Systems Theory with Engineering Applications

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For my dear sons, Răzvan and Horațiu, and their families.

In Memory of Monica, my beloved wife.

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PREFACE

Although the rigorous formalism of the systemic conception has been developed in exact sciences, such as theory of automatic control, theory of electrical circuits or electronics, the basic principles of this new concept have emerged simultaneously in several other scientific fields: biology, neurophysiology, psychology, sociology, economics and so on.

Scientific research in the nineteenth century had a rather analytical character, developed under the influence of Newtonian mechanics, Laplacean spirit and the complexity of phenomena. A considerable amount of knowledge had been gained in all sciences. However, in several scientific fields it was found that no matter how abundant the knowledge about disparate elements, it would not allow understanding of the functionality of an ensemble composed of individual elements. The ensemble or the *system* has new properties that cannot be emphasized only in the component parts taken separately.

Thus, *a system* is an ensemble whose parts support each other and form an organized and coherent whole. Such an ensemble is highlighted by its internal structure and interconnections, and is a unified entity relative to the environment. The behaviour of a system depends not only on the properties of its elements, but rather on the interactions between them. What is remarkable in this respect is the fact that similar ideas and concepts have been reached simultaneously and quasi-independently in various scientific fields. Progress in these areas has led to the concept of *system*, the formulation of laws governing *systems*, and to the discovery of isomorphisms between various *systems* existing in the real or physical world.

Physical systems process substance, energy and information. They are connected to the environment through *causes* (inputs) and *effects* (outputs). Between inputs and outputs there exist *causal relations*, abstractly represented by a *transfer operator*. The experimental study of physical systems involves some interaction with the object under consideration and has, in some situations, limited applicability. Typically, alongside experimental procedures, modelling methods are used to explicitly describe the transfer operator using a *mathematical model*.

It follows that the *mathematical model*, as a perfectible image of the physical (real) system, is itself a system, i.e. an *abstract system*. *Abstract systems* are the most effective way to develop a deep knowledge of physical (real) systems. Such abstract systems are the object of study of *mathematical systems theory*, a theory that enables a large degree of abstraction and generalization. Unlike conventional disciplines, which traditionally treat entities in specific instances according to the different disciplines, systems theory treats large classes of entities according to their general characteristics, and groups them into classes that give them a certain degree of generality. Hence the inter- trans- and multi-disciplinary nature of systems theory.

In this context, the *systems theory* is a symbiosis between (applied) mathematics and *systems science* and *systems engineering*. Its origins lie in the theory of electric circuits, electronics and telecommunication, and especially in the theory of automatic control.

This book aims to present within its six chapters, in a rigorous and comprehensible way, the main results on the mathematical description of linear dynamic systems, the input – state – output transfer, the input – output transfer, the controllability and observability of linear dynamic systems, the stability of linear dynamic systems, the stability of automatic control systems, the stability of linear and nonlinear dynamic systems (including the results obtained using the flow invariance method), and the optimal control of dynamic systems. The treatment is both systemic and theoretic in order to achieve rigorous and generally applicative solutions and is illustrated with many engineering examples disseminated throughout the book.

Accordingly, the themes of the book are dealt with through synthetic treatment – being structured on definitions and theorems (mostly demonstrated) – and through many solved examples and, as the case may be, many remarks. These, as well as the mathematical relations, are numbered with two Arabic numbers: the first indicates the subchapter and the second the order number within the subchapter. The meanings of the symbols and abbreviations used are explained after the contents. Six annexes containing some basic mathematical results used throughout the book are included before the bibliography. For the fluency of exposure, bibliographic references are only introduced if necessary. At the same time, the bibliographic list includes a general part and five divisions by domains: linear dynamic systems, multivariable automatic control systems, nonlinear control systems, componentwise asymptotic stability, optimal control systems and, at the end, an actualized list of references for componentwise asymptotic stability. A subject index is added at the end of the book, intended to facilitate the reader's in-depth study.

Through the conception and the range of problems, the book is addressed to scientists in systems engineering, theory and practice of automatic control, computer science, electrical engineering, electronics, and mathematics applied in biology and economics, and also to scientists working in education, in research, in design and in industry. At the same time, for students in these fields the book opens a way to the advanced knowledge of systems theory, which makes possible the theoretical synthesis and effective practical applications based on products offered by science and information technology.

Iaşi, March 2008; August 2020

Mihail Voicu

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Iaşi, Romania, December 2020
Mihail Voicu

LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbb{R}$ – set of real numbers $\mathbb{R}_+$ – set of real non-negative numbers $\mathbb{Z}$ – set of integer numbers $\mathbb{N}$ – set of natural numbers $i=\overline{1,n}$ – index i takes values $1,2,\ldots,n$ $\mathbb{C}$ – set of complex numbers $\mathbb{C}_-$ – set of complex numbers with strictly negative real part $\mathbb{C}_+$ – set of complex numbers with non-negative real part $\Rightarrow$ – it implies $\operatorname{int} X$ – interior of set X $\dim X$ – dimension of space X $\inf S$ – infimum of subset S $\sup S$ – supremum of subset S $\oplus$ – direct sum of subspaces $\varnothing$ – empty set $j \triangleq \sqrt{-1}$ $\operatorname{Re} s$ – real part of $s \in \mathbb{C}$ $\operatorname{Im} s$ – imaginary part of $s \in \mathbb{C}$ $\lvert s \rvert$ – modulus of $s \in \mathbb{C}$ $\arg s$ – argument of $s \in \mathbb{C}$ $\lVert \bullet \rVert$ – norm of entity $\bullet$ (vector or matrix) I_n – unity matrix of order n M^T – transpose of matrix M	M^* – conjugate transpose of matrix M $\operatorname{rank} M$ – rank of matrix M $\operatorname{Im} M$ – imagine of matrix M $\operatorname{Ker} M$ – kernel of matrix M $\dim M$ – dimension (order) of square matrix M $\det M$ – determinant of square matrix M $\operatorname{adj} M$ – adjoint of square matrix M $\operatorname{Tr} M$ – trace of square matrix M $\operatorname{diag}\{\lambda_1,\ldots,\lambda_n\}$ – diagonal matrix $\operatorname{dgr} \Delta(s)$ – degree of polynomial $\Delta(s)$ $p \mid q$ – p divides q $\operatorname{grad} V(x)$ – gradient vector of scalar function $V(x)$ ***Proof*** – beginning of a proof $\square$ – ending of a statement (excepting theorems followed by proofs), of a proof, of a remark or of an example a.e. – almost everywhere $\mathscr{L}$ – direct Laplace transform $\mathscr{L}^{-1}$ – inverse Laplace transform l.c.m. – lowest common multiple g.c.d. – greatest common divisor

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CHAPTER I

MATHEMATICAL MODELS OF LINEAR DYNAMIC SYSTEMS

1. Systems

1.1. Introductory terminology

a. Definition of systems

A system is an ensemble of parts, which support each other and form an organized and coherent whole. A system constitutes an entity, relatively delimited from the medium by the structure and the inter-connections of its parts. The behaviour of a system depends rather on the interactions between the parts.

In this respect, similar ideas and concepts were quasi-independently developed at the same time in various scientific fields: automatic control systems, theory of electrical circuits and electronics, but also in biology, neurophysiology, psychology, sociology, and economics. Progresses made in these areas marked the beginning of the conceptual evolution to consider and describe systems, the laws that govern them, and the isomorphisms between the real systems, which can be very different in nature.

We note that real systems, natural or technical, are studied in:

- System sciences – dedicated to physical-chemical, biological, organismic, economic, socio-ecological, socio-cultural and organizational systems.
- System engineering – dedicated to technical systems.

Within the real systems area, *direct scientific knowledge* is based on two categories of methods: *experimental methods* and *modelling methods*.

Experimental methods involve direct interaction with the object of study and have, in some situations, limited applicability. Experiments can be performed in physics, chemistry, biology, psychology, and engineering, but they are limited or impossible in astronomy, economics, and sociology. Generally, there are systems for which certain experiments cannot be performed. In these cases, modelling methods are used. At the same time, the real systems on which experimental observations and experiments can be performed are also modellable. Experiments that cannot be performed on real systems can be virtually performed on their models.

Any real system processes substance, energy and / or information. Any system is connected to the environment through two categories of variables:

- Causes, called *inputs* and denoted by the vector u, and
- Effects, called *outputs* and denoted by the vector y.

