Dynamical Systems and Differential Geometry via MAPLE

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By

Constantin Udriste and Ionel Tevy

Cambridge Scholars Publishing

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This book first published 2021

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-5275-7223-4
ISBN (13): 978-1-5275-7223-2

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Preface

Our book "Dynamical Systems and Differential Geometry via MAPLE" is designed starting from the course of the same name taught (by the first author) to engineering students, year III, from the Faculty of Applied Sciences, the Mathematical-Computer Section, in the University Politehnica of Bucharest, since 2005. It condenses the offer and the importance of the mutual impact and interconditioning of the three mathematical disciplines. Some subjects are taught at other faculties or universities in the world, especially in the master or doctorate courses, being un-missed from the papers that can be published in journals now categorized as good or very good.

The subject of the course includes twelve Chapters: Dynamical systems; Invariant sets, conservation and dissipation; The stability of equilibrium positions and Hopf's bifurcation; Gyroscopic forces and collision avoidance; Nambu dynamical systems; Geometric dynamics and wind theory; Motions of a curve; Motions of a surface; Evolutive Riemannian metrics; Evolution of Tzitzeica hypersurfaces; Dynamic of evolutive optimization problems; Tasks with Maple. The problems (self-evaluation or research) in each Chapter, while not at all trivial, tremendously enhance one's understanding of the material.

All Chapters were structured by importance, accessibility and impact of theoretical notions capable of shaping a future mathematician-computer scientist possessing knowledge of evolutionary dynamical systems. The intermediate variants of the manuscript have extended over the years, leading to the selection of the most important and manageable notions and reaching maturity through this version of the book that we have decided to publish.

The scientific authority we have has allowed us to impose the point of view on the thematic and on the types of rations that deserve to be offered to readers of texts of dynamical systems, differential geometry and Maple. Everything was structured for the benefit of optimizing evolutionary geometric aspects that describe significant physical or engineering phenomena. It is known that to avoid blocking the students' minds, a simplified language is preferred, specific to the applied mathematics, as we have done as a rule, sending the pure mathematics readers to bibliography. Our long experience as a mathematics professors at a technical university has helped us to fluently expose ideas stripped of excessive formalization, with great impact on students' understanding of natural phenomena encountered in engineering and economics as well as their dressing in mathematical clothes for the intellectual holiday. In this sense, we preferred evolutionary models that intelligently present the real world and we have avoided the hashish of abstract notions specific to theorists.

The novelty problems were finalized following repeated discussions with the mathematics and physics professors from The Faculty of Applied Sciences, especially with those interested in dynamical systems and differential geometry, with Maple simulations, to whom we thank and thoughts of appreciation. We are open to any kind of remarks or criticisms that bring didactic benefits to this course of dynamic evolutionary systems.

The area of Dynamical Systems and Differential Geometry via MAPLE is a mixed field which has become exceedingly technical in recent years. Everything was structured for the benefit of optimizing evolutionary geometric aspects that describe significant physical or engineering phenomena. Also, Maple has a large number of subroutine libraries which have been expressly written to provide solutions to standard homework problems for students.

With this book, the authors provide a self-contained and accessible introduction for graduate or advanced undergraduate students in mathematics, engineering, physics, and economic sciences. Both classical and modern methods used in the field are described in detail, concentrating on the model cases that simplify the presentation without compromising the deep technical aspects of the theory, thus allowing students to learn the material in a short period of time. This book is suitable for self-study for students and professors with a background in differential geometry both, and for teaching a semester-long introductory graduate course in Dynamical Systems and Differential Geometry via MAPLE. Copious exercises are included in each Chapter, and applications of the theory are also presented to connect dynamical systems and differential geometry with the more general areas of dynamical systems, differential geometry and mathematical physics, in large.

The mottos used at Chapters are stanzas from George Cosbuc's poems (in Romanian), selected by the authors and translated in English by Associate Prof. Dr. Brandusa Prepelita-Raileanu, University Politehnica of Bucharest. George Cosbuc (20 September 1866 - 9 May 1918) was a Romanian poet, translator, teacher, and journalist, best remembered for his verses describing, praising and eulogizing rural life, its many travails but also its occasions for joy.

The authors thank Miss Associate Prof. Dr. Oana-Maria Pastae, "Constantin Brancusi" University of Tg-Jiu, for the English improvement of the manuscript.

February 2021

Prof. Emeritus Dr. Constantin Udriste[1], Prof. Dr. Ionel Tevy

Department of Mathematics and Informatics, Faculty of Applied Sciences, University Politehnica of Bucharest, Splaiul Independentei 313, RO-060042, Bucharest, 6, Romania; udriste@mathem.pub.ro ; vascatevy@yahoo.fr

[1] Academy of Romanian Scientists, Ilfov 3, RO-050044 Bucharest 5, Romania.

