Further Insights into Oscillation Theory

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By

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FOREWORD

The history of understanding of the phenomenon of synchronization begins in 1665, with the famous experiment of H. Huygens with a clock hanging on one beam. Observing the course of a wall clock located on a beam, Huygens noticed an extraordinary coherence of the rhythms of their movement, whereas without a common beam, the coherence of the clock's course disappeared. He made the correct conclusion that the reason for this was the beam, which played the role of a coupling, leading to the interaction of objects and, as a consequence, to the coherence of their movements. Unfortunately, it is not known whether the genius scientist and inventor (among his inventions, there is a pendulum clock with a trigger, invented in 1657) realized the global nature of the phenomenon that he observed and its determining role in animate and inanimate nature.

The next benchmark case in the history of synchronisation is the capture of organ tube vibrations by vibrations of the tuning fork (forced synchronisation), which was observed by D. Raleigh (1878), who later constructed the Theory of Sound (1878).

The systematic character of experimental studies of synchronization is acquired only at the beginning of the 20th century resulting from the origination and rapid development of new areas of engineering knowledge: radio engineering and radiolocation.

The first experimental work on synchronization of triode generators is the work of E. V. Appleton. In 1922, studying the influence of periodic electromotive force (EMF) on the lamp generator, Appleton found forced synchronization of oscillations of this generator. Since then, radio generators have been an extremely convenient tool for experimental research demonstrating not only the phenomenon of synchronization, but also general properties of dynamic systems.

The lack of an adequate mathematical apparatus at the beginning of the 20th century did not allow generalizing numerous experimental results in the form of mathematical models, analysing them and explaining them analytically. Therefore, the essence of synchronization as a purely nonlinear phenomenon has long been considered terra incognita.

With regard to synchronization (and nonlinear physics in general), a revolutionary event was a creation of the qualitative theory of A. Poincaré's differential equations [1] and A. Lyapunov's theory of stability of motion [2]. The combination of these theories served as a basis for the development of all modern nonlinear dynamics, including the theory of synchronization.

Analytical studies of synchronization of periodic oscillations begin with pioneering papers of Van der Pol (1927) [3], A. A. Andronov and A. A. Vitt (1930) [4]. Van der Pol formulated the problem of forced synchronization of a local oscillator in the form of a non-autonomous nonlinear differential equation of the second order, which is nowadays known as the Van der Pol oscillator and became one of the canonical equations of nonlinear dynamics. Van der Pol also proposed an original method to investigate the equation, motivating his actions (averaging) only by physical considerations of the different orders of magnitude (by a small parameter) of changes of the variables: amplitude and phase of oscillations. For a long time, this method and its results were considered at best as "approximate" and "engineering". These were considered as such by A. A. Andronov and A. A. Vitt, who proposed a solution to the problem based on the Poincaré method, in a more general statement, with a mathematically rigorous justification of all "details" of the study. It was then surprising that in the particular case of cubic nonlinearity (Van der Pol's nonlinearity), the results of these two studies coincided. Nowadays, when the meaning of Van der Pol's intuitive averaging procedures has long been known, one can only be surprised by its ingenious discovery.

Next in terms of importance and chronology are the works of L. I. Mandelshtam and N. D. Papaleksi [5], K. F. Teodorchik [6, 7], W. V. Lyon and H. E. Edgerton [8], L. D. Goldstein [9] and other authors. An exceptional contribution to the theory of synchronization of various systems was made by A. A. Andronov's colleagues and students: A. A. Vitt, S. E. Khaikin, N. A. Zheleztsov [10, 11], A. G. Mayer [12], N. N. Bautin, E. A. Leontovich [13], Yu. I. Neimark [14], N. V. Butenin, N. A. Fufaev [15] and subsequent generations of this research school.

A defining event in the development of the theory of dynamical systems in general and synchronization theory in particular, was the discovery of H. M. Krylov and N. N. Bogolyubov of the method of averaging (1934) [16]. As a set of theorems and algorithms, this method did not only justify the procedure of Van der Pol (a side result), but also initiated a whole direction of research of invariant manifolds of dynamical systems directly related to the theory of synchronization [17–20].

