $\delta(x/a) = a\delta(x)$ in Eq. (2.43) we were able to rearrange the terms to introduce the dimensionless structure

$$2m_q W_1^{\text{pontual}}\left(\nu, Q^2\right) = \frac{Q^2}{2m_q\nu}\delta\left(1 - \frac{Q^2}{2m_q\nu}\right) \tag{2.44}$$

$$\nu W_2^{\text{pontual}} = \delta\left(1 - \frac{Q^2}{2m_q\nu}\right) , \tag{2.45}$$

These functions are just functions expressed in terms of $Q^2/2m_q\nu$ and not $Q^2$ and $\nu$, so if the virtual photons in $Q^2 \to \infty$ solve the point constituents within the proton, we can get the following expressions for point particles

$$MW_1^{\text{point}}\left(\nu, Q^2\right) \quad \longrightarrow \quad F_1(w) \tag{2.46}$$

$$\nu W_2^{\text{point}}\left(\nu, Q^2\right) \quad \longrightarrow \quad F_2(w)\ , \tag{2.47}$$

where

$$w = 2\left(\frac{p.q}{Q^2}\right) = 2\left(\frac{M.\nu}{Q^2}\right)\ . \tag{2.48}$$

Note that in Eq. (2.48) there is no scaling. The proton mass $M$ is used, instead of the quark mass $m_q$, to define the dimensionless variable $w$. The presence of free quarks is signaled by the fact that the inelastic structure function is independent of $q^2$ to a fixed $w$ value in Eq. (2.48).

In the parton model, a connection is created between fundamental particles (quarks) and hadrons. Basically, in the parton model, we have

- The proton is made up of a group of partons.
- In deep inelastic scattering, the photon interacts with a parton.
- Partons are elementary particles whose interactions we can calculate.
- These partons are identified as quarks and gluons. The patron has negligible or null transverse momentum.

So the proton is made up of other particles, the partons; these are elemental. Several types of point protons make up the proton. They can carry different fractions $x$ of the energy and total momentum of the proton. We illustrate the momentum distribution of the pattern in Figure (2.7).

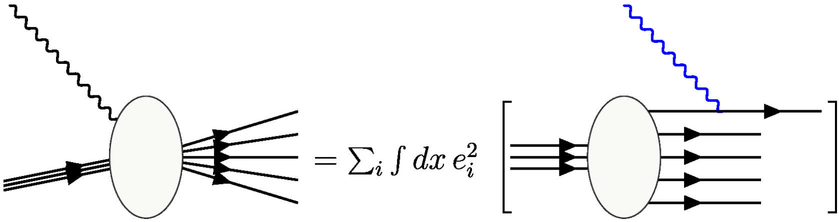


Figure 2.6: The proton consists of point quarks.

The distribution describes the probability of a constituent $i$ carrying a fraction $x$ of the momentum of the proton $p$. So the sum of the fractions $x$ is equal to 1

$$\sum_{i'} \int dx\, x\, f_{i'}(x) = 1. \tag{2.49}$$

where the sum over $i'$ denotes summation over all partons (quarks and gluons included).

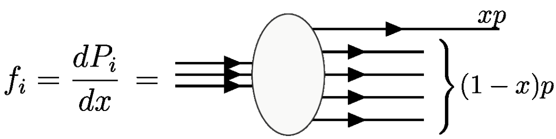


Figure 2.7: Distribution of Momentum of Partons.

| | Proton | Parton |
|---|---|---|
| Energy | $E$ | $xE$ |
| Momentum | $p_L$<br>$p_T = 0$ | $xp_L$<br>$xp_T = 0$ |
| Mass | $M$ | $m = \sqrt{x^2E^2 - x^2p_L^2} = xM$ |