Chapter 1

DYNAMICAL SYSTEMS

Motto:
Some people hold dear
What's vain in the other men's eye
But He who can scan both the earth and the sky
And set up a bridge between life and death.
George Coşbuc - *The Poet*

In this Chapter we present some concepts regarding the exponential matrix, flows and related dynamical systems that have direct applications in science and engineering. Explicit links: (1) the connection between the exponential matrix and the solutions of linear systems of differential equations with constant coefficients; (2) the relationship between flows and vector fields; (3) the correlation between dynamical systems and vector fields. Suggestions for further reading: [1]-[9].

1.1 Exponential matrix

Let $\mathcal{M}_{n\times n}(\mathbb{R})$ be the set of square matrices of order n, endowed with a norm $\|\cdot\|$. Let I be the unit matrix of order n. For each matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, the power series

$$\sum_{k=0}^{\infty} \frac{A^k}{k!} = I + \frac{A}{1!} + \frac{A^2}{2!} + ...$$

is absolutely convergent and hence convergent. For a matrix of functions, A, the series is uniform convergent on each subset $\{A \mid ||A|| \leq a,\ a \in \mathbb{R}_+\}$. The sum of the previous series is called the *exponential matrix* and is denoted by e^A or $\exp A$.

If $\mathcal{O}$ means zero matrix, then $e^{\mathcal{O}} = I$. Also, the absolute convergence implies the fact that if $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are commutable matrices, i.e., $AB = BA$, then $e^A e^B = e^{A+B}$. It follows that the matrix e^A is invertible and its inverse is e^{-A}.

Here we are interested by the matrix function

$$\varphi : \mathbb{R} \to \mathcal{M}_{n\times n}(\mathbb{R}),\ \varphi(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!},$$

also called the *exponential matrix.* The Cayley-Hamilton Theorem ensures that the exponential matrix e^{tA} is a polynomial of degree $n-1$, where n is the order of the matrix A. The coefficients of the polynomial in A are series of functions. Explicitly

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{n-1} c_k(t) A^k.$$

Theorem 1.1.1. *The matrix function $\varphi(t) = e^{tA}$ is the solution of the Cauchy problem*

$$\frac{d\varphi}{dt}(t) = A\varphi(t),\ \varphi(0) = I.$$

Proof. The series

$$\varphi(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \text{ and } \sum_{k=0}^{\infty} \frac{d}{dt}\frac{t^k A^k}{k!} = A\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

converge absolutely and uniformly in any domain of $\mathbb{R} \times \mathcal{M}_{n\times n}(\mathbb{R})$ characterized by $|t| \leq b$ and $||A|| \leq a$. That is why there exists $\frac{d\varphi}{dt}(t)$ (derivative of the sum of initial series) and it is equal to the sum of the series formed deriving term by term. □

The matrices A and e^{tA} are commutable. This remark allows us to give a nice proof for

Theorem 1.1.2. *The matrix e^{tA} is invertible and its inverse is e^{-tA}.*

Proof. Let us consider the matrix $E(t) = e^{tA}e^{-tA}$, $t \in \mathbb{R}$. By derivation, we find

$$\frac{dE}{dt}(t) = Ae^{tA}e^{-tA} + e^{tA}(-A)e^{-tA} = \mathcal{O},\ \forall t \in \mathbb{R}.$$

It follows $E(t) = E(0) = I$. □

The direct calculation of the matrix e^{tA} is cumbersome or even impossible. That is why we explain some helpful techniques.

Case of a diagonal matrix For a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix},$$

we have

$$D^k = \begin{pmatrix} \lambda_1^k & 0 & ... & 0 \\ 0 & \lambda_2^k & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n^k \end{pmatrix}$$

and hence

$$e^{tD} = \begin{pmatrix} e^{\lambda_1 t} & 0 & ... & 0 \\ 0 & e^{\lambda_2 t} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & e^{\lambda_n t} \end{pmatrix}.$$

Case of a Jordan cell Let J_p be a Jordan cell of order n_p, i.e.,

$$J_p = \begin{pmatrix} \lambda_p & 1 & 0 & ... & 0 \\ 0 & \lambda_p & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & \lambda_p \end{pmatrix} = \lambda_p I_p + E_p,$$

where I_p is the unit matrix of order n_p, and the matrix

$$E_p = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}$$

is nilpotent of order n_p. Taking into account that the matrices I_p and E_p are commutable and computing the powers of E_p, it follows

$$e^{tJ_p} = e^{\lambda_p t I_p} e^{tE_p} = e^{\lambda_p t}\left[I_p + \frac{tE_p}{1!} + \frac{t^2E_p^2}{2!} + ... + \frac{t^{n_p-1}E_p^{n_p-1}}{(n_p-1)!}\right]$$

$$= e^{\lambda_p t}\begin{pmatrix} 1 & \frac{t}{1!} & \frac{t^2}{2!} & ... & \frac{t^{n_p-1}}{(n_p-1)!} \\ 0 & 1 & \frac{t}{1!} & ... & \frac{t^{n_p-2}}{(n_p-2)!} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}.$$

Generally, a Jordan matrix has the form

$$J = \begin{pmatrix} J_1 & 0 & ... & 0 \\ 0 & J_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & J_k \end{pmatrix},$$

where J_p, $p = 1, ..., k$ are Jordan cells of orders n_p and $n_1 + ... + n_k = n$. It follows

$$e^{tJ} = \begin{pmatrix} e^{tJ_1} & 0 & ... & 0 \\ 0 & e^{tJ_2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & e^{tJ_k} \end{pmatrix}.$$

Case of diagonalizable and Jordanizable matrices First, we formulate a general result regarding the similar matrices. This gives rise to a consequence that is helpful to us in the calculations.