The exceptional efficiency of the method of the averaging, as well as its relation to the method of point mappings, together with the simplicity of the interpretation of results, have led to the massive appearance of works on various aspects of synchronization of periodic oscillations. Significant contributions to the theory of synchronization of systems with direct couplings, as well as its practical application were made by the works of N. N. Moiseev [21], I. I. Blekhman [22, 23], R. V. Khokhlov [24], G. M. Utkin [25], P. S. Landa [26], L. V. Postnikova and V. I. Korolev [27], V. V. Migulin [28], I. I. Minakova [29], Yu. M. Romanovsky [30], M.F. Dimentberg [31], L. Cesari [32], N. Levinson [33] and many others.

As for the directly related to the synchronization analytical studies of nonlocal bifurcations, the destruction of invariant tori and the formation on this base of chaotic attractors, the works of the Nizhny Novgorod mathematical school of L. P. Shilnikov [34–36] are fundamental in this area.

The above refers to the case of synchronization of directly coupled dynamical systems.

Simultaneously with the beginning of research on synchronization of directly coupled oscillators (generators) in the field of radio communication, a new direction emerged: systems for automatic frequency tuning of one source (tuneable generator) to the frequency of another ("reference") oscillator: the automatic frequency control system (AFC); and the same sense of the phase-locked loop system (PLL). Together they are called the systems of phase synchronization (SPS).

The first phase-locked loop system was proposed by B.P. Terent'ev in 1930 [37], while the theory of these systems originates from the works of de Belsiz [38], F. Tricomi [39], and C. Travis [40].

At present, none of the means of television and radio communication can do without systems of phase synchronization, as well as none of the means of remote control of complex technical systems. The modern state of the art of systems of phase synchronization technology is due to the work of a large team of international researchers: the list of references is huge and most of it can be found in monographs [41–44].

For the theory of oscillations as a scientific discipline that studies the dynamics of mathematical models common to various areas of natural science, it is important that PSS models are simultaneously mathematical models of a large number of physical systems. These models will be the subject of study in this monograph: dynamical systems of phase or, in other words, pendulum type

$$I\ddot{\varphi}_i + \lambda_i\left(1 + f_{1i}\left(\varphi_i\right)\right)\dot{\varphi}_i + f_{2i}\left(\varphi_i\right) = \gamma_i + F_i\left(\varphi_j, \dot{\varphi}_j, \psi, \mathbf{x}, \dot{\mathbf{x}}\right),$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mu\mathbf{X}\left(\mathbf{x}, \boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}, \psi\right), \qquad \text{(F1)}$$

$$\dot{\psi} = \omega_0.$$

Here $i,\ j = \overline{1, n},\ \ \varphi_i \in S,\ \ \psi \in S,\ \ \mathbf{x} \in R^m;\ I,\ \lambda_i,\ \gamma_i$ are the constant parameters, $\mathbf{A}$ is $\left(m \times m\right)$ constant stable Hurwitz matrix; $F_i, \mathbf{X}$ are the functions of couplings. All functions entering (F1) are periodic according to the phase variables. The system is defined in a toroidal phase space $G\left(\boldsymbol{\varphi},\ \dot{\boldsymbol{\varphi}},\ \psi,\ \mathbf{x}\right) = T^{n+1} \times R^{n+m}$.

Among the physical systems that have mathematical models that belong to the (F1) class, there are autonomous and non-autonomous systems of coupled superconducting Josephson junctions [45, 46]; systems of coupled Froude pendulums [47]; coupled electrical machines [13, 48, 49]; vibrational mechanisms for various purposes [22, 50–52]; unbalanced shafts that are flexible to bending and torsion [50, 53, 54] and many other systems. We call equations of the form (F1) *systems of coupled rotators*.

The development of new and adaptation of existing methods to study specific dynamical systems is an independent problem that presents one of the main tasks of the theory of oscillations. It must be said that the priority of this problem was determined by A. A. Andronov at the beginning of the creation of the theory of oscillations as a new scientific direction. As for systems of the class (F1), analytical and qualitative-numerical methods are most developed for the study of limited motions of pendulum systems. They are developed mainly for solving problems of the systems of phase synchronization and problems of automatic control, which is covered by papers of A. I. Lurie [55], E. A. Barbashin, N. N. Krasovsky, V. A. Tabueva [56, 57], V. M. Popov [58], V. A. Yakubovich, A. H. Gelig, G. A. Leonov [42, 59, 60], V. V. Shakhgildyan [41, 61], Yu. N. Bakaeva [62], V. N. Belykh, V. I. Nekorkin [42, 63–65], V. P. Ponomarenko, V. D. Shalfeev, L. A. Belyustina [42, 65, 66] and other authors. In this monograph, an adaptation of the method of the averaging will be proposed for the effective study of synchronization, dynamical chaos in the class of rotational motions of systems of coupled rotators.