Both the proton and proton move along the z axis, (i.e. transverse moment $p_\perp = 0$) with longitudinal momentum $p_L$ and $xp_L$. As for dot-type protons, we have

$$M^2 \to M_i^2 = p_i^2 = (x_iE, \vec{p}_T \sim 0, x_ip_L)^2 = x_i^2M^2\ , \tag{2.50}$$

so the momentum of a patron is: $p_i = x_i P$. So for an electron colliding with a fraction with momentum $x$ and unit charge $e$, from Eq. (2.45) and Eq. (2.47) we have the functions of dimensionless structure

$$F_1(w) = \frac{Q^2}{4m_q \nu x}\delta\left(1-\frac{Q^2}{2m_q\nu}\right) = \frac{1}{2x^2 w}\delta\left(1-\frac{1}{xw}\right) \quad (2.51)$$

$$F_2(w) = \delta\left(1-\frac{Q^2}{2m_q\nu}\right) = \delta\left(1-\frac{1}{xw}\right)\ , \quad (2.52)$$

where we use the kinematics of Eq. (2.51) and $w$ is the dimensionless variable defined in Eq. (2.48). The $F_{1,2}$ structure function is for a proton in Eq. (2.51), now we add over all the proton constituent protons, Fig. (2.6) and Fig. (2.7) so we get

$$F_2(w) = \sum_i \int dx\, e_i^2\, f_i(x)\, \delta\left(x - \frac{1}{w}\right) \quad (2.53)$$

$$F_1(w) = \frac{w}{2}\, F_2(w)\ . \quad (2.54)$$

It is conventional to redefine $F_{1,2}(w)$ as $F_{1,2}(x)$ and express the result in terms of $x$. When compared to Eq. (2.47), Eq. (2.53) takes the same expression when $Q^2 \to \infty$. So to sum all the partons,

$$\nu W_2^{pontual}(\nu, Q^2) \overset{Q^2\to\infty}{\longrightarrow} F_2(x) = \sum_i e^2\, x\, f_i(x) \quad (2.55)$$

$$M W_1^{pontual}(\nu, Q^2) \overset{Q^2\to\infty}{\longrightarrow} F_1(x) = \frac{1}{2x}\, F_2(x)\ , \quad (2.56)$$

being

$$x = \frac{1}{w} = \frac{Q^2}{2M\nu}\ . \quad (2.57)$$

*The moment fraction is identical to the $x$ kinematic variable of the virtual photon, so the virtual photon must have exactly the value of the $x$ variable to be absorbed by a parton with a moment fraction $x$* . Due to the "delta $\delta$ function" in Eq. (2.53) we can equate these two distinct physical quantities.

Thus, the structure function for a parton with momentum $p_i = x_i$

$$F_1(x_i) = \frac{Q^2}{4Mx_i^2\nu}\delta\left[1 - \frac{Q^2}{2Mx_i\nu}\right] = \frac{1}{2}\frac{x}{x_i}\delta\left(x_i - x\right) \tag{2.58}$$

$$F_2(x_i) = \delta\left[1 - \frac{Q^2}{2mx_i\nu}\right] = x_i\delta\left(x_i - x\right)\ , \tag{2.59}$$

where $F_1$ the approach $MW_1 \to (x_iM)\,W_1$ is used.

We can add the result of a proton over all the protons, in this case for the proton

$$F_1(x) = \sum_i \int_0^1 dx_i\, e_i^2 f_i(x_i)\,\frac{1}{2}\,\frac{x}{x_i}\delta\left(x_i - x\right) = \frac{1}{2}\sum_i e_i^2\, f_i(x) \tag{2.60}$$

$$F_2(x) = \sum_i \int_0^1 dx_i\, e_i^2\, f_i(x_i)\, x_i\delta\left(x_i - x\right) = \sum_i e_i^2\, x\, f_i(x) \tag{2.61}$$

These structure functions for $1/2$ spinning partons are related by the *Callan - Gross Relation,*[1] where

$$2xF_1(x) = F_2(x) = \sum_i e_i^2\, x\, f_i(x)\ . \tag{2.62}$$

The inelastic structure functions $F_{1,2}$ from Eq. (2.55) and Eq. (2.56) are functions of the $x$ variable only. They *are independent from* $Q^2$ *to a fixed* $x$. So, it is said that they satisfy the Bjorken scale.

In short we have

*In deep elastic scattering, we have*

- The proton is characterized by form factors that are independent of the energy scale. $Q^2$.