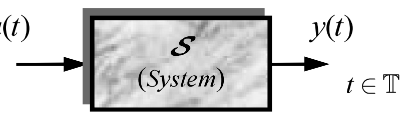

Fig. 1.1. The symbolic representation of a system

Between u and y, which are real vectors defined on *the time set* $\mathbb{T}$, there exists *cause-to-effect relations*, symbolized by *the transfer operator* $\mathcal{S}$. Its symbolic image is given in Fig. 1.1. Time set $\mathbb{T}$ is *ordered*, i.e. the ongoing sequence of events is taking place in order from the past, to the present and towards the future.

b. Systems modelling

The mathematical expression of the relationship between u and y, symbolized by the transfer operator $\mathcal{S}$, is the *mathematical model* of the system. It is more or less the perfect image of the real system. In a simple form, the *input – output* transfer performed by the system is expressed by:

$$y(t) = \mathcal{S} \circ u(t), \quad t \in \mathbb{T}, \tag{1.1}$$

where "$\mathcal{S} \circ$" symbolizes the *operation* of the transformation of u into y through $\mathcal{S}$.

System modelling is based on the knowledge and application of the general laws of nature and on the description of observations and experiments and the interpretation of the measurements of real system phenomena. This *image*, sometimes idealized or simplified, which has a form that can be improved and efficiently used for knowledge and applications, is called the *mathematical model*.

According to the definition of the system notion, the mathematical model is itself a system, i.e. an *abstract system*, which is *representing* the *real system*. Conceptually, the notions of *real system* and *abstract system* are distinct. The abstract system is a more or less idealized and / or simplified, but perfectible, *image* of the real one. The distinction between the real system and its image is illustrated in Fig. 1.2. This figure highlights the modelling process (accompanied by the identification of the structure and parameters) and, implicitly, the role of the system analyst. The analyst develops the model (DM), conceives the experiments (CE), performs experiments (E_{RS}, E_{AS}), validates the model (VM) based on E_{RS} and E_{AS}, and repeats CE, E_{RS}, E_{AS}, VM and DM whenever necessary and / or possible. In this way, it is possible to improve the abstract system according to the purposes of mathematical modelling: knowledge, synthesis of monitoring structures and / or control structures of the real system, etc.

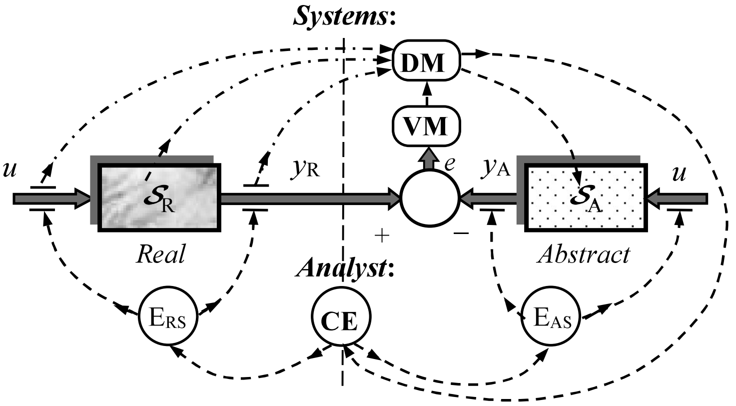

Fig. 1.2. Modelling process; $\mathcal{S}_R$ – *real system*, $\mathcal{S}_A$ – *abstract system*,
u – *causes*, y_R, y_A – *effects*, $e = y_R - y_A$ – *error*; DM – *developing of model*,
CE – *conceiving experiments*, E_{RS}, E_{AS} – *experiments*, VM – *validation of model*;
–·–·–► *information retrieval*, – – – –► *modelling operations*

c. Object of systems theory

Abstract systems are entities that are perfectible (whenever necessary and / or possible) and efficient for knowing the real systems. Abstract systems, grouped in classes that give them a degree of generality, are the object of ***systems theory*** study.

The origins of systems theory lie in the theory of automatic control systems, of electrical circuits, and of electronics. It can be said that the automatic control system, by its intrinsic systemic vision, by the inter-, trans- and multidisciplinary character and by the natural use of the mathematical instrument and language, is the main domain in which the (mathematical) theory of systems evolved.

In the framework of systems theory, a system is a model of a general and abstract nature, that is, a conceptual analogy between certain universal features of the observed facts. The difference from conventional disciplines lies in the degree of generalization and abstraction: the abstract system refers to the general characteristics of some classes of entities traditionally treated by different disciplines. It is from this fact that the inter-, trans- and multidisciplinary nature of systems theory follows, as well as its applicability in various concrete fields.

d. Continuous-time systems and non-anticipative systems

Definition 1.1

A. A system is called *continuous-time* if $\mathbb{T} = \mathbb{R}$ or $\mathbb{T}$ is isomorphic with $\mathbb{R}$.

B. A system is called *discrete-time* if $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T}$ is isomorphic with $\mathbb{Z}$. □

We note that the system variables are defined in terms of physical time (as an intrinsic category of reality) as follows: in the case of continuous-time systems, for every $t \in \mathbb{T}$, while in the case of discrete-time systems, variables are defined only for certain instants of time, usually equidistant.

Any real system (natural or technical) satisfies the *principle of non-anticipation* that has concisely, the following form: *the effect does not anticipate the cause*. In other words, for each moment $t \in \mathbb{T}$ the evolution of the output $y(t)$ depends on the evolution of the input $u(\tau)$ for $\tau \leq t$, $\tau \in \mathbb{T}$.

Definition 1.2

A. A system is called *non-anticipative* (i.e., it satisfies the *principle of non-anticipation*) if the past and present evolution of y does not depend on the future evolution of u.

B. A system is called *anticipative* if the evolution of output y in the *past* and *present* depends on the *future* evolution of u. □

Let $\boldsymbol{\mathcal{S}}^{\tau}$ be the *temporal truncation operator*, defined as follows:

$$w(\tau)=\boldsymbol{\mathcal{S}}^{t}\circ v(\tau)\triangleq\begin{cases}v(\tau), & \tau\le t,\\ 0, & \tau>t, \quad \tau, t\in\mathbb{T}.\end{cases} \tag{1.2}$$

For the input – output transfer (1.1), the *non-anticipation principle* has the following analytic forms:

$$\boldsymbol{\mathcal{S}}^{t}\circ(\boldsymbol{\mathcal{S}}\circ u(\tau))=\boldsymbol{\mathcal{S}}^{t}\circ(\boldsymbol{\mathcal{S}}\circ(\boldsymbol{\mathcal{S}}^{t}\circ u(\tau))),\quad \boldsymbol{\mathcal{S}}^{t}\circ\boldsymbol{\mathcal{S}}=\boldsymbol{\mathcal{S}}^{t}\circ(\boldsymbol{\mathcal{S}}\circ\boldsymbol{\mathcal{S}}^{t}), \tag{1.3}$$

Remark 1.1

Although no realistic anticipative systems have been identified in the real macroscopic world currently known, this concept is especially useful in applications. For example, it creates the conceptual contrast necessary for the appropriate evaluation of mathematical models that may be sometimes anticipative.

Obviously, it is desirable that the mathematical model of a real system be non-anticipative. However, this property is not automatically ensured. The reason for this is that during the mathematical modelling of a real system, we use simplifications / idealizations that might lead to anticipative parts of the obtained mathematical model. (For further details see Remarks 2.1 and 4.1.) □

Remark 1.2

There are also situations where, for reasons of simplicity of the treatment, we work with anticipative mathematical models. The results obtained in such cases should be appropriately interpreted, namely in line with the idealizations / simplifications that led to those mathematical models. □

1.2. Definition of dynamic systems

A dynamic system is characterized by the fact that the output $y(t)$, where $t\in\mathbb{T}$ is the current instant of time, depends on the entire internal evolution (history) of the system under the action of the input u over the time interval $[t_0,t]\subseteq\mathbb{T}$; t_0 is the initial instant of the observation interval. In this context, a fundamental concept is that of the *system state*, denoted by the vector x, which describes the *internal evolution* of $x(t)$ generated by the action of the *input segment* $u_{[t_0,t]}\triangleq\{u(\theta);\theta\in[t_0,t]\}$. It follows that $y(t)$ depends on $x(t)$ and $u(t)$. But state $x(t)$ determines itself (through its own evolution) under the action of $u_{[t_0,t]}\triangleq\{u(\theta);\theta\in[t_0,t]\}$ (i.e. it depends on the entire history of the state evolution on $[t_0,t]$), and eventually directly on $u(t)$.

Definition 1.3, [*apud* 27]

A. A system is called *dynamic* if the input – output transfer (1.1) is expressed as:

$$x(t)=\varphi\left(t;\, t_0,\, x_0,\, u_{[t_0,t]}\right),\ t\ge t_0,\ x\in\mathbb{X},\ u\in\mathbb{U}, \tag{1.4}$$

$$y(t)=g\left(t,\, x(t),\, u(t)\right),\qquad y\in\mathbb{Y}, \tag{1.5}$$

where

(a) φ is the *transition function*, i.e. the map through which the *transition* from the *initial state*

$$x_0=x(t_0) \tag{1.6}$$

to the *current state* $x(t)$, $t\ge t_0$, takes place under the action of the *input segment*

$u_{[t_0,t]}\triangleq\{u(\theta);\,\theta\in[t_0,t]\}$;

(b) g is the map which transforms *current state* $x(t)$ and *current input* $u(t)$ into *current output* $y(t)$, $t\ge t_0$.