Theorem 1.1.3. *If S is a nonsingular matrix and if $S^{-1}AS = B$, then $e^{tA} = Se^{tB}S^{-1}$.*

Proof. The relation $A = SBS^{-1}$ implies $A^k = SB^kS^{-1}$. That is why

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^kA^k}{k!} = \sum_{k=0}^{\infty} \frac{t^kSB^kS^{-1}}{k!} = S\left(\sum_{k=0}^{\infty} \frac{t^kB^k}{k!}\right)S^{-1} = Se^{tB}S^{-1}.$$

□

Corollary 1.1.4. *(1) If A is a diagonalizable matrix, D is the diagonal matrix associated to A, and S is the diagonalizing matrix (the matrix of eigenvectors placed on columns), then $e^{tA} = Se^{tD}S^{-1}$.*

(2) If the matrix A has multiple eigenvalues and is reduced to the Jordan canonical form J, and S is the jordanizing matrix (the matrix of eigenvectors and principal vectors placed on columns), then $e^{tA} = Se^{tJ}S^{-1}$.

1.1.1 Connection with systems of linear differential equations

First, we refer to systems of linear differential equations with constant coefficients. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a real square matrix of order n.

Theorem 1.1.5. *The solution of the Cauchy problem $\dot{x}(t) = Ax(t)$, $x(0) = x_0$ is $x(t) = e^{tA}x_0$, $t \in \mathbb{R}$.*

Proof. The existence and uniqueness theorem together with the extension theorem show that the solution exists, is unique and defined on the whole set $\mathbb{R}$. The rest is just checks by calculations. □

Second, we refer to systems of linear differential equations with nonconstant coefficients.

Theorem 1.1.6. *The matriceal differential equation $\dot{X}(t) = A(t)X(t)$, $X(0) = X_0$ has the solution*

$$X(t) = \exp\left(\int_0^t A(\tau)d\tau\right) X_0$$

iff the matrices $A(t)$ and $\int_0^t A(\tau)d\tau$ commute.

Theorem 1.1.7. *The solution of Cauchy problem $\dot{X}(t) = A(t)X(t)$, $X(0) = X_0$ can be written in the form*

$$X(t) = \exp(\Omega(t))\, X_0\,,$$

where $\dot{\Omega}(t)$ and $A(t)$ are similar and $\Omega(0) = 0$. If the matrices $A(t)$ and $\Omega(t)$, or $\dot{\Omega}(t)$ and $\Omega(t)$, commute, then $\dot{\Omega}(t) = A(t)$.

Proof. Looking for a solution as $X(t) = \exp(\Omega(t))\, X_0$, we compute,

$$\dot{X}(t) = \left(\frac{d}{dt}\exp(\Omega(t))\right) X_0 = \int_0^1 \exp(\alpha\Omega(t))\,\dot{\Omega}(t)\,\exp((1-\alpha)\Omega(t))\; d\alpha\, X_0.$$

But $X_0 = \exp(-\Omega(t))\, X(t)$. Then,

$$\dot{X}(t) = \int_0^1 \exp(\alpha\Omega(t))\, \dot{\Omega}(t)\, \exp((-\alpha)\Omega(t))\; d\alpha\, X(t).$$

Comparing with the given system and applying the mean-value Theorem, we find a function $\alpha : [0,1] \to \mathbb{R}$ such that

$$A(t) = \exp(\alpha(t)\Omega(t))\, \dot{\Omega}(t)\, \exp(-\alpha(t)\Omega(t))\,.$$

This means that the matrices $\dot{\Omega}(t)$ and $A(t)$ are similar.

If the matrices $A(t)$ and $\Omega(t)$ commute, then $A(t)$ and $\exp(\Omega(t))$ commute and there results $\dot{\Omega}(t) = A(t)$, as in the previous theorem. Similarly, if $\dot{\Omega}(t)$ and $\Omega(t)$ commute. □

Example 1.1.8. Find a fundamental matrix of the system of differential equations $\dot{x}(t) = x(t) + ty(t), \dot{y}(t) = tx(t) + y(t)$ making sure that the coefficient matrix

$$A(t) = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix},\ t > 0.$$

commutes with its integral.

