Relativistic Forces in Special and General Relativity

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By

Adrian Sfarti

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This book is dedicated to my mentors: Jana Rancu, Eliza Haseganu, Elena Kreindler-Wexler, Mikayel Sarian, Marius Preda, Alexandru Fransua, Ovidiu Lupas, Laurentiu Lupas, Paul Cristea and Vlad Ionescu.

CONTENTS

PREFACE

The notion of force has a very important meaning in physics, from very early one we learn how to describe the particle trajectories by solving the equation $m\frac{d^2\mathbf{r}}{dt^2}=\mathbf{F}$ in the framework of Newtonian mechanics. Things get a lot more complicated at high speeds, in the framework of special relativity, where the equation of motion takes a much more complicated form of $m\frac{d}{dt}(\frac{\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}})=\mathbf{F}$ where $\mathbf{u}=\frac{d\mathbf{r}}{dt}$. We dedicate the first one third of the book to these cases by studying different forms of the equations of motion as a result of the different expressions for the force $\mathbf{F}$. Much effort is dedicated to the case of the general Lorentz force, $\mathbf{F}=q(\mathbf{v}\times\mathbf{B}+\mathbf{E})$ that intervenes so often in the design of particle accelerators. We present a few new derivations for Thomas precession and Thomas Wigner rotation as well as applications to the Compton effect. As we will see later on in the book, the situation is even more complicated in the case of the fictitious forces (d'Alembert, centrifugal, Coriolis and Euler) that appear only in non-inertial frames (accelerated linearly, uniformly rotating and in accelerated rotation). It is interesting to note that the equations of motion in this case fall out directly from the double integration with respect to time of the fictitious accelerations. The second third of this book is dedicated to these forces. The last third deals with forces in a roundabout way, since in General Relativity gravitation is not a force, so, we solve the equations of motion by deriving the Euler-Lagrange equations directly from the different metrics (Schwarzschild, Reissner-Nordstrom).

Adrian Sfarti, 2021

Biographical Note

Mr Sfarti received his PhD from the Polytechnic Institute of Bucharest, Romania, and has now accumulated over 30 years of teaching and research experience. Dr Sfarti was a Professor from the Industry at the University of California Berkeley between 1989 and 2004. He has published over 50 research papers and has 32 patents awarded.

Covariant Treatment of Collisions in Particle Physics

Synopsis

The use of relativistic frame invariants is very well established, especially when it comes to the energy-momentum. In the current paper we show how the conservation of the energy and momentum applies to collisions of particles moving at relativistic speeds, like the ones encountered in nuclear accelerators. We derive the equations for two main types of collisions: elastic and inelastic. The starting point in both cases is the well known theorems of conservation of total energy and conservation of momentum for isolated systems [1-3]. The covariance, once proven, becomes a very useful tool due to the fact that researchers can use any inertial frame in solving the particle collision problems, thus greatly simplifying the solutions.

1. Fundamental notions

You should know by now the definition of proper time:

$d\tau = dt\sqrt{1-(u/c)^2}$ where u is the **coordinate speed** and t is the **coordinate time**. Coordinate time is the time measured by a clock in an arbitrary inertial frame. Proper time is the time measured on a clock commoving with the observer. The **coordinate velocity** is defined as a 3-vector:

$$\mathbf{u}=(dx/dt,dy/dt,dz/dt) \quad (1.1)$$

Now, **proper velocity**, by contrast, is a 4-vector defined as:

$$\mathbf{U}=(dx/d\tau,dy/d\tau,dz/d\tau, d(ct)/d\tau) \quad (1.2)$$

It is easy to show that:

$$\mathbf{U} = \gamma(u)(\mathbf{u}, c)$$
$$\gamma(u) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{1.3}$$

Since τ can be viewed as a proper parameter of a wordline in 4 space, it follows that, by the way it was defined, **U** is the tangent to that wordline.