(c) $\mathbb{U}, \mathbb{X}, \mathbb{Y}$ are the *input space*, the *state space*, and the *output space*, respectively.

(d) $x(t)$ depends on input segment $u_{[t_0,t]}$ belonging to the *input functions class*:

$$\Omega\triangleq\{\omega:\mathbb{T}\to\mathbb{U}\}. \tag{1.7}$$

This includes the *admissible evolutions* of the input u, described by:

$$\omega\triangleq\{u(t);\, t\in\mathbb{T},\, u(t)\in\mathbb{U}\}. \tag{1.8}$$

The value of the evolution (1.8) at instant t is obtained by:

$$u(t) = \pi_t \, \omega \triangleq \omega(t), \tag{1.9}$$

where π_t is the extracting operator at instant t.

e) The *transition function* φ has the following four properties:

1^0 *Orientability*: $\varphi(t;\, t_0,\, x_0,\, u_{[t_0,t]})$ is defined for any $t \ge t_0$.

2^0 *Consistency*: for every $t_0 \in \mathbb{T}$ and for $t = t_0$ the following holds:

$$\varphi(t_0;\, t_0,\, x_0,\, u_{[t_0,\, t_0]}) = x_0. \tag{1.10}$$

3^0 *Composability*: for every $t_1,\, t_2,\, t_3 \in \mathbb{T}$ with $t_1 < t_2 < t_3$, it follows that:

$$\varphi(t_3;\, t_1,\, x(t_1),\, u_{[t_1,\, t_3]}) = \varphi(t_3;\, t_2,\, \varphi(t_2;\, t_1,\, x(t_1),\, u_{[t_1,\, t_2]}),\, u_{[t_2,\, t_3]}). \tag{1.11}$$

4^0 *Causality*: for every $t_0, t \in \mathbb{T}, t \ge t_0$, and every $u_{[t_0,\, t]},\ \tilde{u}_{[t_0,\, t]} \in \Omega$ it follows that:

$$u_{[t_0,\, t]} = \tilde{u}_{[t_0,\, t]} \ \Rightarrow\ \varphi(t;\, t_0,\, x_0,\, u_{[t_0,\, t]}) = \varphi(t;\, t_0,\, x_0,\, \tilde{u}_{[t_0,\, t]}). \tag{1.12}$$

B. A system is called *static* if the input – output transfer occurs instantly, in the sense that $y(t)$ depends only on $u(t)$ for all $t \ge t_0,\ t_0, t \in \mathbb{T}$. □

Definition 1.4

A. A system is called *finite dimensional* if $\mathbb{U}$, $\mathbb{X}$, $\mathbb{Y}$ are finite dimensional and linear (vector) spaces (see Annex A).

B. A system is called *infinite dimensional* if there exist infinite dimensional variables in the system. □

In the finite-dimensional case, usually $\mathbb{U} = \mathbb{R}^m, \mathbb{X} = \mathbb{R}^n, \mathbb{Y} = \mathbb{R}^p$, with m, n, p being natural numbers. n, called the *system / model order*, is the number of independent and relevant storage elements of substance, energy and information contained in the system.

Definition 1.5

A dynamic system of form (1.4), (1.5) is called *smooth* if $\mathbb{T} = \mathbb{R}$, $\mathbb{U}, \mathbb{X}, \mathbb{Y}$ are topological spaces, and φ has a continuous derivative in t and is continuous in t_0, x_0, u. □

Theorem 1.1

The transition function $\varphi(t;\, t_0,\, x_0,\, \omega)$ of a smooth dynamic system having the form (1.4), (1.5), where $u(t)$ is piecewise continuous on $[t_0, t]$ and $\mathbb{U}, \mathbb{X},\ \Omega$ are normed spaces, is the solution of a differential equation of the form:

$$\frac{dx}{dt} = f\big(t,\, x(t),\, u(t)\big),\ t \ge t_0. \tag{1.13}$$

Proof. The topology of spaces $\mathbb{U}, \mathbb{X}$ and Ω is induced by the norms defined on these spaces (see Annex A). Let $F_t : \mathbb{T} \times \mathbb{X} \times \Omega \to \mathbb{X}$ be the map defined by

$$F_t(t_0,\, x_0,\, \omega) \triangleq \frac{d}{dt}\varphi(t;\, t_0,\, x_0,\, \omega), \tag{1.14}$$

where the input has the form $\omega = u_{[t_0,\, t]}$ and $x_0 = x(t_0)$. Let $t_0 = t$ and let $f\big(t,\, x(t),\, u(t)\big) \triangleq$ $\triangleq F_t\big(t,\, x(t),\, u_{[t,\, t]}\big)$ by definition. In this situation, from (1.14) it follows that:

$$\frac{d}{dt}\varphi\big(t;\, t,\, x(t),\, u_{[t,\, t]}\big) = F_t\big(t,\, x(t),\, u_{[t,\, t]}\big) = f\big(t,\, x(t),\, u(t)\big). \tag{1.15}$$

f is continuous with respect to t, x, u because the system is smooth. From (1.15), using the property of consistency, i.e. $\varphi\big(t;\, t,\, x(t),\, u_{[t,\, t]}\big) = x(t)$ (obtained from (1.10) for $t_0 = t$), the differential equation (1.13) follows immediately. □

Therefore, *smooth dynamic systems are described by differential equations.*

The mathematical model (1.13), (1.5) is a *recursive model* because the differential equation (1.13) actually expresses *only the derivative of state evolution in time* (i.e. *the tendency of state evolution*), depending on $x(t)$ (i.e. on itself) and on $u(t)$. In contrast, the mathematical model of $\mathcal{S}$ (see (1.1)), i.e. (1.4), (1.5), where (1.4) is the solution of the differential equation (1.13), is a *non-recursive model* that describes *only the evolution of the system state over time*.

Usually, in the case of smooth dynamic systems, the modelling process shown in Fig. 1.2 is finalized using recursive models. This is easily justified by the fact that the general laws that are used in the mathematical modelling process, are usually formulated in terms of the *time-evolutionary tendencies of the phenomena.* It follows that a fundamental problem is that of the transition from the recursive model (1.13), (1.5) to the non-recursive model (1.4), (1.5). This amounts to showing the existence and uniqueness of the solution of the *Cauchy problem* described by the differential equation (1.13) and the initial condition (1.6). In this respect, a classic result is the following.

Theorem 1.2 (of existence and uniqueness, [3])

If for any $u(t)$, known on $[t_0, t]\subseteq\mathbb{T}$, f is continuous and satisfies the boundary and Lipschitz conditions (in a norm $\|\bullet\|$ defined on $\mathbb{X}$), respectively:

$$\begin{cases} \left\| f\left(t,\, x(t),\, u(t)\right)\right\| \le M, & t\in,\quad x\in\mathbb{X}, \\ \left\| f(t,\, x,\, u)-f(t,\, \tilde{x},\, u)\right\| < L\left\|x-\tilde{x}\right\|, & t\in,\quad x,\, \tilde{x}\in\mathbb{X}, \end{cases} \tag{1.16}$$

where M and L are two positive constants, then there exists a unique solution (1.4) of the Cauchy problem (1.13), (1.6). □

1.3. Time-invariant dynamic systems

Definition 1.6

A. A system is called *time-invariant* or *constant* if its evolution is invariant with respect to any temporal translations.

B. A system is called *time-variant* if the definition above does not hold. □

Let $\mathcal{S}_\tau,\ \tau\in\mathbb{T}$, be the *temporal translation operator*. By this operator, v is transformed into w as follows: $w(t)=\mathcal{S}_\tau\circ v(t)\triangleq v(t+\tau)$.