The integral of the matrix $A(t)$ is found by elementwise integration. For simplicity, we take the lower boundary of integration to be zero. Then,

$$\int_0^t A(\tau)d\tau = \begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{pmatrix}$$

By computation, we verify $A(t) \cdot \int_0^t A(\tau)d\tau = \int_0^t A(\tau)d\tau \cdot A(t)$. So, the commutative property of the matrix product is true. Therefore, the fundamental matrix is given by

$$\Phi(t) = \exp\left(\int_0^t A(\tau)d\tau\right) = \exp\begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{pmatrix}.$$

We compute the matrix exponential by converting the matrix to the diagonal form. In this case, the eigenvalues and the eigenvectors depend on the variable t. Since the matrix of reduction to the diagonal form is given by

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

we find

$$\Phi(t) = \exp\begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{pmatrix} = e^t \begin{pmatrix} \cosh\frac{t^2}{2} & \sinh\frac{t^2}{2} \\ \sinh\frac{t^2}{2} & \cosh\frac{t^2}{2} \end{pmatrix}.$$

Example 1.1.9. Consider the homogeneous linear differential system $\dot{x}(t) = A(t)x(t)$, where

$$A(t) = \begin{pmatrix} \frac{4}{t} & -\frac{4}{t^2} \\ 2 & -\frac{1}{t} \end{pmatrix}, \quad t > 0.$$

Here the vectors $x^1(t) = {}^T[2t^2, t^3]$, $x^2(t) = {}^T[1, t]$ are particular solutions. Since

$$\begin{vmatrix} 2t^2 & 1 \\ t^3 & t \end{vmatrix} = t^3 > 0,$$

the solutions $x^1(t)$ and $x^2(t)$ are linearly independent. The general solution of the system is $x(t) = c_1x^1(t) + c_2x^2(t)$.

Let us write the solution using the exp matrix. Here, the idea of solving is based on knowing the solution

$$x(t) = \begin{pmatrix} 2t^2 & 1 \\ t^3 & t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Identifying, we obtain

$$\exp(\Omega(t)) = \begin{pmatrix} 2t^2 & 1 \\ t^3 & t \end{pmatrix}, \quad t > 0.$$

1.2 Flows

Let D be an open subset in $\mathbb{R}^n$. A family of functions $\{T^t : D \to \mathbb{R}^n,\ t \in (-\epsilon, \epsilon)\}$ is called *local flow* on D or *local group with one parameter* of diffeomorphisms on D if the following conditions are satisfied:

(1) The function $T : (-\epsilon, \epsilon) \times D \to \mathbb{R}^n$, $(t, x) \to T^t(x)$ is of class C^∞.

(2) For each $t \in (-\epsilon, \epsilon)$, the function $T^t : D \to T^t(D)$ is a diffeomorphism.

(3) For each pair $t, s \in (-\epsilon, \epsilon)$, with $t+s \in (-\epsilon, \epsilon)$ and $T^s(x) \in D$, we have $T^{t+s}(x) = T^t(T^s(x))$, for each $x \in D$. It follows $T^0(x) = x$.

The adjective "local" refers to $t \in (-\epsilon, \epsilon)$. If $(-\epsilon, \epsilon) = \mathbb{R}$, i.e., $\epsilon = \infty$, then this adjective is replaced by "global".

A flow T^t defines a vector field $X(x) = \frac{d}{dt}T^t(x)|_{t=0}$ on D. Conversely, any vector field of class C^∞ defines a flow by its field lines. This thing will be proved in the following Section.

Let us consider the *affine group of transformations*

$$x^{i'} = a^i_j(t)x^j + a^i(t),\ \det(a^i_j(t)) \neq 0,\ t \in (-\epsilon, \epsilon),\ a^i_j(0) = \delta^i_j,\ a^i(0) = 0$$

on $\mathbb{R}^n$. The associated affine vector field is defined by

$$\frac{dx^{i'}}{dt}(0) = \frac{da^i_j}{dt}(0)x^j + \frac{da^i}{dt}(0).$$

Denoting $A = \left(\frac{da^i_j}{dt}(0)\right)$, $a = \left(\frac{da^i}{dt}(0)\right)$, we find $X(x) = Ax + a$. The flow generated by the vector field $X(x) = Ax + a$ consists in certain affine transformations.

1.3 Dynamical systems

The evolution over time of some physical systems can be modelled as dynamical systems.

Definition 1.3.1. *A semigrup G which acts on the space $\mathbb{R}^n$, in the sense that there exists a function $\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi_s \circ \Phi_t = \Phi_{s \circ t}$ is called dynamical system. The function Φ is called flow.*

If G is a group, then we say that the dynamical system is *invertible*.

The discrete dynamical systems correspond to $G = \mathbb{N} \setminus \{0\} = \mathbb{N}_0$ or $G = \mathbb{Z}$ and they have the prototype a recursive sequence described by $\Phi : G \to G$, $\Phi^n = \Phi \circ \Phi^{n-1}$, $n \in \mathbb{N}_0$ or $x(t+1) = X(x(t)),\ t \in \mathbb{N}_0$.