Discovery in 1983 of synchronization of chaotic oscillations for identical (Yamada T. and Fujisaka H. [67]) and, independently in 1986, for non-identical self-oscillating systems with chaotic dynamics (V. S. Afraimovich, N. N. Verichev, M. I. Rabinovich [68]) changed the existing to date understanding of the phenomenon of synchronization to a large extent. In contrast to classical synchronization (synchronization of periodic oscillations), due to its prevalence and familiarity seeming almost obvious, chaotic synchronization, on the contrary, seemed unlikely and even impossible. The reason for this is the prevailing belief at that time that the interaction of internally unstable systems can only generate an increase in the instability of a coupled system. However, it turned out that this is not entirely true: as a result of the "interaction of strange attractors", a new strange attractor (an image of synchronization) can be born, such that the motions of the individual systems, while remaining chaotic, become synchronized when a coupled system moves on this attractor. In the course of studying the chaotic synchronization (1985), it became clear that the very definition of synchronization, which had previously been reduced to the commensurability of frequencies, needs an updating. At that time, it was already clear that chaotic synchronization, unique in its properties, which can be implemented using simple technical solutions (in particular, radio circuits), will find the widest application, which has been confirmed over time.

One of the areas of application of chaotic synchronization is modern information technology. The first experiments on information transmission based on chaotic synchronization were carried out by A. S. Dmitriev, A. I. Panas, and S. O. Starkov [69]; L. Kocarev, K. S. Halle, K. Eckert, L. Chua, U. Parlitz [70]; H. Dedieu, M. Kennedy, M. Hasler [71]. The use of PLL systems in the transmission of information with chaotic signals was investigated by V. V. Matrosov [72].

The attractiveness of the dynamical chaos and chaotic synchronization for the transmission of information is explained by several reasons. First, by the broadband signal of the carrier, and, consequently, by the large information capacity. Second, by the possibility of confidentiality (secrecy) of information transfer. Communication confidentiality is achieved due to the fact that the "reference" signal generator (oscillator), which determines the chaotic carrier of the information signal, can be synchronized only with the signal generator (oscillator) that has analogous dynamics.

Another area of application is modelling of the biological neural tissues and artificial neuron-like networks. Numerous physiological observations of the activity of various parts of the brain show the chaotic nature of their dynamics. It can be such as a reflection of a normal life activity or arise as a result of a crisis state of the object [73, 74]. Therefore, modelling neural networks in the form of interconnected dynamical systems with chaotic dynamics seems plausible and provides quite adequate results. The number of publications devoted to modelling of such systems is large. Some of them are covered in reviews [75, 76]. Basically, the classical models of Hodgkin – Hasley [77], Fitz Hugh – Nagumo [78], Kolmogorov – Petrovsky – Piskunov [79] and their modifications are chosen as a base of the networks.

Chaotic synchronization is a complex phenomenon, and research into its various aspects is still ongoing. Significant contributions were made by the studies of L. M. Pecora, T. L. Carroll [80], N. F. Rulkov, A. R. Volkovsky [81], P. Ashwin, J. Buescu, I. Stewart [82], S. C. Venkataramani, B. R. Hunt, E. Ott, D. J. Gaunthier, J. S. Bienfang [83, 84], A. S. Pikovsky, J. Kurths [85], V. S. Anischenko [86], etc. Due to the large number of papers, providing a somewhat complete and ordered list of publications is simply impossible. However, despite the large number of studies, the problem of chaotic synchronization remains relevant to this day, and this primarily relates to research methods. Below we describe in detail both the phenomenon itself and its asymptotic theory.