In the input – output transfer (1.1), time invariance is expressed by:

$$y(t)=\mathcal{S}\circ u(t)=\mathcal{S}_\tau\circ(\mathcal{S}\circ(\mathcal{S}_{-\tau}\circ u(t))),\quad \mathcal{S}=\mathcal{S}_{\mathrm{T}}\circ(\mathcal{S}\circ\mathcal{S}_{-\mathrm{T}}). \tag{1.17}$$

Theorem 1.3

Consider the time-invariant smooth dynamic system (1.4), (1.5), where $u(t)$ is piecewise continuous, and $\mathbb{U}$, $\mathbb{X}$ and Ω are normed spaces. The transition function φ is the solution of the differential equation (1.13) and the transformation of $x(t)$ and $u(t)$ into $y(t)$ is described by (1.5), with f and g independent of t, i.e.:

$$\frac{dx}{dt}=f\left(x(t),\, u(t)\right),\ t\ge t_0, \tag{1.18}$$

$$y=g\left(x(t),\, u(t)\right). \tag{1.19}$$

Proof. According to Definition 1.6, for every $t,\, t_0,\tau\in\mathbb{R}$ it follows that:

$$x(t)=\varphi\left(t;\, t_0,\, x_0,\, u_{[t_0,\, t]}\right)=\varphi\left(t+\tau;\, t_0+\tau,\, x_0,\, \mathcal{S}_{-\tau}\circ u_{[t_0+\tau,\, t+\tau]}\right). \tag{1.20}$$

Since $\mathcal{S}_{-\tau}\circ u_{[t_0+\tau,\, t+\tau]}=u_{[t_0,\, t]}$, from (1.14), (1.20), for $\tau=-t_0$ it follows that:

$$F_t\left(t_0,\, x_0,\, u_{[t_0,\, t]}\right)=\frac{d}{dt}\varphi\left(t;\, t_0,\, x_0,\, u_{[t_0,\, t]}\right), \tag{1.21}$$

$$F_t\left(t_0,\, x_0,\, u_{[t_0,\, t]}\right)=\frac{d}{dt}\varphi\left(t+\tau;\, t_0+\tau,\, x_0,\, u_{[t_0,\, t]}\right)\Big|_{\tau=-t_0}=\frac{d}{dt}\varphi\left(t-t_0;\, 0,\, x_0,\, u_{[t_0,\, t]}\right). \tag{1.22}$$

For $t_0=t$ it immediately follows that $F_t(t,\, x(t),\, u_{[t,\, t]})$ does not explicitly depend on t because from (1.21), (1.22) one can write:

$$\frac{d}{dt}\varphi\left(t;\, t,\, x(t),\, u_{[t,\, t]}\right)=F_t\left(t,\, x(t),\, u_{[t,\, t]}\right)\equiv F_t\left(x(t),\, u_{[t,\, t]}\right)=\frac{d}{dt}\varphi\left(0;\, 0,\, x(t),\, u_{[t,\, t]}\right). \tag{1.23}$$

We define $f\left(x(t),\, u(t)\right) \triangleq F_t\left(x(t),\, u_{[t,\, t]}\right)$. From (1.23) it follows that:

$$\frac{d}{dt}\varphi\left(t;\, t,\, x(t),\, u_{[t,\, t]}\right) = F_t\left(x(t),\, u_{[t,\, t]}\right) = f\left(x(t),\, u(t)\right). \tag{1.24}$$

f is continuous with respect to x and u since the dynamic system is smooth. From (1.24) and the property of consistency, $\varphi\left(t;\, t,\, x(t),\, u_{[t,\, t]}\right) = x(t)$ (obtained from (1.10) for $t_0 = t$), the differential equation (1.18) follows.

Proceeding similarly for (1.5) we obtain $g(t,\, x,\, u) \equiv g(x,\, u)$, i.e. (1.19). □

Example 1.1

The scheme of the dc electric motor with double control is shown in Fig. 1.3. By modelling the electrical, magnetic and mechanical phenomena only the corresponding interactions are considered. The thermal phenomena are neglected. Using the variables given in Fig. 1.3, the equations of this system are the following:

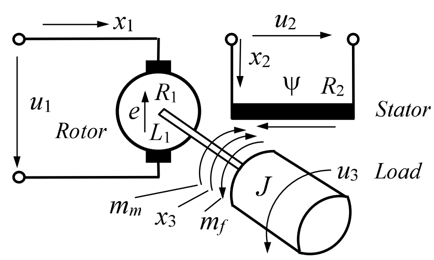

Fig. 1.3. Scheme of d.c. electric motor; u_1, u_2 – control voltages, x_1, x_2– rotor and stator currents, e –counter-electromotive voltage, R_1, L_1 – rotor resistance and inductance, R_2 – stator resistance, x_3 – angular speed, u_3 – load torque, m_m – motor torque, m_f –friction torque, J – rotor moment of inertia, ψ – electromagnetic flux

- On the rotor circuit:

$$u_1 = e + R_1 x_1 + L_1 \dot{x}_1,$$
$$e = c_1 \psi x_3;$$

- On the stator circuit:

$$u_2 = R_2 x_2 + \dot{\psi},$$
$$\psi = h(x_2);$$

- On the electro-mechanical part:

$$J\dot{x}_3 = m_m - m_f - u_3,$$
$$m_m = c_2 \psi x_1,$$
$$m_f = c_3 x_3.$$

Function h represents the first magnetization curve of the magnetic circuit; $c_{1,2,3}$ are constants of the electrical motor.

The system is of order three, equal to the number of energy storage elements contained in the system: two electromagnetic ones (represented by inductances $L_1,\, L_2$) and a mechanical storage element (represented by the moment of inertia J). The state variables may be naturally chosen as $x_1,\, x_2,\, x_3$. The input vector is composed of the external variables $u_1,\, u_2,\, u_3$. By eliminating the variables $e,\, \psi,\, m_m,\, m_f$ between the above equations and solving with respect to derivatives $\dot{x}_1,\, \dot{x}_2,\, \dot{x}_3$, we obtain finally a system of three differential equations, input – state of the form (1.13), namely:

$$\dot{x}_1 = -\frac{R_1}{L_1}x_1 - \frac{c_1}{L_1}h(x_2)x_3 + \frac{1}{L_1}u_1,$$

$$\dot{x}_2 = -\frac{R_2}{h'(x_2)}x_2 + \frac{1}{h'(x_2)}u_2,\quad \dot{x}_3 = \frac{c_2}{J}f(x_2)x_1 - \frac{c_3}{J}x_3 - \frac{1}{J}u_3.$$

To these equations, the output equation must be added. For example, when the operator is interested in the angular speed, the output equation is:

$$y = x_3. \;\square$$

1.4. Linear dynamic systems

Definition 1.7

A. A system is called *linear* if any linear combination of inputs is transformed into a similar linear combination of outputs.

B. A system is called *nonlinear* if the definition above is not necessarily satisfied. □

In the linear case, for every u^i with $y^i(t) = \mathcal{S} \circ u^i(t)$, $i \in I$ (I is a set of finite / countable indices), and every constant $c_i \in \mathbb{R}$, $i \in I$, it holds that

$$\mathcal{S} \circ \left(\sum\nolimits_{i\in I} c_i u^i(t)\right) = \sum\nolimits_{i\in I} c_i (\mathcal{S} \circ u^i(t)) = \sum\nolimits_{i\in I} c_i y^i(t). \tag{1.25}$$

Relation (1.25) illustrates the well-known *principle of superposition effect*.

Theorem 1.4

Equations (1.13), (1.5) of a smooth, finite-dimensional and linear dynamic system have the form:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in \mathbb{R}, x \in \mathbb{R}^n, u \in \mathbb{R}^m, \tag{1.26}$$

$$y(t) = C(t)x(t) + D(t)u(t), \quad y \in \mathbb{R}^p, \tag{1.27}$$

where $A(t),\ B(t),\ C(t),\ D(t)$ are matrices of adequate dimensions.

Proof. The system is smooth; usually $\mathbb{T} = \mathbb{R}$. The finiteness and the linearity (1.25) imply that $\mathbb{X}, \mathbb{U}, \mathbb{Y}$ are identical or isomorphic spaces respectively with $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p$; and n, m, p are appropriately defined during the mathematical modelling process.

The transition function φ is linear on $\mathbb{X} \times \Omega$. This means that for

$$x_0 = \sum\nolimits_{i \in I} c_i x_0^i, \quad u_{[t_0, t]} = \sum\nolimits_{i \in I} c_i u_{[t_0, t]}^i, \text{ for any } (x_0^i,\ u_{[t_0, t]}^i) \in \mathbb{X} \times \Omega,$$

and for every constant $c_i \in \mathbb{R}, i \in I,$ the transition function φ (see Definition 1.3) has the property:

$$\varphi(t; t_0, \sum\nolimits_{i \in I} c_i x_0^i, \sum\nolimits_{i \in I} c_i u_{[t_0, t]}^i) = \sum\nolimits_{i \in I} c_i \varphi(t; t_0, x_0^i, u_{[t_0, t]}^i).$$

In particular, for $x_0 = x_0 + 0,\ u_{[t_0, t]} = 0_{[t_0, t]} + u_{[t_0, t]},$ it follows that:

$$\varphi(t; t_0, x_0, u_{[t_0, t]}) = \varphi(t; t_0, x_0, 0_{[t_0, t]}) + \varphi(t; t_0, 0, u_{[t_0, t]}).$$

Therefore, φ consists of the following two components:

- $x_L(t) = \varphi(t; t_0, x_0, 0_{[t_0, t]})$ – the component of the *free-regime*, determined by *internal causes*, i.e. by the *initial state* x_0;
- $x_F(t) = \varphi(t; t_0, 0, u_{[t_0, t]})$ – the component of the *forced-regime*, determined by *external causes*, i.e. by the *input segment* $u_{[t_0, t]}$.