The continuous dynamical systems correspond to $G = \mathbb{R}_+$ or $G = \mathbb{R}$ and they have the prototype described by a first order differential system of the form $\dot{x}(t) = X(x(t))$, where X is a vector field on $\mathbb{R}^n$ of class C^∞. The maximal curve that satisfies the Cauchy problem $\dot{x}(t) = X(x(t))$, $x(0) = x_0$ is defined on the interval $I_{x_0} = (T_-(x_0), T_+(x_0))$ and is called field line. Abandoning the index 0 and entering the set $W = \cup_{x \in \mathbb{R}^n} I_x \times \{x\} \subset \mathbb{R} \times \mathbb{R}^n$, we define the flow $\Phi : W \to \mathbb{R}^n$, $(t, x) \to \Phi(t, x)$, where $t \to \Phi(t, x)$ is an integral curve starting from the point x. In other words, a solution $\alpha = \alpha(t), \alpha(0) =$

x, $t \in (-\epsilon, \epsilon)$ of the differential system $\dot{x}(t) = X(x(t))$ generates a local diffeomorphism

$$T^t : U \subset \mathbb{R}^n \to \mathbb{R}^n,\, T^t(x) = \Phi(t, x),\, t \in I,$$

solution of the operatorial differential equation

$$\frac{dT^t}{dt} = X \circ T^t,\, T^0 = id,$$

called *local flow on* $\mathbb{R}^n$ generated be the vector field X (*local group with one parameter of diffeomorphisms*). The linear approximation $x' = x + tX(x)$ of $T^t(x)$ is called *infinitesimal transformation generated by the vector field* X.

The image set in $\mathbb{R}^n$ of the curve (field line, integral curve) $\alpha_x(t) = \Phi(t, x)$ is called *orbit* passing through the point $x \in \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ which verifies $\Phi(t, x) = x$, for any $t \in I$, is called *equilibrium point* for the system. In this case the orbit passing through the point x is reduced to this point, $\{x\} = \alpha_x$.

If any solution $\alpha = \alpha(t), \alpha(0) = x$ is defined for each $x \in \mathbb{R}^n$ and each $t \in \mathbb{R}$, then the vector field X is called *complete*, and the flow is called *global* (*group with a parameter* of diffeomorphisms).

A graphical representation of the equilibrium points and of some generic orbits of the dynamical system is called the *phase portrait.*

To point things out, let's take the example $X(x) = x^3$, $\dot{x}(t) = x^3(t)$ defined on $\mathbb{R}$. Here we find the flow

$$\Phi(t, x) = \frac{x}{\sqrt{1 - 2x^2 t}},\, W = \{(t, x) \,|\, 2tx^2 < 1\},$$

$$T_-(x) = -\infty,\, T_+(x) = \frac{1}{2x^2}.$$

Example 1.3.1. We consider the linear vector field

$$X(x, y, z) = (x - y + z, 2y - z, z).$$

The field lines are solutions of the linear differential system (with constant coefficients)

$$\frac{dx}{dt} = x - y + z,\ \frac{dy}{dt} = 2y - z,\ \frac{dz}{dt} = z.$$

The third equation has the general solution $z = ce^t$, $t \in \mathbb{R}$. Replacing in the second equation, we find a linear differential equation $\frac{dy}{dt} = 2y-$

ce^t, with the general solution $y = be^{2t} + ce^t$, $t \in \mathbb{R}$. Introducing y and z in the first equation, we get $\frac{dx}{dt} = x - be^{2t}$, with the general solution $x = ae^t - be^{2t}$, $t \in \mathbb{R}$. Obviously $(0,0,0)$ is the only equilibrium point.

A modified flow generated by the previous vector field is

$$T^t : \mathbb{R}^3 \to \mathbb{R}^3,\ T^t(x,y,z) = (xe^t - ye^{2t}, ye^{2t} + ze^t, ze^t),\ t \in \mathbb{R}.$$

In order to study the effect of the flow on the volume, we use the evolution velocity, i.e., the divergence of the vector field X,

$$\operatorname{div}(X) = \frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y} + \frac{\partial X^3}{\partial z} = 1 + 2 + 1 = 4.$$

Since $\operatorname{div}(X) > 0$, the flow associated to this vector field increases the volume.

Example 1.3.2. Study the completeness of the non-linear vector field

$$X(x,y,z) = (xz, yz, -(x^2+y^2)).$$

The orbits of the vector field X, solutions of the symmetric differential system

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{-(x^2+y^2)},$$

are $x = c_1 y$, $x^2+y^2+z^2 = c_2$. These orbits can thus be parameterized

$$x = \sqrt{c_2}\cos u \cos v, y = \sqrt{c_2}\cos u \sin v, z = \sqrt{c_2}\sin u, u \in \mathbb{R},$$

$\cos v = c_1 \sin v$. We parametrize again $u = u(t)$ such that $\frac{dz}{dt} = -(x^2+y^2)$. Since $\frac{dz}{dt} = \frac{dz}{du}\frac{du}{dt}$, it follows $\sqrt{c_2}\cos u \frac{du}{dt} = -c_2 \cos^2 u$ and hence $\frac{du}{\cos u} = -\sqrt{c_2}\, dt$. So,

$$\ln\left|\operatorname{tg}\left(\frac{\pi}{4} - \frac{u}{2}\right)\right| = -\sqrt{c_2}\, t$$

or

$$\operatorname{tg}\left(\frac{\pi}{4} - \frac{u}{2}\right) = e^{-\sqrt{c_2}\, t},\ t \in \mathbb{R}.$$

Consequently, the vector field X is complete.