Further, we can now define the **proper acceleration:**

$$\mathbf{A} = d\mathbf{U} / d\tau \tag{1.4}$$

We can show that:

$$\mathbf{A} = \gamma(u)\frac{d\mathbf{U}}{dt} = \gamma(u)(\mathbf{u}\frac{d\gamma(u)}{dt} + \mathbf{a}\gamma(u), c\frac{d\gamma(u)}{dt}) \tag{1.5}$$

where **a**=d**u**/dt is the **coordinate acceleration**

We also know that in the proper frame of the particle (the frame commoving with the particle) u=0 so, in the proper frame:

That is, the proper acceleration coincides with the coordinate acceleration in the proper frame of the particle. Thus, **A**=0, if and only if **a**=0. By contrast, **U** can never be equal to zero based on its definition. Based on the definitions of velocity, we can define the 3- and the 4-momentum respectively, as:

$$\mathbf{p} = \gamma(u)m\mathbf{u}$$
$$\mathbf{P} = m\mathbf{U} \tag{1.7}$$

Based on the above definitions, we can define the 3- and the 4-force as:

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = m\frac{d(\gamma(u)\mathbf{u})}{dt}$$
$$\mathbf{F} = \frac{d\mathbf{P}}{d\tau} = m\frac{d\mathbf{U}}{d\tau} = m\mathbf{A} \tag{1.8}$$

Note that the derivatives are taken with respect to different times, coordinate for 3-force and proper for 4-force. Sometimes we see the 3-force defined as:

$$\mathbf{f} = \frac{d\mathbf{p}}{d\tau} = m\frac{d(\gamma(u)\mathbf{u})}{d\tau} \tag{1.9}$$

2. Introduction, Inelastic Collisions

In the present chapter we demonstrate that the equations of conservation have a covariant form, that is, they have the same form in all inertial frames. This conclusion is far from obvious since it needs to be proven mathematically. The existent [2,3,8] literature on the subject does not prove the covariance but rather assumes it from the start. We move from simple to complex, from inelastic collisions to elastic ones.

Consider two particles of proper masses m_1 and $m_2$traveling at speeds $u_1$and u_2 with respect to frame F. The particles collide and travel as **one** body at speed u after collision. The equations of conservation of momentum and energy in frame F are [1-3]:

$$\gamma(u_1)m_1u_1 + \gamma(u_2)m_2u_2 = \gamma(u)mu \tag{2.1}$$

$$\gamma(u_1)m_1c^2 + \gamma(u_2)m_2c^2 = \gamma(u)mc^2$$

where $\gamma(u_i)m_iu_i$ represent the momenta before collision, $\gamma(u_i)m_ic^2$ represent the energies before collision, $\gamma(u)mu$ represents the momentum after collision and $\gamma(u)mc^2$ represents the energy after collision. Obviously, (2.2) can be rewritten as:

$$\gamma(u_1)m_1 + \gamma(u_2)m_2 = \gamma(u)m \tag{2.3}$$

where $\gamma(u)$ is a shorthand for $\dfrac{1}{\sqrt{1-(\frac{u}{c})^2}}$

In other words, while rest mass is not conserved, relativistic mass is. Expression (2.3) will come in handy later on. The question is, are the equations (2.1) and (2.2) frame invariant? Is the equation of conservation of energy and momentum frame invariant? The reason it is important to settle this question is that we always prefer equations that are frame-invariant [4], due to not only their intrinsic elegance but also due to the fact that we may need to switch frames in order to be able to solve the particle trajectories [4] easier. Let a frame F' be another inertial frame moving with speed V with respect to F. Substituting:

$$\gamma(u_i) = \gamma(u'_i)\gamma(V)(1+\frac{u'_i V}{c^2})$$

$$u_i = \frac{u'_i + V}{1+\frac{u'_i V}{c^2}}$$

$$\gamma(u_i)u_i = \gamma(u'_i)\gamma(V)(u'_i + V) \tag{2.4}$$

into (2.1) we obtain:

$$\gamma(u'_1)m_1(u'_1+V)+\gamma(u'_2)m_2(u'_2+V)=\gamma(u')(u'+V)m \tag{2.5}$$

that is:

$$\begin{aligned}&\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2 =\\&=\gamma(u')u'm+V(\gamma(u')m-\gamma(u'_1)m_1-\gamma(u'_2)m_2)\end{aligned} \tag{2.6}$$

$$\gamma(u')m-\gamma(u'_1)m_1-\gamma(u'_2)m_2=$$
$$=\frac{V}{c^2}(\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2-\gamma(u')u'm) \tag{2.7}$$

Substituting (2.7) back into (2.6) we obtain the final result:

$$(1-\frac{V^2}{c^2})(\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2-\gamma(u')u'm)=0 \tag{2.8}$$

In other words:

$$\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2=\gamma(u')u'm \tag{2.9}$$