Further, by virtue of finiteness and linearity, it follows that:

$$x_L(t) = \varphi(t; t_0, x_0, 0) = \Phi(t, t_0)x_0, \tag{1.28}$$

$$x_F(t) = \varphi(t; t_0, 0, u_{[t_0, t]}) = \Psi(t, t_0)u_{[t_0, t]}, \tag{1.29}$$

where $\Phi(t, t_0)$ is the *transition matrix*, which determines the *free-regime* evolution from the *initial state* x_0, and $\Psi(t, t_0)$ is the *transfer operator*, which determines the *forced-regime* evolution under the action of the *input segment* $u_{[t_0, t]}$; in the case of continuous-time systems, $\Psi(t, t_0)$ is an integral operator (see Remark 1.6).

At the same time, $x(t) = \varphi(t; t_0, x_0, u_{[t_0, t]})$ is the solution of a differential equation of form (1.13), where $f(t, x, u)$ has the significance highlighted in the proof of Theorem 1.1. The linearity of function φ also implies the linearity of function f; so, it can be written as:

$$f(t, x, u) \equiv A(t)x + B(t)u,$$

which leads to equation (1.26) with $A(t),\ B(t)$ matrices of appropriate dimensions.

The linearity condition (1.25) also includes equation (1.5). As a result:

$$y = g(t, x, u) \equiv C(t)x + D(t)u,$$

from which (1.27) follows, with $C(t),\ D(t)$ matrices of appropriate dimensions. □

Explicitly, the vectors and matrices of (1.26), (1.27) have the following forms:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \tag{1.30}$$

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} b_{11}(t) & \cdots & b_{1m}(t) \\ \vdots & & \vdots \\ b_{n1}(t) & \cdots & b_{nm}(t) \end{bmatrix}, \tag{1.31}$$

$$C(t) = \begin{bmatrix} c_{11}(t) & \cdots & c_{1n}(t) \\ \vdots & & \vdots \\ c_{p1}(t) & \cdots & c_{pn}(t) \end{bmatrix}, \quad D(t) = \begin{bmatrix} d_{11}(t) & \cdots & d_{1m}(t) \\ \vdots & & \vdots \\ d_{p1}(t) & \cdots & d_{pm}(t) \end{bmatrix}. \tag{1.32}$$

These matrices are called the *system* or the *evolution matrix*, the *input matrix*, the *output matrix*, and the *matrix of the direct input – output connection*.

The problem of establishing the connection between the recursive model (1.26), (1.27) and the non-recursive model (1.4), (1.5) can be solved by showing that the unique solution of the *Cauchy problem* (1.26), (1.6) is the transition function of the system. This problem is addressed in the following results and statements: Theorems 1.5–1.8 and Remarks 1.3–1.5.

a. Fundamental matrix and transition matrix

According to equation (1.26), the homogeneous equation is first considered:

$$\dot{x} = A(t)x, \tag{1.33}$$

for which the following two results from the theory of differential equations are used.

Theorem 1.5 (of existence and unicity, see [3])

If $A(t)$ is continuous for $t \geq 0$, then for the Cauchy problem (1.33), (1.6) with the initial instant $t_0 = 0$ and initial state $x(0)$, there exists the unique solution:

$$x(t) = X(t)x(0), \quad t \geq 0, \tag{1.34}$$

where $X(t)$ is the *fundamental matrix* of equation (1.33). $X(t)$ is non-singular, i.e. $\det X(t) = \; = e^{\int_0^t \mathrm{Tr}\, A(\theta) d\theta} \neq 0.$ $X(t)$ is the unique solution of the equation:

$$\dot{X}(t) = A(t)X(t), \quad t \geq 0, \tag{1.35}$$

with the initial condition:

$$X(0) = I_n, \tag{1.36}$$

where I_n is the unit matrix of order n. □

Theorem 1.6 (Peano – Baker series, see [57])

The sequence of matrices

$$X_0 = I_n, \; X_k(t) = I_n + \int_0^t A(\theta) X_{k-1}(\theta) d\theta, \; t \geq 0, \; k = 1,2,3,...,$$

converges uniformly over $[0, t]$ and the limit is the fundamental matrix $X(t)$. □

Using this result, you can roughly calculate $X(t)$. For the $A(t)$ differentiable, it is also possible to calculate the exact matrix $X(t)$, [113].

Using the solution (1.34), it is easy to verify that, for any other initial instant t_0, the unique solution of the Cauchy problem (1.33), (1.6) has the form:

$$x(t) = X(t)X^{-1}(t_0)x_0, \;\; t \geq t_0; \quad t, t_0 \in \mathbb{R}. \tag{1.37}$$

The following result will be proved for this solution.

Theorem 1.7

The solution (1.37) of the homogeneous equation (1.33) is the transition function of a linear, finite-dimensional and smooth dynamic system, and $X(t)X^{-1}(t_0), \; t \geq t_0,$ is the corresponding *transition matrix*.

Proof. To show that

$$x(t) = \varphi(t; t_0, x_0, 0) \equiv X(t)X^{-1}(t_0)x_0, \;\; t \geq t_0, \tag{1.38}$$

is a transition function, we check the four properties from Definition 1.3, namely:

(i) Orientability: $\varphi(t; t_0, x_0, 0)$ is defined for every $t \geq t_0$.

(ii) Consistency: for every $t_0 \in \mathbb{R}$ and for $t = t_0$ it follows that:

$$\varphi(t_0; t_0, x_0, 0) = x_0, \;\; X(t_0)X^{-1}(t_0)x_0 = x_0.$$

(iii) Composability: for every $t_1, \; t_2, \; t_3 \in \mathbb{R}$, with $t_1 < t_2 < t_3$, it follows that:

$$\varphi(t_3; t_1, x(t_1), \; 0) = \varphi(t_3; t_2, \varphi(t_2; t_1, x(t_1), \; 0), \; 0),$$

$$X(t_3)X^{-1}(t_1)x(t_1) = X(t_3)X^{-1}(t_2)\,[X(t_2)X^{-1}(t_1)x(t_1)].$$

(iv) Causality: It is obvious that in the particular case $u_{[t_0, t]} \equiv \tilde{u}_{[t_0, t]} \equiv 0_{[t_0, t]} \equiv 0.$

φ has a continuous derivative in *t*, is continuous in t_0, x_0 and is linear with respect to x_0. In addition, from (1.37) and (1.28) the following transition matrix is obtained:

$$\Phi(t, t_0) = X(t)X^{-1}(t_0), \quad t \geq t_0. \ \square \tag{1.39}$$

Remark 1.3

Using (1.39) it is immediately shown that the transition matrix itself has properties (*i*)–(*iii*) of Definition 1.3, namely:

(i) Orientability: $\Phi(t, t_0)$ is defined for every $t \geq t_0$.

(ii) Consistency: $\Phi(t_0, t_0) = X(_0 t)X^{-1}(t_0) = I_n, \ t_0 \geq 0$. (1.40)

(iii) Composability: for every $t_1, \ t_2, \ t_3 \in \mathbb{R}$, with $t_1 < t_2 < t_3$, it follows that:

$$\Phi(t_3,t_1) = \Phi(t_3,t_2)\Phi(t_2,t_1), \quad X(t_3)X^{-1}(t_1) = X(t_3)X^{-1}(t_2)X(t_2)X^{-1}(t_1). \ \square \tag{1.41}$$

Remark 1.4

Similar to (1.35), the transition matrix is the solution of the differential equation:

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \quad t \geq t_0, \tag{1.42}$$

with the initial condition (1.40). Equation (1.42) results from equation (1.35) by multiplying to the right by $X^{-1}(t_0)$, and taking into account (1.39). Relation (1.42) expresses at the same time the derivation rule of matrix $\Phi(t, t_0)$. $\square$