Shortcut The orbits of X belong to the spheres $x^2+y^2+z^2 = c_2$, and these surfaces are compact. Therefore the vector field X is complete.

1.4 The almost linear expression of a vector field

We consider a real function f of class C^1 defined on a convex neighbourhood V of a point $x_0 \in \mathbb{R}^n$. Then it appears the almost linear expression

$$f(x) = f(x_0) + \int_0^1 \frac{d}{dt} f(x_0 + t(x - x_0))dt$$

$$= f(x_0) + \int_0^1 f_{x^i}(x_0 + t(x - x_0))(x^i - x_0^i)dt = f(x_0) + g_i(x)(x^i - x_0^i),$$

where $g_i(x_0) = f_{x^i}(x_0)$. This result is known as Hadamard's Lemma.

Extending this remark to a vector field X of class C^1, with the property $X(x_0) = 0$, there is a quadratic matrix $A(x)$ such that $X(x) = A(x)x$, $x \in V$. We combine with the remark that a Cauchy problem $\dot{x}(t) = A(x(t))x(t)$, $x(t_1) = x_1$ is equivalent to an integral equation $x(t) = x(t_1) + \int_{t_1}^t A(x(s))x(s)ds$.

1.5 Self-evaluation problems

Problem 1. *Show that $T(t, x) = e^t(1 + x) - 1$ is a flow and write the dynamical system that corresponds to this flow.*

Solution. We show that $T(0, x) = x$ and $T(t + s, x) = T(t, T(s, x))$ (or, if we denote $T(t, x)$ with $T^t(x)$, then $T^0(x) = id(x) = x$ and $T^{t+s}(x) = (T^t \circ T^s)(x)$). The relationships are obviously true, so the application T is a flow.

Since

$$\frac{d}{dt} T^t(x)|_{t=0} = \frac{d}{dt}\alpha_x(t)|_{t=0} = \dot{\alpha}_x(0) = X(\alpha_x(0)) = X(x),$$

the vector field corresponding to this flow is

$$X(x) = \frac{d}{dt} T^t(x)|_{t=0} = \frac{d}{dt}(e^t(1 + x) - 1)|_{t=0} = 1 + x,$$

and the associated differential equation is $\dot{x}(t) = x(t) + 1$.

Problem 2. *Describe the flow generated by the vector field $X(x) = x^3$ on $\mathbb{R}$.*

Solution. X being a vector field on $\mathbb{R}$, the field lines are described by those functions $\alpha : I \to \mathbb{R}$, $\alpha(t) = x(t)$ which verify the ODE $\dot{x}(t) = X(x(t))$, i.e., $\dot{x}(t) = x^3(t)$. By integration, we find the solution $x^2 = \dfrac{1}{c-2t}$, $c \in \mathbb{R}$. The field lines are the functions

$$\alpha(t) = x(t) = \frac{1}{\pm\sqrt{c-2t}}, \quad c \in \mathbb{R}.$$

Let $x \in \mathbb{R}_+$. We are looking for that field line that "leaves" from x, i.e., $\alpha(0) = x$, hence $c = \dfrac{1}{x^2}$. Therefore, the flow associated with the vector field in the statement is the application $T^t(x) = \dfrac{x}{\sqrt{1-2tx^2}}$.

Problem 3. *Find out the flow generated by the linear vector field*

$$X(x, y, z) = (x + y - z, x - y + z, -x + y + z)$$

and investigate its effect on volume.

Solution. The field lines are the solutions of the differential system

$$\dot{\alpha}(t) = X((\alpha(t)), \quad \begin{cases} \dot{x} = x + y - z \\ \dot{y} = x - y + z \\ \dot{z} = -x + y + z. \end{cases}$$

This is a linear system, that can be solved by eigenvalues and eigenvectors method.

The eigenvalues and eigenvectors of the associated matrix are: $\lambda_1 = 1$, whose eigenvector is $v_1 = (1, 1, 1)$; $\lambda_2 = 2$, with $v_2 = (1, 0, -1)$ and $\lambda_3 = -2$, with $v_3 = (1, -2, 1)$. Then the general solution of the ODE system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Equivalently, the field lines are

$$\alpha(t) = (c_1 e^t + c_2 e^{2t} + c_3 e^{-2t}, c_1 e^t - 2c_3 e^{-2t}, c_1 e^t - c_2 e^{2t} + c_3 e^{-2t}),$$

$c_1, c_2, c_3 \in \mathbb{R}$. The only equilibrium point is $(0, 0, 0)$.

By definition, the flow is the map T^t which associates to each arbitrary point (x,y,z) of $\mathbb{R}^3$ the field line $\alpha_{(x,y,z)}(t)$, satisfying $\alpha_{(x,y,z)}(0) = (x,y,z)$. For $t=0$, we obtain the linear algebraic system

$$c_1 + c_2 + c_3 = x,\ c_1 - 2c_3 = y,\ c_1 - c_2 + c_3 = z,$$

with unique solution $c_1 = \dfrac{x+y+z}{3}$, $c_2 = \dfrac{x-z}{2}$, $c_3 = \dfrac{x-2y+z}{6}$. Hence the flow is

$$T^t(x,y,z) = \left(\frac{x+y+z}{3}\, e^t + \frac{x-z}{2}\, e^{2t} + \frac{x-2y+z}{6}\, e^{-2t}\,,\right.$$

$$\frac{x+y+z}{3}\, e^t - 2\frac{x-2y+z}{6}\, e^{-2t},\ \frac{x+y+z}{3}\, e^t - \frac{x-z}{2}\, e^{2t}$$

$$\left. + \frac{x-2y+z}{6}\, e^{-2t} \right).$$

To determine the effect of the flow on the volume, we calculate the divergence of the vector field X, i.e.,

$$\operatorname{div}(X) = \frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y} + \frac{\partial X^3}{\partial z} = 1 - 1 + 1 = 1.$$

Since $\operatorname{div}(X) > 0$, the flow increases the volume.

Problem 4. *Study the effect of flow generated by the non-linear vector field*

$$X(x,y,z) = (x^2 + y^2, 2xy, -z^2)$$

on the volume. Write the straightening diffeomorphism and the corresponding infinitesimal transformation.

Solution. The field lines are the solutions of differential system

$$\dot{x} = x^2 + y^2,\ \dot{y} = 2xy,\ \dot{z} = -z^2.$$

The dynamical system has only one equilibrium point, $(0,0,0)$.

We write the associated symmetric differential system

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{-z^2},$$

to find the first integrals. The first two quotients defines an homogeneous differential equation, with unknown $y = y(x)$. With the change of unknown function $u = \dfrac{y}{x}$, we obtain a separable variables equation $\dfrac{1+u^2}{u-u^3}du = \dfrac{dx}{x}$, with the unknown $u = u(x)$. The fraction in the left member is broken down into simple fractions. As a result of integration, we find the equality $\dfrac{u}{1-u^2} = xc,\ c \in \mathbb{R}$, whence it follows $\dfrac{y}{x^2-y^2} = c_1,\ c_1 \in \mathbb{R}$. The expression $\dfrac{y}{x^2-y^2}$ defines a first integral.

To find another first integral, we use the method of integrable combinations. By adding the numerators and denominators of the first two reports, we reach $\dfrac{d(x+y)}{(x+y)^2} = \dfrac{dz}{-z^2}$. Denote $x+y=v$. Integrating, we obtain $\dfrac{1}{v}+\dfrac{1}{z} = c,\ c \in \mathbb{R}$, i.e., $\dfrac{1}{x+y}+\dfrac{1}{z} = c_2,\ c_2 \in \mathbb{R}$. In this way, the expression $\dfrac{1}{x+y}+\dfrac{1}{z}$ defines also a first integral.

We study the functional independence of the two first integrals. The Jacobi matrix is

$$\begin{pmatrix} -\dfrac{2xy}{(x^2-y^2)^2} & \dfrac{x^2+y^2}{(x^2-y^2)^2} & 0 \\ -\dfrac{1}{(x+y)^2} & -\dfrac{1}{(x+y)^2} & -\dfrac{1}{z^2} \end{pmatrix}.$$

The rank of the matrix is two on $\mathbb{R}^3 \setminus Oz$. Hence, on this set, the two first integrals are functional independent.

The implicit Cartesian equations of field lines are

$$\begin{cases} \dfrac{y}{x^2-y^2} = c_1 \\ \dfrac{1}{x+y}+\dfrac{1}{z} = c_2, \quad c_1, c_2 \in \mathbb{R}. \end{cases}$$

The straightening diffeomorphism (change of variables) is

$$\phi : \mathbb{R}^3 \to \mathbb{R}^3,\ \phi(x,y,z) = (\tilde{x},\tilde{y},\tilde{z}),\ \begin{cases} \dot{\tilde{x}} = 0 \\ \dot{\tilde{y}} = 0 \\ \dot{\tilde{z}} = 1. \end{cases}$$

We define $\tilde{x} = \dfrac{y}{x^2-y^2}$ and $\tilde{y} = \dfrac{1}{x+y}+\dfrac{1}{z}$ (the right hand members are constant along the field lines, so their derivatives with respect to the parameter t is equal to zero).

We are looking for $\tilde{z}$. From $\dfrac{d\tilde{z}}{dt} = 1$, it follows $\dfrac{d\tilde{z}}{dz} \cdot \dfrac{dz}{dt} = 1$, i.e., $\dfrac{d\tilde{z}}{dz} \cdot (-z^2) = 1$, equation with separable variables which by integration leads to $\tilde{z} = \dfrac{1}{z}$.

Let $A = \{(x, y, z) \,|\, x - y = 0 \text{ or } x + y = 0 \text{ or } z = 0\}$. It follows the straightening diffeomorphism

$$\phi : \mathbb{R}^3 \setminus A \to \mathbb{R}^3,\ \phi(x, y, z) = \left(\frac{y}{x^2 - y^2},\ \frac{1}{x + y} + \frac{1}{z},\ \frac{1}{z} \right).$$

Remark. By a diffeomorphism $x^{i'} = x^{i'}(x^i),\ i = 1, 2, 3$, the components X^i of a vector field are changed after the rule (tensorial law)

$$X^{i'} = \frac{\partial x^{i'}}{\partial x^i} X^i.$$

The infinitesimal transformation determined by the vector field X is

$$(x', y', z') = (x, y, z) + tX(x, y, z),$$

i.e.,

$$x' = x + t(x^2 + y^2),\ y' = y + 2txy,\ z' = z - tz^2, \quad t \in (-\varepsilon, \varepsilon).$$

Problem 5. *Find the straightening diffeomorphism and the infinitesimal transformation for the vector field* $X(x, y, z) = (z, -xy, 2xz)$.

Solution. The field lines are the solutions of the differential system

$$\dot{x} = z,\ \dot{y} = -xy,\ \dot{z} = 2xz.$$

The equilibrium points are the solutions of the algebraic system

$$z = 0,\ xy = 0,\ 2xz = 0,$$

i.e., all the points of the form $(\alpha, 0, 0)$ and $(0, \alpha, 0)$, with $\alpha \in \mathbb{R}$. In other words, the axes Ox and Oy consist of equilibrium points.

To find the first integrals, we use the associated symmetric system

$$\frac{dx}{z} = \frac{dy}{-xy} = \frac{dz}{2xz}.$$

The first ratio and the last one determine a differential equation with separable variables, which, by integration, leads to $x^2 - z = c_1, \quad c_1 \in \mathbb{R}$.

The second and third ratios also determine an equation with separable variables, with the implicit solution $zy^2 = c_2, \quad c_2 \in \mathbb{R}$.

The functional independence of the two first integrals, $x^2 - z$ and zy^2, is established using the rank of the Jacobi matrix

$$\begin{pmatrix} 2xy & 0 & -1 \\ 0 & 2yz & y^2 \end{pmatrix}.$$

This matrix has rank 1 on the axis Oy and for the points in the plane xOz, and in the other points its rank is 2, so the two first integrals found are functionally independent on $\mathbb{R}^3 \backslash Oy \cup xOz$. Cartesian equations of orbits are $x^2 - z = c_1, zy^2 = c_2$.

The straightening diffeomorphism is the map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, $\phi(x, y, z) = (\tilde{x}, \tilde{y}, \tilde{z})$, where $\dot{\tilde{x}} = 0, \dot{\tilde{y}} = 0, \dot{\tilde{z}} = 1$.

Define $\tilde{x} = x^2 - z$ and $\tilde{y} = zy^2$, using the first integrals, which are constant along the field lines, so their derivative with respect to the parameter t equal to zero.

Let us find $\tilde{z}$. From the condition $\dfrac{d\tilde{z}}{dt} = 1$, it follows $\dfrac{d\tilde{z}}{dz} \cdot \dfrac{dz}{dt} = 1$, i.e., $\dfrac{d\tilde{z}}{dz} \cdot 2xz = 1$. We express the variable x in function of z, from the Cartesian equations of field lines (using the first integrals). It follows $x = \sqrt{z + c_1}$, with the condition $z + c_1 > 0$. The equation for finding $\tilde{z}$ becomes $d\tilde{z} = \dfrac{1}{2z\sqrt{z + c_1}}\, dz$, which has separable variables. For the calculation of the integral of the right member, a change of variable is made, $\dfrac{1}{\sqrt{z + c_1}} = u$, which ultimately leads to the result

$$\tilde{z} = \frac{1}{2\sqrt{x^2 - z}} \ln \frac{|x| - \sqrt{x^2 - z}}{|x| + \sqrt{x^2 - z}}.$$

The straightening diffeomorphism is the map

$$\phi : D \subset \mathbb{R}^3 \to \mathbb{R}^3,\ \phi(x, y, z) = \left(x^2 - z, zy^2, \frac{1}{2\sqrt{x^2 - z}} \ln \frac{|x| - \sqrt{x^2 - z}}{|x| + \sqrt{x^2 - z}} \right).$$

The infinitesimal transformation determined by X is $(x', y', z') = (x, y, z) + tX(x, y, z)$, i.e., $x' = x + tz, y' = y - txy, z' = z + 2txz, \quad t \in (-\varepsilon, \varepsilon)$.