So, the equation of conservation of momentum is frame invariant. Substituting (2.9) in (2.6) we obtain that:

$$\gamma(u')m-\gamma(u'_1)m_1-\gamma(u'_2)m_2=0 \tag{2.10}$$

i.e., the conservation of energy is frame invariant as well. The fact that both momentum and total energy conservation equations are frame invariant gives researchers the option to write the equations in whatever frame makes the calculations easier to perform [4]. Often the importance of covariance of the conservation of energy- momentum is ignored or underestimated, due to the fact that neither the energy nor the momentum is covariant as explained in [5]. The use of relativistic frame invariants is very well established, especially when it comes to the energy-momentum. Most traditional treatments use this particular invariant in order to calculate the "equivalent mass" of a system or, the "mass added to a system". The systems under evaluations are a most general hybrid made up of both massive particles and photons. One question that arises is what happens for the case when the direction of the boost is different from the one of the particle trajectory. To answer this question we will study a simplified case when the boost is oriented along the x-axis and the particle collision is along the y-axis, that is: $\mathbf{u_i}=(0,u_{i,y},0)_{i=1,2}$. In other words, we must substitute $u_i=u_{i,y}$ with $i=1,2$ in (2.1)-(2.3):

$$\gamma(u_{1,y})m_1u_{1,y}+\gamma(u_{2,y})m_2u_{2,y}=\gamma(u_y)mu_y \tag{2.11}$$

obtaining the equation of momentum conservation, while the equation for energy conservation becomes:

$$\gamma(u_{1,y})m_1+\gamma(u_{2,y})m_2=\gamma(u_y)m \tag{2.12}$$

On the other hand, (2.4) becomes:

$$0=u_{i,x}=\frac{u'_{i,x}+V}{1+\frac{u'_{i,x}V}{c^2}}$$

$$u_{i,y}=\frac{\frac{u'_{i,y}}{\gamma(V)}}{1+\frac{u'_{i,x}V}{c^2}}=\frac{\frac{u'_{i,y}}{\gamma(V)}}{1-\frac{V^2}{c^2}}=u'_{i,y}\,\gamma(V)$$

$$0=u_{i,z}=\frac{\frac{u'_{i,y}}{\gamma(V)}}{1-\frac{V^2}{c^2}}=u'_{i,z}\,\gamma(V)$$

$$\gamma(u_{i,y})=\frac{1}{\gamma(V)\sqrt{1-\frac{V^2+u_{i,y}^{'2}}{c^2}}}$$

$$\gamma(u_{i,y})u_{i,y}=\frac{u'_{i,y}}{\sqrt{1-\frac{V^2+u_{i,y}^{'2}}{c^2}}} \tag{2.13}$$

where V is the relative speed between frames F and F'. Substituting (2.13) into (2.11)-(2.12) we obtain a very interesting result:

$$\frac{m_1 u'_{1,y}}{\sqrt{1-\frac{V^2+u'^2_{1,y}}{c^2}}}+\frac{m_2 u'_{2,y}}{\sqrt{1-\frac{V^2+u'^2_{2,y}}{c^2}}}=\frac{mu'_y}{\sqrt{1-\frac{V^2+u'^2_y}{c^2}}} \quad (2.14)$$

$$\frac{m_1}{\sqrt{1-\frac{V^2+u'^2_{1,y}}{c^2}}}+\frac{m_2}{\sqrt{1-\frac{V^2+u'^2_{2,y}}{c^2}}}=\frac{m}{\sqrt{1-\frac{V^2+u'^2_y}{c^2}}} \quad (2.15)$$

The equations are not as elegant as (2.11)-(2.12). There is a very profound lesson resulting from this very simple exercise, the covariance of the equations of conservation for energy-momentum is not a **given**, it needs to **be established**. A judicious choice of frames of reference, like in the beginning of the paragraph, results into one (elegant) covariant expression, while choosing a frame orthogonal onto the direction of collision results into a **different-looking**, not as elegant, **still-covariant** expression. In both cases the problem reduces to solving a system of non-linear equations of the form:

$$\gamma(u_1)m_1u_1+\gamma(u_2)m_2u_2=\gamma(u)mu \quad (2.16)$$

$$\gamma(u_1)m_1+\gamma(u_2)m_2=\gamma(u)m \quad (2.17)$$

that, fortunately, has a very nice solution for both the speed of the resulting particle and its rest mass:

$$u=\frac{\gamma(u_1)m_1u_1+\gamma(u_2)m_2u_2}{\gamma(u_1)m_1+\gamma(u_2)m_2}$$
$$m=\frac{\gamma(u_1)m_1+\gamma(u_2)m_2}{\gamma(u)} \quad (2.18)$$

Or, written in frame F’:

$$u'_y = \frac{\dfrac{m_1 u'_{1,y}}{\sqrt{1-\dfrac{V^2+u'^2_{1,y}}{c^2}}} + \dfrac{m_2 u'_{2,y}}{\sqrt{1-\dfrac{V^2+u'^2_{2,y}}{c^2}}}}{\dfrac{m_1}{\sqrt{1-\dfrac{V^2+u'^2_{1,y}}{c^2}}} + \dfrac{m_2}{\sqrt{1-\dfrac{V^2+u'^2_{2,y}}{c^2}}}} \tag{2.19}$$

A second question that often arises is: "what happens for collisions between particles at non-zero angles"? The answer is very simple, we only need to project equations (2.1),(2.3) thrice, once for each axis of coordinates:

$$\begin{gathered}\gamma(u_1)m_1u_{1,w} + \gamma(u_2)m_2u_{2,w} = \gamma(u)mu_w \\ w = \{x, y, z\}\end{gathered} \tag{2.20}$$

$$\gamma(u_1)m_1 + \gamma(u_2)m_2 = \gamma(u)m \tag{2.21}$$

Note that the projection formalism does not affect the $\gamma(u_i)_{i=1,2}$ expressions. Therefore, the proof of covariance of the equations of motion reduces trivially to the previous proof. In a frame S' boosted in the x direction with respect to the original frame S, the equations become:

$$\gamma(u'_1)m_1u'_{1,x} + \gamma(u'_2)m_2u_{2,x} = \gamma(u')mu'_x \tag{2.22}$$

$$\gamma(u'_1)m_1 + \gamma(u'_2)m_2 = \gamma(u')m \tag{2.23}$$

$$\begin{gathered}\frac{m_1 u'_{1,w}}{\sqrt{1-\dfrac{V^2+u'^2_{1,w}}{c^2}}} + \frac{m_2 u'_{2,w}}{\sqrt{1-\dfrac{V^2+u'^2_{2,w}}{c^2}}} = \frac{m u'_w}{\sqrt{1-\dfrac{V^2+u'^2_w}{c^2}}} \\ \frac{m_1}{\sqrt{1-\dfrac{V^2+u'^2_{1,w}}{c^2}}} + \frac{m_2}{\sqrt{1-\dfrac{V^2+u'^2_{2,w}}{c^2}}} = \frac{m}{\sqrt{1-\dfrac{V^2+u'^2_w}{c^2}}}\end{gathered} \tag{2.24}$$

where $w = \{y, z\}$ .So, what about the four-vector formalism? It is well known that four-vectors provide a "shorthand" way of expressing the same information as three-vectors, so recasting the above equations in the four-vector formalism does not add any information, nor does it simplify the proofs[8]. Using (2.13) we can re-write the energy-momentum four vector as:

$$\mathbf{p}_{\mathbf{i}}^{'} = \left(\frac{-m_i V}{\sqrt{1-\frac{V^2+u_{i,y}^{'2}}{c^2}}}, \frac{m_i u_{i,y}^{'}}{\sqrt{1-\frac{V^2+u_{i,y}^{'2}}{c^2}}}, 0, \frac{m_i c^2}{\sqrt{1-\frac{V^2+u_{i,y}^{'2}}{c^2}}}\right)_{i=1,2}$$

$$\mathbf{p}^{'} = \left(\frac{-mV}{\sqrt{1-\frac{V^2+u_y^{'2}}{c^2}}}, \frac{mu_y^{'}}{\sqrt{1-\frac{V^2+u_y^{'2}}{c^2}}}, 0, \frac{mc^2}{\sqrt{1-\frac{V^2+u_y^{'2}}{c^2}}}\right) \qquad (2.25)$$

Armed with the above, we can write the covariant form of the energy conservation theorems in a much more concise form[7]:

$$\sum_{i=1}^{2} \mathbf{p}_{\mathbf{i}}^{'} = \mathbf{p}^{'}$$
$$\sum_{i=1}^{2} \mathbf{p}_{\mathbf{i}} = \mathbf{p} \qquad (2.26)$$

Nevertheless, if we want to derive any measurable information, like the speed of the particle after collision or its mass, we need to go back to the three-vector formulas (2.18)-(2.19).

3. Elastic Collisions

Let's consider now a more complicated case, the case of elastic collisions. After collision the particles have different speeds from each other:

$$\gamma(u_1)m_1u_1 + \gamma(u_2)m_2u_2 = \gamma(U_1)m_1U_1 + \gamma(U_2)m_2U_2 \qquad (3.1)$$

$$\gamma(u_1)m_1 + \gamma(u_2)m_2 = \gamma(U_1)m_1 + \gamma(U_2)m_2 \tag{3.2}$$

$\gamma(u_i)m_i u_i$ represent the momenta before collision, $\gamma(u_i)m_i c^2$ represent the energies before collision, $\gamma(U_i)m_i U_i$ represents the momenta after collision and $\gamma(U_i)m_i c^2$ represent the energies after collision. Inserting (2.4) into (3.1) we obtain:

$$\begin{aligned}&\gamma(u'_1)m_1(u'_1+V)+\gamma(u'_2)m_2(u'_2+V)=\\&=\gamma(U'_1)m_1(U'_1+V)+\gamma(U'_2)m_2(U'_2+V)\end{aligned} \tag{3.3}$$

where V is the relative speed between frames F and F'. After isolating the terms in V:

$$\begin{aligned}&\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2=\\&=\gamma(U'_1)m_1U'_1+\gamma(U'_2)m_2U'_2+V(\gamma(U'_1)m_1+\\&+\gamma(U'_2)m_2-\gamma(u'_1)m_1-\gamma(u'_2)m_2)\end{aligned} \tag{3.4}$$

Inserting (2.4) into (3.2):

$$\begin{aligned}&\gamma(U'_1)m_1+\gamma(U'_2)m_2-\gamma(u'_1)m_1-\gamma(u'_2)m_2=\\&=\frac{V}{c^2}(\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2-\\&-\gamma(U'_1)U'_1\,m_1-\gamma(U'_2)U'_2\,m_2)\end{aligned} \tag{3.5}$$

Substitute the right hand side of (3.5) into the right hand side of (3.4):

$$\begin{aligned}&(1-\frac{V^2}{c^2})(\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2-\\&-\gamma(U'_1)m_1U'_1-\gamma(U'_2)m_2U'_2)=0\end{aligned} \tag{3.6}$$

That means that the equation of momentum conservation is frame-invariant:

$$\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2 = \\ =\gamma(U'_1)m_1U'_1+\gamma(U'_2)m_2U'_2 \tag{3.7}$$

Substituting (3.7) into (3.5) we obtain that the energy conservation equation is frame invariant:

$$\gamma(U'_1)m_1+\gamma(U'_2)m_2=\gamma(u'_1)m_1+\gamma(u'_2)m_2 \tag{3.8}$$

The fact that both momentum and total energy conservation equations are frame invariant gives researchers the option to write the equations in whatever frame makes the calculations easier to perform.

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CONSERVATION LAWS FOR PLASMA SYSTEMS

Synopsis

The use of relativistic frame invariants is very well established, especially when it comes to the energy-momentum. In the following paper we clarify the terms "conserved" vs. "frame invariant" and we explain the differences between the two concepts. Our paper is divided into three main sections. In the first section we explain the notion of frame invariance. In the second section we explain the energy-momentum conservation. We end up by giving a practical example (a hybrid plasma gas) of an open system, whereby energy and momentum are added from outside the system. We will show the interesting effects caused by adding photons to a system of massive particles. The new approach is extremely important in applications like particle accelerators where we can only work with directly measurable quantities, the kinetic energy KE and the momentum **p**.

1. Relativistic Frame Invariance

Frame-invariance is one of the most important properties in special relativity. As physicists, we try to express the laws of physics in frame invariant quantities in order to take advantage of the important property of such quantities remaining unchanged when passing from one inertial frame to another. It is well known that in relativity, the total energy (E) and the three-vector momentum (**p**) of a single particle are frame variant:

$$E = \gamma(u)mc^2 \tag{1.1}$$

$$\mathbf{p} = \gamma(u)m\mathbf{u} \tag{1.2}$$

The kinetic energy: $KE = \gamma(u)mc^2 - mc^2$ is also frame variant.

By contrast, the norm of the energy-momentum four-vector $\tilde{\mathbf{P}} = (E, \mathbf{p}c)$ is frame invariant:

$$\tilde{\mathbf{P}}.\tilde{\mathbf{P}} = E^2 - (\mathbf{p}c)^2 = m^2c^4 \qquad (1.3)$$

In the present paper we will make extensive use of the frame variant quantities E and **p** as well as the frame invariant norm of the energy-momentum.

2. Transformation of Energy and Momentum between Frames

We have already shown that neither energy, nor the momentum is frame invariant, therefore it becomes interesting to derive the mathematical transformations when passing from one inertial frame to another. In the general case of arbitrary orientation between the axes of S and S' moving with the relative velocity **v**:

$$E = \gamma(v)(E' + \mathbf{p}'.\mathbf{v}) \qquad (2.1)$$

$$\mathbf{p} = \mathbf{p}' + \gamma(v)((1-\gamma^{-1}(v))\mathbf{p}'.\mathbf{v} + \frac{v^2}{c^2}E')\frac{\mathbf{v}}{v^2} \qquad (2.2)$$

Since the velocity **v** between S and S' is constant, by differentiating (2.1)-(2.2) we obtain:

$$dE = \gamma(v)(dE' + d\mathbf{p}'.\mathbf{v}) \qquad (2.3)$$

$$d\mathbf{p} = d\mathbf{p}' + \gamma(v)((1-\gamma^{-1}(v))d\mathbf{p}'.\mathbf{v} + \frac{v^2}{c^2}dE')\frac{\mathbf{v}}{v^2} \qquad (2.4)$$

Both (2.3) and (2.4) are instrumental in the computations involved in the next section.

3. The Theorems of Energy-Momentum Conservation for Closed Systems of Massive Particles

Let the total energy of a system of particles with arbitrarily distributed velocities $\mathbf{v}_i$ in a frame of reference S be:

$$E = c^2 \Sigma \gamma_i m_i \tag{3.1}$$

$$\gamma_i = \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}}$$

The total momentum in frame S is:

$$\mathbf{p} = \Sigma \gamma_i m_i \mathbf{v}_i \tag{3.2}$$

Let us calculate:

$$E^2 - (\mathbf{p}c)^2 = c^4 (\Sigma \gamma_i m_i)^2 - c^2 \Sigma (\gamma_i \gamma_j m_i m_j \mathbf{v}_i \mathbf{v}_j) \tag{3.3}$$

We can always find M and **V** such that:

$$E = c^2 \Sigma \gamma_i m_i = c^2 \gamma(V) M \tag{3.4}$$

$$\mathbf{p} = \Sigma \gamma_i m_i \mathbf{v}_i = \gamma(V) M \mathbf{V} \tag{3.5}$$

so $E^2 - (\mathbf{p}c)^2 = M^2 c^4$ (3.6)

is clearly invariant. Obviously from (3.1),(3.2),(3.4) and (3.5) we obtain:

$$\mathbf{V} = \frac{\Sigma \gamma_i m_i \mathbf{v}_i}{\Sigma \gamma_i m_i} \tag{3.7}$$

$$M = \frac{\Sigma \gamma_i m_i}{\gamma(V)} \tag{3.8}$$

Expression (3.8) provides the relativistic equivalent mass of the system of massive particles while (3.7) represents the average relativistic speed. In classical mechanics energy and momentum conservation are independent of each other. Not so in relativity, courtesy of expression (3.6). Differentiating (3.6) we obtain:

$$EdE - c^2\mathbf{p}d\mathbf{p} = c^4 MdM \tag{3.9}$$

A closed system is defined by dM=0, or its equivalent:

$$EdE - c^2\mathbf{p}d\mathbf{p} = 0 \tag{3.10}$$

Theorem1: A closed system that exhibits conservation of three-momentum **p** will also exhibit conservation of energy.

Proof: $d\mathbf{p} = 0 => dE = 0$ (3.11)

Theorem2: If energy is conserved with respect to an inertial frame S', then it is conserved with respect to any other inertial frame S.

Proof: We start with:

$$dE = \gamma(v)(dE' + d\mathbf{p}'.\mathbf{v}) \tag{3.12}$$

From (3.10) we infer that $d\mathbf{p}' = 0 => dE' = 0 => dE = 0$ so

$$dE' = 0 => dE = 0 \tag{3.13}$$

Theorem3: A closed system that exhibits conservation of energy will exhibit conservation of momentum.

Proof: From theorem2 we obtain

$$dE = 0 => dE' = 0 => \mathbf{v}.d\mathbf{p}' = 0 => d\mathbf{p}' = 0 \;. \tag{3.14}$$

Theorem4: If momentum is conserved with respect to an inertial frame S', then it is conserved with respect to any other inertial frame S.

Proof: We start with:

$$d\mathbf{p} = d\mathbf{p}' + \gamma(v)((1-\gamma^{-1}(v))d\mathbf{p}'.\mathbf{v} + \frac{v^2}{c^2}dE')\frac{\mathbf{v}}{v^2} \tag{3.15}$$

We already know that $d\mathbf{p}' = 0 => dE' = 0$ so (3.15) implies immediately that $d\mathbf{p}' = 0 => d\mathbf{p} = 0$.

Theorem5: If the four-vector momentum is conserved then the total energy and the total three-vector momentum are also conserved:

Proof:

If $d\sum\tilde{\mathbf{P}}=0$ (3.16)

then:

$d\sum E=0$ **and** $d\sum\mathbf{p}=0$ (3.18)

Consequence: since $d\sum\mathbf{p}=0$ it follows trivially that $\frac{d}{dt}\sum\mathbf{p} = 0$, that is:

$$\sum\mathbf{f}=0 \quad (3.19)$$

4. Open Systems: Hybrid Plasma Gasses Composed of a Mix of Massless and Massive Particles

Imagine that we add a photon to the system of massive particles described in the previous paragraph. Such hybrid systems made up of photons injected into plasma form the object of statistical [8] or of kinematic treatments [9]. By contrast, we will show a relativistic-invariant based treatment, similar to the one shown in [7] while using the theory developed in the preceding paragraphs. Obviously, since the system is not closed, the energy and momentum will vary due to the addition of the photon to the existent system. To fix the ideas, let's assume that we add a photon of energy e and momentum **p** to a system of massive particles of total energy E and total three-vector momentum **P**. This is a common application in the study of plasma systems where electromagnetic energy is injected gradually. By using (3.3) we can derive a very interesting result. Let us calculate:

$$(\Sigma E)^2 - c^2(\Sigma \mathbf{p})^2 = (E+e)^2 - c^2(\mathbf{P}+\mathbf{p})(\mathbf{P}+\mathbf{p}) =$$
$$= E^2 - (cP)^2 + 2(Ee - c^2\mathbf{Pp}) \quad (4.1)$$

$$\mathbf{Pp}_{\max} = Pp \quad (4.2)$$

$$(Ee - c^2\mathbf{Pp})_{\min} = Epc - c^2\mathrm{Pp{=}pc(E\text{-}Pc)} \geq 0 \quad (4.3)$$

$$(E+e)^2 - c^2(\mathbf{P}+\mathbf{p})(\mathbf{P}+\mathbf{p}) \geq E^2 - (cP)^2 \quad (4.4)$$

The above shows that the addition of the photon results into an increase of the value of the expression (4.1).

Adding a system of photons having the total energy Ee and the total three-vector momentum $\Sigma\,\mathbf{p}$ to the system of massive particles produces an interesting situation:

$$(E + \Sigma e)^2 - c^2(\mathbf{P}+\Sigma\mathbf{p})(\mathbf{P}+\Sigma\mathbf{p}) =$$
$$= E^2 + 2E\Sigma e + (\Sigma e)^2 - (cP)^2 - c^2\Sigma p^2 - 2c^2\mathbf{P}\Sigma\mathbf{p} =$$
$$= E^2 - (cP)^2 + 2(E\Sigma e - c^2\mathbf{P}\Sigma\mathbf{p}) \quad (4.5)$$

$$(\mathbf{P}\Sigma\mathbf{p})_{\max} = P\Sigma p \quad (4.6)$$

$$(E\Sigma e - c^2\mathbf{P}\Sigma\mathbf{p})_{\min} = E\Sigma e - c^2 P\Sigma p = c(E - Pc)\Sigma p \geq 0 \quad (4.7)$$

$$(E+\Sigma e)^2 - c^2(\mathbf{P}+\Sigma\mathbf{p})(\mathbf{P}+\Sigma\mathbf{p}) \geq E^2 - (cP)^2 \quad (4.8)$$