Remark 1.5

From (1.39), the inversion of the transition matrix is immediately obtained as:

$$\Phi^{-1}(t, t_0) \equiv \Phi(t_0, t). \ \square \tag{1.43}$$

b. Solution of the non-homogeneous input – state equation

The following theorem will show that the solution of the *Cauchy problem* (1.26), (1.6) is a transition function. Taking into account the linearity of the differential equation (1.26), it follows that its solution has the form:

$$x(t) = x_h(t) + x_p(t), \tag{1.44}$$

$$x_h(t) = \Phi(t, t_0)x(t_0), \quad t \geq t_0, \tag{1.45}$$

$$x_p(t) = \Phi(t, t_0)z(t), \quad t \geq t_0; \tag{1.46}$$

- $x_h(t)$ is the *solution of the homogeneous equation* (1.33) ((1.26) with $u(t) \equiv 0$);
- $x_p(t)$ is the *particular solution* of equation (1.26), with $x_0 = 0$; (1.46) is of the form (1.45), but $x(t_0)$ has been replaced by $z(t)$ (variation of constants). The unknown $z(t)$ is determined such that $x_p(t)$ satisfies (1.26), i.e.:

$$\dot{x}_p(t) = A(t)x_p(t) + B(t)u(t). \tag{1.47}$$

By replacing (1.46) into (1.47) and taking into account (1.42) and (1.43), from (1.47) and then from (1.46) it can be successively written that:

$$\Phi(t, t_0)\dot{z}(t) = B(t)u(t),$$

$$z(t) = \int_{t_0}^{t} \Phi^{-1}(\theta, t_0)B(\theta)\, u(\theta)\, d\theta, \quad t \geq t_0,$$

$$x_p(t) = \int_{t_0}^{t} \Phi(t, \theta)B(\theta)\, u(\theta)\, d\theta, \quad t \geq t_0. \tag{1.48}$$

To obtain this result, according to (1.39), we used the fact that

$$\Phi(t, t_0)\Phi^{-1}(\theta, t_0) = \Phi(t, t_0)\Phi(t_0, \theta) = X(t)X^{-1}(t_0)X(t_0)X^{-1}(\theta) = \Phi(t, \theta).$$

Therefore, according to (1.44)–(1.48), the unique solution of the Cauchy problem (1.26), (1.6) has the expression:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \theta)B(\theta)\, u(\theta)\, d\theta, \quad t \geq t_0. \tag{1.49}$$

The input – output transfer, represented by *non-recursive* transfer operator $\mathcal{S}$ (see (1.1)), is now obtained from (1.49) and (1.27) as:

$$y(t) = C(t)\Phi(t, t_0)x(t_0) + C(t)\int_{t_0}^{t} \Phi(t, \theta)B(\theta)u(\theta)d\theta + D(t)u(t), \quad t \geq t_0. \tag{1.50}$$

Theorem 1.8

The solution (1.49) of the non-homogeneous equation (1.26) is the transition function of a linear, finite-dimensional and smooth dynamic system.

Proof. We first show that the solution (1.49) of equation (1.26), respectively:

$$x(t) = \varphi(t; t_0, x_0, u_{[t_0, t]}) \equiv \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \theta)B(\theta)u(\theta)d\theta, \ t \geq t_0, \tag{1.51}$$

is a transition function. We check the four properties from Definition 1.3, namely:

(i) Orientability: $\varphi(t; t_0, x_0, u_{[t_0, t]})$ is defined for every $t \geq t_0$.

(ii) Consistency: for every $t_0 \in \mathbb{R}$ and for $t = t_0$ it follows:

$$\varphi(t_0; t_0, x_0, u_{[t_0, t_0]}) = \Phi(t_0, t_0)x(t_0) + \int_{t_0}^{t_0} \Phi(t_0, \theta)B(\theta)u(\theta)d\theta = x(t_0) = x_0.$$

(*iii*) *Composability*: for every $t_1, t_2, t_3 \in \mathbb{R}$, with $t_1 < t_2 < t_3$, it follows:

$$\varphi(t_3; t_1, x(t_1), u_{[t_1, t_3]}) = \Phi(t_3, t_1)x(t_1) + \int_{t_1}^{t_3} \Phi(t_3, \theta)B(\theta)u(\theta)d\theta =$$

$$= \Phi(t_3, t_2)\Phi(t_2, t_1)x(t_1) + \int_{t_1}^{t_2} \Phi(t_3, t_2)\Phi(t_2, \theta)B(\theta)u(\theta)d\theta +$$

$$+ \int_{t_2}^{t_3} \Phi(t_3, \theta)B(\theta)u(\theta)d\theta = \Phi(t_3, t_2)\left(\Phi(t_2, t_1)x(t_1) + \int_{t_1}^{t_2} \Phi(t_2, \theta)B(\theta)u(\theta)d\theta\right) +$$

$$+ \int_{t_2}^{t_3} \Phi(t_3, \theta)B(\theta)u(\theta)d\theta = \varphi(t_3; t_2, \varphi(t_2; t_1, x(t_1), u_{[t_1, t_2]}), u_{[t_2, t_3]}).$$

(*iv*) *Causality*: for every $t_0, t \in \mathbb{R}$ and for every $u_{[t_0, t]}, \tilde{u}_{[t_0, t]} \in \Omega$, the equality

$u_{[t_0, t]} = \tilde{u}_{[t_0, t]}$ implies

$$\varphi(t; t_0, x_0, u_{[t_0, t]}) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \theta)B(\theta)u(\theta)d\theta =$$

$$= \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \theta)B(\theta)\tilde{u}(\theta)d\theta = \varphi(t; t_0, x_0, \tilde{u}_{[t_0, t]}).$$

φ given by (1.51) has a continuous derivative in t and is continuous in t_0, x_0, u.

φ given by (1.51) is linear in x_0, u. Indeed, for

$$x_0 = \sum_{i \in I} c_i x_0^i, \quad u_{[t_0, t]} = \sum_{i \in I} c_i u_{[t_0, t]}^i, \text{ for any } (x_0^i, u_{[t_0, t]}^i) \in \mathbb{X} \times \Omega,$$

and for every constant $c_i \in \mathbb{R}, i \in I$ (I is a set of finite / enumerable indices), it follows that:

$$\varphi(t; t_0, \sum_{i \in I} c_i x_0^i, \sum_{i \in I} c_i u_{[t_0, t]}^i) = \Phi(t, t_0)\sum_{i \in I} c_i x_0^i + \int_{t_0}^{t} \Phi(t, \theta)B(\theta)\sum_{i \in I} c_i u_{[t_0, t]}^i d\theta =$$

$$= \sum_{i \in I} c_i \left[\Phi(t, t_0)x_0^i + \int_{t_0}^{t} \Phi(t, \theta)B(\theta)\, u^i(\theta)d\theta\right] = \sum_{i \in I} c_i\, \varphi(t; t_0, x_0^i, u_{[t_0, t]}^i). \ \square$$

Now, based on Theorems 1.4–1.8 it is easy to conclude that the results can be summarized as in the following statement.

Theorem 1.9

A necessary and sufficient condition for the finite-dimensional, smooth and dynamic system (1.4), (1.5) to be linear is to be described by input – state – output equations (1.26), (1.27). □

Remark 1.6

Based on the above result, it can be seen that (1.45), i.e. the *solution of the homogeneous equation*, coincides with the *free-regime component* (1.28), while the *particular solution* (1.48) coincides with the *forced-regime component* (1.29). At the same time, it is clear that the transfer operator $\Psi(t, t_0)$ from (1.29) is the integral given in (1.48). □

1.5. Linear time-invariant dynamic systems

Theorem 1.10

A necessary condition for the linear finite-dimensional smooth and dynamic system (1.26), (1.27) to be time-invariant (constant) is to have the form:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in \mathbb{R}, x \in \mathbb{R}^n, u \in \mathbb{R}^m, \tag{1.52}$$

$$y(t) = Cx(t) + Du(t), \quad y \in \mathbb{R}^p, \tag{1.53}$$

where A, B, C, D are real and constant matrices of appropriate dimensions.