Another area of application of chaotic synchronization is dissipative structures, the study of which was initiated by I. R. Prigogine [87]. Chaotic synchronization adds a new aspect to the development of this trend, which can be expressed by the phrase “ordered chaos from universal chaos”, which in its meaning complements the dictum fixed in the title of the famous monograph by I. Prigogine and

I. Stengers "Order out of chaos" [88]. In lattices of dynamical systems (oscillators), as discrete analogues of an active continuous medium, such structures are called "cluster" structures. In addition to modelling processes in a continuous medium, there is also an independent interest in the study of cluster structures in lattices of oscillators: a large number of objects of animate and inanimate nature have or may have similar structures [89–93].

It should be noted that most of the results of studies of "cluster" dynamics were obtained by computer simulation of systems. Analytical results are not so numerous and mainly refer to isotropic lattices [94–101]. Our interest will be in ordering the theory of cluster structures in oscillator lattices and bringing the mechanisms of their formation in accordance with generally accepted concepts of the phenomenon of synchronization.

CHAPTER 1

OSCILLATORS AND ROTATORS WITH CHAOTIC DYNAMICS

In this chapter, we discuss classical and well-known oscillators with chaotic dynamics, which we will use as subjects of chaotic synchronization to illustrate analytical results. Here we consider autonomous and non-autonomous models with a cylindrical phase space. Interest in such models has two aspects. First, along with oscillators, they are of interest as subjects of synchronization. Secondly, these systems are much less studied and described in the scientific and technical literature, despite being of considerable interest for their applications.

1.1. Oscillators with chaotic dynamics

Lorenz oscillator. In 1963, while studying convection in a layer of liquid heated from below, E. Lorentz presented a visually simple dynamic system of the form

$$\dot{x} = -\sigma(x - y),$$
$$\dot{y} = -y + rx - xz,$$
$$\dot{z} = -bz + xy. \tag{1.1}$$

The system describes the dynamics of a fluid in the form of convective rolls. To construct it from the equations of hydrodynamics, the Galerkin method and the Boussinesq approximation [102] were used. The physical meaning of variables and parameters is as follows: x is the velocity of rotation of rolls; y, z is the temperature of the fluid in horizontal and vertical directions, σ is the Prandtl number, r is the normalized Rayleigh number, and b is the convection cell scale parameter. In a numerical study of equations (1.1), E. Lorentz discovered a complex non-periodic behavior of the system, which corresponds to a nontrivial attracting set in the phase space of the model, called the strange attractor or the Lorentz attractor (Fig. 1.1). Subsequently, the chaotic nature of this attractor was proved in [103].

There exist numerous publications devoted to the study of various properties of this system. The derivation of the Lorentz equations from the Navier – Stokes equations, as well as the properties of this system, can be learned from [104 – 108]. Note that in addition to thermal convection in the layer, the Lorentz system also simulates fluid convection in an annular tube, the dynamics of a single-mode laser, and dynamics of a water wheel [106]. As it will be shown below, this system is also related to the dynamics of vibrational mechanisms, superconductive junctions, as well as bending vibrations of shafts.

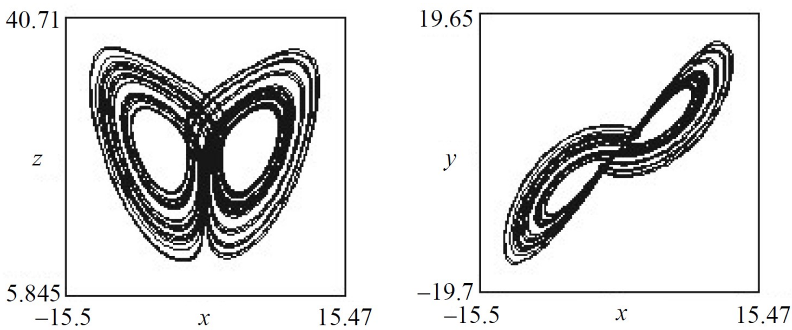

Fig. 1.1. Projections of Lorentz attractor onto coordinate planes for: $\sigma = 10, r = 25, b = 8/3$.

In what follows, let us list certain the properties of the Lorentz system that will be required by us later on.

1. *Dissipativity*. Consider a quadratic form

$$V = \frac{1}{2}\left(x^2 + y^2 + \left(z - \sigma - r\right)^2\right).$$

The derivative of this function, calculated by virtue of system (1.1), has the form

$$\dot{V} = -\sigma x^2 - y^2 - bz^2 + b\left(\sigma + r\right)z$$

and is negative outside the ball $V \le 2(\sigma + r)^2$ (it is assumed that $\sigma > b$). That is, all limit sets of trajectories in the phase space $G(x, y, z)$ of Lorenz system are limited by a dissipation ball. Due to the invariance of the system to a change of the form: $x \to -x$, $y \to -y$, any limit set of trajectories is symmetric with respect to the plane $x = y$ or has a symmetrical "twin".

2. *Equilibria*. In a general case, there exist three equilibria: $O(0, 0, 0)$, $O_1\left(\sqrt{r-1}, \sqrt{r-1}, r-1\right)$ and $O_2\left(-\sqrt{r-1}, -\sqrt{r-1}, r-1\right)$.

a) For $r < 1$, a single equilibrium $O(0, 0, 0)$ is globally asymptotically stable (GAS). This is established using the Lyapunov function

$$V = \frac{1}{2\sigma}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2,$$

the derivative of which by virtue of system (1.1)

$$\dot{V} = -x^2 + (r+1)xy - y^2 - bz^2$$

is negative in the entire phase space for $0 < r < 1$.

b) For $r > 1$, equilibrium $O(0, 0, 0)$ represents a saddle with two-dimensional stable and one-dimensional unstable manifolds, $\dim u^s = 2$, $\dim u^u = 1$. For $1 < r < \dfrac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$, equilibria O_1 and

O_2 ("twins") are stable knots or focuses, while for $r > r_c = \dfrac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$, they represent saddle-focuses with $\dim u^s = 1$, $\dim u^u = 2$. Loss of stability occurs through the reverse Andronov-Hopf bifurcation ("sticking" into the equilibrium of an unstable limit cycle).

3. For $r = r^* < r_c$, a loop of the saddle separatrix is formed such that for $r = r^* + 0$ a pair of saddle cycles and a strange Lorenz attractor are simultaneously born out of it [103]. For the values $r^* < r < r_c$, either a strange attractor or stable equilibria are realized depending on the initial conditions: the region of "metastable" chaos. For $r > r_c$ (but not too large ones), the chaotic Lorentz attractor is the only attracting limit set of phase trajectories of the system.

4. For large *r* (for $\sigma = 10,\ \ b = 8/3,\ \ r > 300$), a symmetric limit cycle exists in the system. For reduced *r*, this limit cycle loses its stability with the birth of a pair of stable "twin" cycles. With a further decrease of parameter *r*, these cycles undergo a period-doubling cascade with the development of the chaotic Feigenbaum attractor [109] (one of the scenarios).

Generalized Lurie Oscillator. This oscillator describes the dynamics of a nonlinear automatic control system [55] and is governed by differential equations of the form

$$\dot{x} = -f(x) + \mathbf{a}^T \mathbf{y},$$
$$\dot{\mathbf{y}} = \mathbf{B}\mathbf{y} + \mathbf{b}x. \qquad (1.2)$$

Here $x \in R^1$, $\mathbf{y} = (y_1, y_2, ..., y_n)^T$, $y_i \in R^1$, $\mathbf{B}$ is the constant stable Hurwitz matrix; **a**, **b** are $(n \times 1)$ constant vectors.

It must be said that equations (1.2) were not investigated by the author in relation to chaotic dynamics. For this, the system was discovered too early: in 1951. However, at present it represents a generalization of many known models of physical systems with chaotic dynamics.

It is assumed in equations (1.2) that nonlinear function $f(x)$ is a continuous function and has a form similar to a cubic parabola with three zeros. For this reason, we will assume that the following inequality is fulfilled for all x:

$$xf(x) \geq mx^2 - l, \qquad (1.3)$$

where m, l are some positive constants. The order of their choice will be discussed below. This condition is satisfied for a large number of applied problems (for the case of local oscillators, it is sufficient to recall the form of the current-voltage characteristics (CVC) of tunnel diodes and of other nonlinear active elements).

Of all the properties (1.2), here we point out only the existence of a dissipation ball, and we provide rest of information about the dynamics of the system using an example of another oscillator, which represents a particular case of (1.2).

Let us introduce an auxiliary linear system of the form

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u},$$
$$\mathbf{u} = (x, \mathbf{y})^T, \quad \mathbf{A} = \begin{pmatrix} -m & \mathbf{a}^T \\ \mathbf{b} & \mathbf{B} \end{pmatrix}. \tag{1.4}$$

With respect to system (1.4), we will assume:

a) equilibrium $\mathbf{u} = 0$ is asymptotically stable;

b) derivative of the Lyapunov function of the form $V = \frac{1}{2}\left(x^2 + \mathbf{y}^T \mathbf{H} \mathbf{y}\right)$, calculated along the trajectories of system (1.4), $\dot{V} = -mx^2 + (\mathbf{H}\mathbf{b} + \mathbf{a})^T x\mathbf{y} + \mathbf{y}^T \mathbf{H}\mathbf{B}\mathbf{y} = -Q(x, \mathbf{y})$, where $\mathbf{H}$ is a certain positive symmetric matrix, is negative in the entire phase space.

The values of parameter *m* will be chosen as the minimum of those values for which properties of system (1.4) are satisfied. This choice has a simple physical meaning: *m* is the minimum active resistance, replacing the nonlinear element, at which the corresponding linear system acquires the property of absolute stability.

In the language of phase space, such a choice defines a surface without contact for linear system (1.4): $V = \text{const} = V\left(m_{\min}\right)$, which represents the boundary of the dissipation ball of nonlinear system (1.2).

Let us show that under condition (1.3) and conditions of system (1.4), system (1.2) is dissipative.

Consider the following quadratic form: $V = \frac{1}{2}\left(x^2 + \mathbf{y}^T\mathbf{Hy}\right)$. Taking its derivative by virtue of system (1.2) and taking into account inequality (1.3), we obtain the following form and estimate:

$$\dot{V} = -xf\left(x\right) + \left(\mathbf{Hb} + \mathbf{a}\right)^T x\mathbf{y} + \mathbf{y}^T\mathbf{HBy} \le -Q\left(x, \mathbf{y}\right) + l.$$

Obviously, the last expression is non-positive outside some ball $x^2 + |\mathbf{y}|^2 \le r^2$. In turn, the negativity of the derivative outside this ball determines the dissipation ball $x^2 + \mathbf{y}^T\mathbf{Hy} \le R^2$ of system (1.2).

Chua's Oscillator. One of the electric circuit diagrams of a local oscillator of chaotic oscillations, proposed by L. Chua [110], is shown in Fig. 1.2. In this Figure, G^* denotes nonlinear element with current-voltage characteristic $I(V)$.

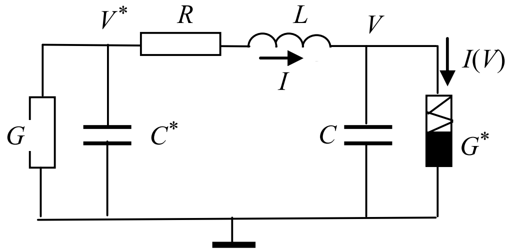

Fig. 1.2. Electric circuit diagram of Chua's oscillator.

This circuit is quite simple as for practical implementation as for a physical experiment. In addition, all the "necessary" properties of this oscillator are found with a piecewise linear volt-ampere characteristic, which makes its mathematical model as simple as possible for analytical research. For these reasons, the dynamics of this local oscillator (and its analogues) has been well studied experimentally, numerically, and analytically [111]. At present, the Chua's oscillator has actually become a classical object of nonlinear dynamics.

In physical variables and parameters, the meaning of which is reflected in Fig. 1.2, the dynamics of the circuit is described by the following equations:

$$C\dot{V} = I - I(V),$$
$$L\dot{I} = -RI + V^* - V,$$
$$C^*\dot{V}^* = -I - GV^*,$$

where $I(V)$ is the volt-ampere characteristic of the nonlinear element with nonlinear conductivity G^*.