In other words, the addition of photons to a system of particles always results into an increase of the expression evaluated by (4.5). Finally, from the above formalism we can easily compute [6] the “equivalent mass” of the system as a function of its directly measurable kinetic energy KE and the three-vector total momentum **P**:

$$M = \frac{(\mathbf{P}c)^2 - KE^2}{2KE * c^2} \tag{4.9}$$

From (4.9) it follows that when photons are injected, the system mass can also be expressed as a function of directly measurable quantities like the total kinetic energy KE and its total momentum **P**. The equivalent mass variation for such an open system as a function of the variation of the total kinetic energy d(KE) and the variation of total momentum dP (also as a scalar) is:

$$dM = \frac{P}{KE} dP - \frac{(Pc)^2 + KE^2}{2KE^2 c^2} d(KE) \tag{4.10}$$

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PRACTICAL AND THEORETICAL METHODS FOR DETERMINING THE TRAJECTORIES FOR PARTICLES INVOLVED IN ELASTIC COLLISIONS

Synopsis

The present chapter shows how to use the conservation of the energy-momentum in order to determine the trajectories of two particles after they are subjected to an elastic collision. While the problem is studied in existent literature, there are severe limitations in the solutions, like the fact that the solution only determines the angle between the particles after the collision and not their exact trajectories. This is not very satisfactory when it comes to setting up experiments aimed at verifying the theoretical predictions. In the following paper, we will show how to obtain a much more detailed fix of the trajectories of the particles post collision by determining their exact angles with respect to the trajectory of the particles before the collision. The new approach is extremely important in applications like particle accelerators where we can only work with directly measurable quantities, the kinetic energy KE and the momentum **p**.

1. Elastic collision of two arbitrary mass particles

Consider two particles of rest masses m_1 and m_2. In the most general case $m_1 \neq m_2$. The case $m_1 = m_2$ is well represented in literature [1,2] and we will show later on how our solution reduces in the limit to the existent ones. It is well known that in relativity, the total energy (E) and the three-vector momentum (**p**) of a system of particles involved in a collision are conserved [3]:

$$E_1 + E_2 = E_3 + E_4 \tag{1.1}$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$$

In the above, $E_1, E_2, \mathbf{p}_1, \mathbf{p}_2$ are the total energies / momenta of the two particles before the collision, $E_3, E_4, \mathbf{p}_3, \mathbf{p}_4$ are their total energies / momenta after collision.

$$
\begin{aligned}
&E_i = \gamma(v_i) m_i c^2, i = 1,2,3,4 \\
&\mathbf{p}_i = \gamma(v_i) m_i \mathbf{v}_i \\
&E_i^2 - (\mathrm{p_i c})^2 = (m_i c^2)^2 \\
&\gamma(v_i) = \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}}
\end{aligned}
\tag{1.2}
$$

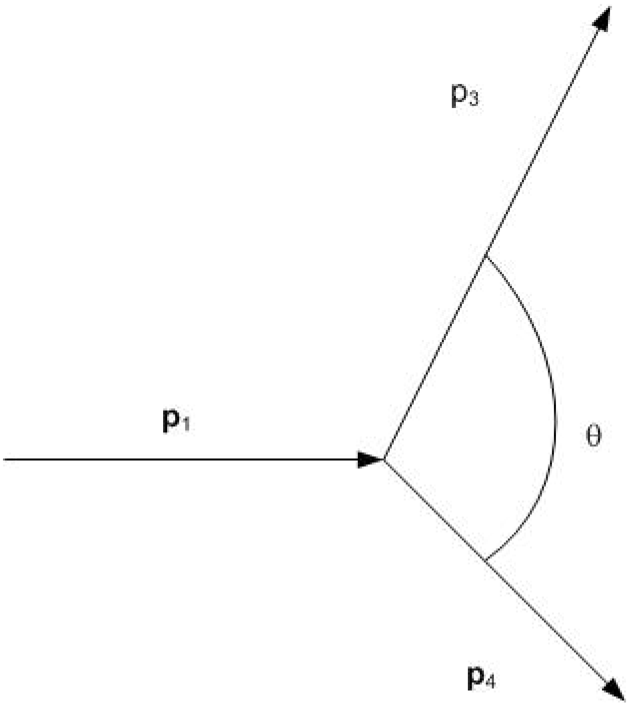

Fig. 1 The collision of two particles