Proof. For $u_{[t_0, t]} \equiv 0$ the transition function (1.51) is reduced to the free-regime component (1.28), which is identical to the solution of the homogeneous equation (1.33) (see Remark 1.6):

$$x(t) = \varphi(t; t_0, x_0, 0) = \Phi(t, t_0)x_0, \quad t \ge t_0. \tag{1.54}$$

The system being time-invariant, for every $t, t_0, \tau \in \mathbb{R}$, and every $x_0 \in \mathbb{R}^n$, according to Definition 1.6, φ and Φ from (1.54) satisfy the relations:

$$x(t) = \varphi(t; t_0, x_0, 0) = \varphi(t+\tau; t_0+\tau, x_0, 0), \quad t \ge t_0, \ \forall x_0 \in \mathbb{R}^n;$$

$$x(t) = \Phi(t, t_0)x_0 \quad = \Phi(t+\tau, t_0+\tau)x_0, \qquad t \ge t_0, \ \forall x_0 \in \mathbb{R}^n;$$

$$\Phi(t, t_0) \qquad = \Phi(t+\tau, t_0+\tau), \qquad t \ge t_0.$$

For $\tau = t_0$ from the above relations it follows that:

$$\Phi(t, t_0) = \Phi(t - t_0, 0), \quad t \ge t_0, \tag{1.55}$$

$$x(t) = \Phi(t - t_0, 0)x_0, \quad t \ge t_0. \tag{1.56}$$

Relation (1.56) highlights the solution of the homogeneous equation (1.33), i.e.

$$\frac{dx}{dt} = \frac{d}{dt}\Phi(t - t_0, 0)x_0. \tag{1.57}$$

For $t_0 = t$, by comparing (1.57) to (1.33), it follows that:

$$A(t) = \frac{d}{dt}\Phi(t - t_0, 0)\Big|_{t=t_0} \equiv A = \text{constant}, \quad t \in \mathbb{R}. \tag{1.58}$$

For $x_0 = 0$ the transition function (1.51) is reduced to the forced-regime component (1.29), which is identical to the particular solution (1.48) (see Remark 1.6)):

$$x(t) = \varphi(t; t_0, 0, u_{[t_0, t]}) = \int_{t_0}^{t} \Phi(t-\theta, 0)B(\theta)u(\theta)d\theta, \quad t \ge t_0, \tag{1.59}$$

where (1.55) has been taken into account.

Since the system is time-invariant, it follows that for every $t, t_0, \tau \in \mathbb{R}$ and every input segment $u_{[t_0, t]} \in \Omega$ – appropriately translated, according to Definition 1.6, function (1.59) satisfies:

$$x(t) = \varphi(t; t_0, 0, u_{[t_0, t]}) \qquad = \varphi(t+\tau; t_0+\tau, 0, \mathcal{S}_{-\tau} \circ u_{[t_0+\tau, t+\tau]}), \text{ i.e.,}$$

$$x(t) = \int_{t_0}^{t} \Phi(t-\theta, 0)B(\theta)u(\theta)d\theta = \int_{t_0+\tau}^{t+\tau} \Phi(t-\theta+\tau, 0)B(\theta)u(\theta-\tau)d\theta, \ t \ge t_0.$$

By making the change of variables $\theta - \tau \to \theta$ in the integrals, one may continue with:

$$\int_{t_0}^{t} \Phi(t-\theta, 0)B(\theta)u(\theta)d\theta = \int_{t_0}^{t} \Phi(t-\theta, 0)B(\theta+\tau)u(\theta)d\theta, \text{ i.e.,}$$

$$\int_{t_0}^{t} \Phi(t-\theta, 0)\left[B(\theta) - B(\theta+\tau)\right]u(\theta)d\theta = 0, \ t \ge t_0, \tag{1.60}$$

With $\Phi(t-\theta, 0) \not\equiv 0, u(\theta) \not\equiv 0, \theta \in [t_0, t] \subseteq \mathbb{R}$, according to the Titchmarsh theorem [77], on the vanishing of the convolution product, from (1.60) it follows that

$$B(\theta) = B(\theta+\tau), \quad \theta \in [t_0, t] \subseteq \mathbb{R}, \ \tau \in \mathbb{R}.$$

For $\theta = t, \ \tau = -t$, from the previous relation it follows that:

$$B(t) = B(0) \equiv B = \text{constant}, \quad t \in \mathbb{R}, \tag{1.61}$$

Finally, the time invariance (Definition 1.6), applied to the output equation (1.27), obviously implies that:

$$C(t) \equiv C = \text{constant}, \quad t \in \mathbb{R}, \tag{1.62}$$

$$D(t) \equiv D = \text{constant}, \quad t \in \mathbb{R}. \ \square \tag{1.63}$$

a. Fundamental matrix and transition matrix

We show next that Theorem 1.10 is also a sufficient condition for a system to be linear time-invariant. First, we give some preparatory results in Theorems 1.11–1.15 and Remark 1.7.

Theorem 1.11

The fundamental matrix of equation (1.52) is expressed as:

$$X(t) = e^{At}, \ t \geq 0. \tag{1.64}$$

Proof. According to Theorem 1.5, for $A = \text{constant}$, $X(t)$ is obtained from:

$$\dot{X}(t) = AX(t), \ t \geq 0, \quad X(0) = I_n, \quad \text{with } \det X(t) = e^{t \operatorname{Tr} A} \neq 0, \tag{1.65}$$

where $\operatorname{Tr} \bullet$ denotes the *trace of matrix* $\bullet$.

We are looking for a solution expressed by the power series (see Theorem 1.6):

$$X(t) = \sum_{k=0}^{\infty} X_k t^k, \tag{1.66}$$

where X_k, $k = 0,1,2,...$, are unknown matrices. These matrices are determined from (1.65) by taking successive derivatives with respect to time and evaluating for $t = 0$. We can obtain:

$$X(0) = I_n, \ \dot{X}(0) = A, \ \ddot{X}(0) = \frac{1}{2!}A^2, \ \dddot{X}(0) = \frac{1}{3!}A^3, ..., \ \overset{(k)}{X}(0) = \frac{1}{k!}A^k, ... \ .$$

With these and with the derivative of series (1.66), calculated for $t = 0$, we can write:

$$X(t) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k \triangleq e^{At}. \tag{1.67}$$

The exponential matrix e^{At} is the notation for the power series (1.67). $\square$

This notation naturally extends the scalar exponential $e^{at} = \sum_{k=0}^{\infty} \frac{1}{k!} a^k t^k$.

Remark 1.7

From (1.67), multiplying by A to left and to right, we obtain:

$$Ae^{At} \equiv e^{At}A, \quad AX(t) \equiv X(t)A. \tag{1.68}$$

From (1.65), (1.68) we obtain the rule of the exponential matrix e^{At} derivation:

$$\frac{d}{dt} e^{At} = Ae^{At} \equiv e^{At}A, \quad \dot{X}(t) = AX(t) \equiv X(t)A. \ \square \tag{1.69}$$

Theorem 1.12

The inverse of the fundamental matrix (1.64) has the expression:

$$\tilde{X}(t) = \sum_{k=0}^{\infty} \frac{1}{k!} (-A)^k t^k \triangleq e^{-At}. \tag{1.70}$$

Proof. With (1.69) and (1.70) (derived with (1.69)), we successively write:

$$\frac{d}{dt}\left(X(t)\tilde{X}(t)\right) = \dot{X}(t)\tilde{X}(t) + X(t)\dot{\tilde{X}}(t) = AX(t)\tilde{X}(t) + X(t)\left(-A\tilde{X}(t)\right) =$$

$$= AX(t)\tilde{X}(t) - AX(t)\tilde{X}(t) = 0 \ \Rightarrow \ X(t)\tilde{X}(t) = I_n.$$

The final result was obtained by integration, and the constant on the right-hand side was determined from the initial conditions $X(0) = I_n$, $\tilde{X}(0) = I_n$. Thus,

$$X^{-1}(t) \equiv e^{-At}. \ \square \tag{1.71}$$

Remark 1.7 and Theorem 1.12 highlight the similarity with the rules of derivation and inversion of the scalar exponential e^{at}. This confirms the consistency of the notation adopted for the power series (1.67).

Let the homogeneous equation corresponding to the equation (1.52) be:

$$\dot{x} = Ax, \ t \geq t_0. \tag{1.72}$$

It is known that the solution of the Cauchy problem (1.72), (1.6) is (1.45).

To give an explicit form for the transition matrix $\Phi(t, t_0)$, the following matrix analysis result will be used [25], [26].

Theorem 1.13

Let E, F be any two square matrices of the same order. The equality

$$e^{Et}e^{Ft}=e^{(E+F)t} \tag{1.73}$$

holds if and only if $EF=FE$. □

Theorem 1.14

The transition matrix of system (1.52) has the expression:

$$\Phi(t,t_0)\equiv\Phi(t-t_0,0)=e^{A(t-t_0)},\ t\geq t_0;\ \det\Phi(t-t_0,0)=e^{(t-t_0)\mathrm{Tr}\,A}\neq 0, \tag{1.74}$$

where $\mathrm{Tr}\,\bullet$ denotes the *trace of matrix* $\bullet$.