By introducing dimensionless time and dimensionless variables and parameters, of the form

$$V = V_0 x,\ I = I_0 y,\ V^* = V_0^* z,\ \frac{R}{L}t = \tau,$$
$$\frac{L}{CR^2} = \alpha,\quad \frac{\alpha I(V_0 x)}{I_0} = \alpha h(x),\quad \frac{L}{C^* R^2} = \beta,\quad \frac{GL}{C^* R} = \gamma,$$

where $V_0,\ I_0,\ V_0^*$ are the scale factors, we obtain the following dynamical system:

$$\begin{aligned} \dot{x} &= \alpha\big(y - h(x)\big), \\ \dot{y} &= -y + z - x, \\ \dot{z} &= -\beta y - \gamma z. \end{aligned} \tag{1.5}$$

The idealized volt-ampere characteristic of the nonlinear element has the form

$$h(x) = m_1 x + \frac{m_0 - m_1}{2}\big(|x+1| - |x-1|\big),$$

where $m_0,\ m_1$ are the constant parameters. Comparing equations (1.5) with (1.2), we see that the Chua oscillator is a special case of the Lurie oscillator. To pass from (1.5) to (1.2), one should assume that $\mathbf{y} = (y, z)^T$, $f(x) = \alpha h(x)$, $\mathbf{a}^T = (\alpha, 0)$, $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ -\beta & -\gamma \end{pmatrix}$, $\mathbf{b} = (-1, 0)^T$.

Fig. 1.3 graphically illustrates the fulfilment of condition (1.3) for the nonlinear function $f(x) = \alpha h(x)$. It follows from the figure that the value of the parameter m can also be chosen to be arbitrarily small.

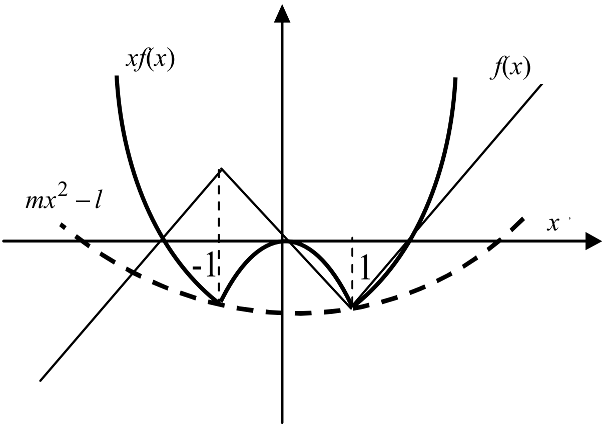

Fig. 1.3. Graphical illustration of the property of a nonlinear function (1.3).

Let us discuss some properties of dynamical system (1.5).

1. *Dissipativity*. Let us show that conditions of system (1.4) are satisfied when its parameters are the parameters of the Chua's oscillator. Let $f(x) = mx.$ In this case, system (1.5) transforms into equations (1.4) and $O(0, 0, 0)$ is it's only equilibrium. Consider the Lyapunov function of the form $V = \frac{1}{2}x^2 + \frac{\alpha}{2}y^2 + \frac{\alpha}{2\beta}z^2.$ It's derivative taken along trajectories of system (1.5) has the form

$\dot{V} = -\left(mx^2 + \alpha y^2 + \frac{\alpha\gamma}{\beta} z^2 \right)$. Since the derivative is negative in the entire phase space, the equilibrium $O(0, 0, 0)$ is generally asymptotically stable. In this case $\mathbf{H} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha/\beta \end{pmatrix}$, and $Q = mx^2 + \alpha y^2 + \frac{\alpha\gamma}{\beta} z^2 \geq 0.$

Let us find a dissipation ball of system (1.5). From Fig. 1.3 and inequality (1.3), we find that maximum $m = \frac{\alpha(m_1 + m_0)}{2}$, wherein $l = \frac{\alpha(m_1 - m_0)}{2}$.

Suppose that $\max(1, \alpha, \alpha/\beta) = \lambda_1$, and $\min(m, \alpha, \alpha\gamma/\beta) = \lambda_2$. In this case $x^2 + \alpha y^2 + \frac{\alpha}{\beta} z^2 \leq \frac{\lambda_1}{\lambda_2} l$ is the dissipation ball.

2. *Equilibria of system (1.5)*. Note that due to oddness of the function $h(x)$, the system is invariant to the change of the variables $x \to -x$, $y \to -y$, $z \to -z$, i.e. possesses central symmetry and all of its limit sets of trajectories inside of the dissipation ball either symmetric with respect to the origin, or have a “twin” symmetric with respect to zero. Naturally, bifurcations of “twins” occur for the same values of parameters.