---

[1] This relationship is recurring from the details of the disturbing shock section, confirming between the patron the existence of Dirac particles, spin $1/2$

*The parton model*

- Reproduces Bjorken's scaling behavior.
- Is a model for the proton structure with structural factors (which are measured experimentally) interpreted as distribution functions of quarks and gluons within the proton, which are independent of the energy scale.
- Predicts the Callan-Gross relationship: $xF_1(x) = F_2(x)$
- Considers the partons as free particles inside the hadron (inelastic structure function is independent of $q^2$ for a value of $x$)

*In QCD, some important features*

- Asymptotic freedom (quarks are treated as free particles inside high energy hadrons).
- Confinement, not observing free quarks and gluons, which guarantees the existence of colorless hadrons.
- Gauge theory (predicts a massless particle which carries the strong interaction: the gluon). Therefore, QCD fits perfectly into the standard model.

### 2.4.1 Distribution function with partons

Measurements of the large inelastic structure functions for $Q^2$ reveal the hadron structure to quarks. The sum in Eq. (2.55) is overall protons in the proton, so

$$\begin{aligned}\frac{1}{x}F_2^{ep}(x) &= \left(\frac{2}{3}\right)^2\left[u^p(x)+\bar{u}^p(x)\right]+\left(\frac{1}{3}\right)^2\left[d^p(x)+\bar{d}^p(x)\right]+\\ &+\left(\frac{1}{3}\right)^2\left[s^p(x)+\bar{s}^p(x)\right],\end{aligned} \tag{2.63}$$

where $u^p(x)$ e $\bar{u}^p(x)$ are the probability distributions of the "$u$" quarks and antiquarks in the proton and neglect the presence of "$c$" quark charm and heavier quarks.

The inelastic structure function for neutrons is performed experimentally by scattering electrons by a deuterium target.

$$\frac{1}{x}F_2^{en} = \left(\frac{2}{3}\right)^2\left[u^n(x)+\bar{u}^n(x)\right]+\left(\frac{1}{3}\right)^2\left[d^n(x)+\bar{d}^n(x)\right]+\left(\frac{1}{3}\right)^2\left[s^n(x)+\bar{s}^n(x)\right], \tag{2.64}$$

and since proton and neutron are members of an isospin doublet, their quarks are related by

$$u^p(x) \quad = d^n(x) \quad \equiv u(x) \tag{2.65}$$

$$d^p(x) \quad = u^n(x) \quad \equiv d(x) \tag{2.66}$$

$$s^p(x) \quad = s^n(x) \quad \equiv s(x)\ . \tag{2.67}$$

The proton consists of valence quarks with the combination $uud \equiv u_v u_v d_v$, accompanied by quark-antiquark pairs known as "*sea quarks*". Fig. (2.8) illustrates this process in detail.

If we assume that the sea is symmetrical in the flavors of the $SU(3)$ quarks, then for each $q(x)$ quark we have a valence quark $q_v$ and a sea antiquark $q_s$ listed as follows.

$$S(x) = u_s(x) = d_s(x) = s_s(x) = \bar{u}_s(x) = \bar{d}_s(x) = \bar{s}_s(x), \tag{2.68}$$

therefore, we have

$$u(x) \quad = \quad u_v(x) + u_s = u_v + S(x)\ , \tag{2.69}$$

$$d(x) \quad = \quad d_v(x) + d_s = d_v + S(x)\ , \tag{2.70}$$

where $S(x)$ is the distribution of the sea common to all flavors, if we assume that *sea is symmetrical*. So we have $S(x)$ has 06 antiquarks from the sea.

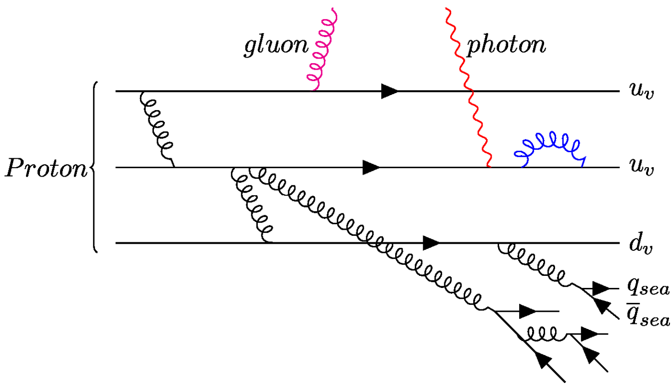


Figure 2.8: Proton with valence quarks $q_v$, gluons and sea quark-antiquark pairs $q_{sea}, \overline{q}_{sea}$