$\Phi(t,t_0)$ is the solution of the differential equation:

$$\dot{\Phi}(t-t_0,0)=A\Phi(t-t_0,0),\ t\geq t_0;\ \ \Phi(0,0)=I_0, \tag{1.75}$$

and has the commutative property:

$$Ae^{A(t-t_0)}\equiv e^{A(t-t_0)}A,\quad A\Phi(t-t_0,0)\equiv\Phi(t-t_0,0)A,\quad t\geq t_0. \tag{1.76}$$

Using (1.76), the solution of the homogeneous equation (1.72) is obtained as:

$$x(t)=e^{A(t-t_0)}x_0,\quad t\geq t_0. \tag{1.77}$$

Proof. According to (1.39), (1.64), (1.71), and (1.73) the transition matrix is:

$$\Phi(t,t_0)=e^{At}e^{-At_0}=e^{A(t-t_0)}\equiv\Phi(t-t_0,0),\quad t\geq t_0.$$

Equation (1.75) (which provides the derivation rule), property (1.76) and solution (1.77) are immediately obtained from (1.65), (1.68) and (1.45) respectively. □

Remark 1.8

The transition matrix $\Phi(t-t_0,0)$ (see (1.74)) has the properties specified in Remark 1.3, namely:

(*i*) *Orientability*: $\Phi(t,t_0)=\Phi(t-t_0)=e^{A(t-t_0)}$ is defined for every $t\geq t_0$.

(*ii*) *Consistency*: from (1.74) for $t=t_0$ it follows that:

$$\Phi(t_0,t_0)=\Phi(t_0-t_0,0)=\Phi(0,0)=e^{A0}=I_n, \tag{1.78}$$

which assures the consistency of the solution (1.77) for $t=t_0$.

(*iii*) *Composability*: For every $t_1,\ t_2,\ t_3\in\mathbb{R}$, with $t_1<t_2<t_3$, the following holds:

$$\Phi(t_3-t_1,0)=\Phi(t_3-t_2,0)\Phi(t_2-t_1,0),\quad e^{A(t_3-t_1)}=e^{A(t_3-t_2)}e^{A(t_2-t_1)}. \tag{1.79}$$

For solution (1.77) of the homogeneous equation (1.72), relation (1.79) has the following meaning: the solution

$$x(t_3)=\Phi(t_3-t_2,0)x(t_1),\ t_3\geq t_1,$$

is uniquely composed from the following concatenated solutions:

$$x(t_3)=\Phi(t_3-t_2,0)x(t_2),\ t_3\geq t_2 \text{ and } x(t_2)=\Phi(t_2-t_1,0)x(t_1),\ t_2\geq t_1.\ \square$$

b. Solution of the non-homogeneous input – state equation

Taking into account (1.49) and (1.74), the solution of the *Cauchy problem* (1.52), (1.6), which is also the transition function of system (1.52), (1.53), has the expression:

$$x(t)=\varphi(t;t_0,x_0,u_{[t_0,t]})\equiv e^{A(t-t_0)}x(t_0)+\int_{t_0}^{t}e^{A(t-\theta)}Bu(\theta)d\theta,\ t\geq t_0. \tag{1.80}$$

Theorem 1.15

A sufficient condition for the linear finite-dimensional and smooth dynamic system (1.26), (1.27) to be time-invariant is to have the form (1.52), (1.53).

Proof. Since the solution (1.80) is a transition function (a result of Theorem 1.8), it remains to be shown that the system is time-invariant (see Definition 1.6). Indeed, for every $t,t_0,\tau\in\mathbb{R}$, , every $x_0\in\mathbb{R}^n$ and every $u_{[t_0,t]}\in\Omega$, the solution (1.80) satisfies:

$$\varphi(t;t_0,x_0,u_{[t_0,t]})=\varphi(t+\tau;t_0+\tau,x_0,u_{[t_0-\tau,t-\tau]}),$$

$$e^{A(t-t_0)}x_0+\int_{t_0}^{t}e^{A(t-\theta)}Bu(\theta)d\theta=e^{A(t+\tau-t_0-\tau)}x_0+\int_{t_0+\tau}^{t+\tau}e^{A(t+\tau-\theta)}Bu(\theta-\tau)d\theta.$$

Collecting the like terms and making the substitution $\theta-\tau\to\theta$ in the integrals, we obtain an obvious equality.

The proof is completed taking into account the fact that (1.53) is time-invariant in the sense of Definition 1.6. Therefore, C, D are constant matrices. □

The results in Theorems 1.9–1.15, can be stated in a unified manner as in the following statement.

Theorem 1.16

A necessary and sufficient condition for the linear finite-dimensional and smooth dynamic system (1.26), (1.27) to be time-invariant is to be described by the input – state – output equations of the form (1.52), (1.53). □

The input – output transfer represented by the *non-recursive* transfer operator $\mathcal{S}$ (see (1.1)) and equations (1.52) and (1.53) are now obtained from (1.80) and (1.53):

$$y(t)=Ce^{A(t-t_0)}x(t_0)+C\int_{t_0}^{t}e^{A(t-\theta)}Bu(\theta)d\theta+Du(t),\ \ t\geq t_0. \tag{1.81}$$

Note that systems (1.52), (1.53) with constant parameters are also called *constant linear dynamic systems* (see Definition 1.6), the obvious reason being that the elements of the matrices A, B, C, D (i.e. the system parameters) are constant.

c. Linearization of a nonlinear, smooth, finite-dimensional and time-invariant dynamic system

Let the nonlinear system be described by equations (1.18) and (1.19) and let the constant vectors $\overline{u}\in\mathbb{R}^m$, $\overline{x}\in\mathbb{R}^n$, $\overline{y}\in\mathbb{R}^p$ define a *stationary regime* characterized by

$$\dot{\overline{x}}=0,\ \ f(\overline{x},\overline{u})=0, \tag{1.82}$$

$$\overline{y}=g(\overline{x},\overline{u}). \tag{1.83}$$

There are certain situations where the system evolves through *small variations* $\tilde{u}$, $\tilde{x}$, $\tilde{y}$ of u, x, y around the stationary regime defined by constant vectors $\overline{u}$, $\overline{x}$, $\overline{y}$. In this case, we simplify the treatment – without exceeding the limits of *admissible modelling errors*, and from model (1.18), (1.19) we obtain its *linearization* by using the *Taylor formula*. Assuming that functions f, g have continuous second-order derivatives, we write:

$$f(x,u)=\underbrace{f(\overline{x},\overline{u})}_{=\dot{\overline{x}}=0}+f_x(\overline{x},\overline{u})(x-\overline{x})+f_u(\overline{x},\overline{u})(u-\overline{u})+\underbrace{F_1(v)}_{\approx 0}, \tag{1.84}$$

$$g(x,u)=\underbrace{g(\overline{x},\overline{u})}_{=\overline{y}}+g_x(\overline{x},\overline{u})(x-\overline{x})+g_u(\overline{x},\overline{u})(u-\overline{u})+\underbrace{G_1(v)}_{\approx 0}, \tag{1.85}$$

where, for the first terms on the right-hand side, we used (1.82) and (1.83). f_x, f_u, g_x, g_u are the first-order partial derivatives of f, g.

In the formulas (1.84) and (1.85), the constant matrices

$$f_x(\overline{x},\overline{u})\triangleq\overline{A},\quad f_u(\overline{x},\overline{u})\triangleq\overline{B}, \tag{1.86}$$

$$g_x(\overline{x},\overline{u})\triangleq\overline{C},\quad g_u(\overline{x},\overline{u})\triangleq\overline{D}, \tag{1.87}$$

are the *Jacobian matrices* with respect to vectors x and u of vector functions f and g respectively. They are calculated for the stationary regime considered, represented by $\overline{x}$ and $\overline{u}$. At the same time, $F_1(v)$ and $G_1(v)$ are the first-order residuals, where $v\triangleq[x^T\ u^T]^T$ is the vector composed of x and u. These residuals can be estimated. They are negligible (as outlined in the formulas (1.84) and (1.85)) for the small variations of u, x, respectively around the stationary values $\overline{u}$, $\overline{x}$, according to the properties:

$$\lim_{v\to\overline{v}}\frac{\|F_1(v)\|}{\|v-\overline{v}\|^2}=0,\ \ \lim_{v\to\overline{v}}\frac{\|G_1(v)\|}{\|v-\overline{v}\|^2}=0,\ \text{where}\ \overline{v}\triangleq[\overline{x}^T\ \overline{u}^T]^T.$$

We already denoted with:

$$\tilde{u}=u-\overline{u}\ ,\ \tilde{x}=x-\overline{x}\ ,\ \tilde{y}=y-\overline{y} \tag{1.88}$$

the small variations of u, x, y around the values $\overline{u}$, $\overline{x}$, $\overline{y}$ respectively.

Thus, by appropriately replacing (1.86)–(1.88) into (1.84), (1.85) and then appropriately substituting the results into (1.18), (1.19) with (1.88), we obtain the following linear time-invariant and smooth dynamic system:

$$\dot{\tilde{x}}=\overline{A}\tilde{x}+\overline{B}\tilde{u},\ \ t\geq t_0, \tag{1.89}$$

$$\tilde{y}=\overline{C}\tilde{x}+\overline{D}\tilde{u}. \tag{1.90}$$

These equations adequately represent the system (1.18), (1.19) only as long as the errors of the approximation, i.e. the first-order residuals, satisfy the conditions: