# Loop-like Solitons in the Theory of Nonlinear Evolution Equations

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**Cambridge Scholars** Publishing

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This book first published 2022

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library



ISBN (10): 1-5275-8146-2
ISBN (13): 978-1-5275-8146-3

# Contents

# Preface

The physical phenomena and processes that take place in nature generally have complicated nonlinear features. This leads to nonlinear mathematical models for the real processes. There is much interest in the practical issues involved, as well as in the development of methods to investigate the associated nonlinear mathematical problems including nonlinear wave propagation. An early example of powerful mathematical methods was the development of the inverse scattering method for the Korteweg-de Vries (KdV) giving rise to the persistent interest in soliton theory as applied to many branches of science.

The modern physicist should be aware of the key aspects of nonlinear wave theory developed over the past few decades. This monograph focuses on the interconnections between a variety of different approaches and methods. The application of the theory of nonlinear evolution equations to study a new equation is always an important and sometimes rather nontrivial step. Based on our experience of the study of the Vakhnenko equation (VE), we acquaint the reader with a series of methods and approaches tnat can be applied to a wide class of nonlinear equations. We outline a way in which an uninitiated reader could investigate a new nonlinear equation.

Loop-like solitons are a class of interesting wave phenomena, which have been involved in some nonlinear systems. One remarkable feature of the VE is that it possesses loop-like soliton solutions.

It is our pleasure to thank and acknowledge our colleague and co-author of joint researches Dr. A.J. Morrison.

# Chapter 1
# Introduction

A variety of methods for examining the properties and solutions of nonlinear evolution equations are explored by using the Vakhnenko equation (VE) as an example. It is shown (Chapter 2) how the KdV equation arises in modeling the propagation of low-frequency waves in a relaxing medium. While in high-frequency cases the waves in a relaxing medium are described by an equation called now in scientific literature as the Vakhnenko equation (VE). The consideration of the VE has an interest not only from the viewpoint of the investigation of the propagation of high-frequency perturbations, but more specifically from the viewpoint of the study of methods and approaches that may be applied in the theory of nonlinear evolution equations.

By studying the VE in Chapter 3, it is traced a way in which an uninitiated reader could investigate even early unknown nonlinear equations. As a first step for a new equation, it is necessary to consider the linear analogue and its dispersion relation (these steps for the equation considered here are described in Chapter 2). The next step is, where possible, to link the equation with the known nonlinear equations, as it is carried out for the VE, for example.

The solution procedure, which was used for the Vakhnenko equation (see Chapter 4), can be successfully adopted to find implicit periodic and solitary travelling-wave solutions of the Degasperis–Procesi equation in [69], the Camassa–Holm equation [71], the transformed Hirota–Satsuma-type 'shallow water wave' equation [72] and special cases thereof, namely the generalised Vakhnenko equation and the modified generalised Vakhnenko equation, the short-pulse equation [103] and other equations. An important feature of the method is that it delivers solutions in which both the dependent variable and the independent variable are given in terms of certain parameters.

In Chapter 5 the VE has been written in an alternative form, now known as the Vakhnenko-Parkes equation (VPE), by a change of independent variables. One of the main results of this chapter stated in Section 5.3 is that we have obtained a way of applying the IST method to the VPE. Keeping in mind that the IST is the most appropriate way of tackling the initial value problem, one has to formulate the associated eigenvalue problem. We have proved that the system of equations for the IST problem associated with the VPE does not contain the isospectral Schrödinger equation. Nevertheless, the analysis of the VPE in the context of the isospectral Schrödinger equation allowed us to obtain the two-soliton solution to the VPE even though, in contrast to the KdV equation, the VPE's spectral equation is not the second-order one. These results may be useful in the investigation of a new equation for which the spectral problem is unknown. Historically, once this investigation was completed, we were able to make some progress in the formulation of the IST for the VPE. In Section 7.1 it has been proved that the spectral problem associated with the VPE is of the third order.

The VPE can be written in Hirota bilinear form, as this has been carried out in Chapter 6. It is then possible to show that the VPE satisfies the '$N$-soliton condition', in other words that the equation has an $N$-soliton solution. This solution is found by using a blend of the Hirota method and ideas originally proposed by Moloney & Hodnett. This solution is discussed in detail, including the derivation of phase shifts due to interaction between solitons. Individual solitons are hump-like. However, when transformed back into the original variables, the corresponding solution to the VE comprises $N$ loop-like solitons. It is established that a dissipative term, with a dissipation parameter less than some limit value, does not destroy these loop-like solutions.

The Hirota method not only gives the $N$-soliton solution, but enables one to find a way from the Bäcklund transformation through the conservation laws and associated eigenvalue problem to the inverse scattering method. Thus the Hirota method, which can be applied only for finding solitary wave solutions or traveling wave solutions, allows us to formulate the inverse scattering method which is the most appropriate way of tackling the initial value problem (Cauchy problem). Consequently, in this case, the integrability of an equation can be regarded as proved.

Chapter 7 deals with the inverse scattering method. The inverse scattering transform (IST) method is arguably the most important discovery in the theory of solitons. The method enables one to solve the initial value problem for a nonlinear evolution equation. Moreover, it provides a proof of the complete integrability of the equation.

The idea of the inverse scattering method was first introduced for the KdV equation [94] and subsequently developed for the nonlinear Schrödinger equation [28], the mKdV equation [126, 127], the sine-Gordon equation [25, 128] and the equation of motion for a one-dimensional lattice with an exponential form of inter-site interaction (Toda lattice) [129]. It is to be remarked that the inverse scattering method is a unique theory whereby the initial value problem for the nonlinear differential equations can be solved exactly. For the KdV equation this method was expressed in general form by Lax [130].

The essence of the application of the IST is as follows. The equation of interest for study (in our case the VPE) is written as the compatibility condition for two linear equations (the Lax pair). It is significant that the spectral equation in Lax pair for the VPE is third-order. The whole Lax pair is given by Eq. (7.1.2) and Eq. (7.1.3). First, based on the ideas of Kaup and Caudrey, the initial condition $W(X, 0)$ is mapped into the spectral data $S(0)$ (the direct spectral problem). It is important that, since the variable $W(X, T)$ contained in the spectral equation evolves according to the VPE, the spectral parameter $\lambda$ always retains constant values (i.e. demonstrates the isospectrality). The time evolution of the spectral data $S(T)$ is simple and linear. From a knowledge of $S(T)$ the relationships obtained by Caudrey should be invoked to reconstructed $W(X, T)$ (the inverse spectral problem). The procedure outlined allows solving the Cauchy problem for the VPE.

In Chapter 8 the standard IST method for third-order spectral problems is used to investigate solutions corresponding to bound states of the spectrum and a continuous spectrum. This leads to $N$-soliton solutions and $M$-mode periodic solutions respectively. Interactions between these types of solutions are investigated. Sufficient conditions have been proven so that the solutions become the real functions.

In Chapter 9, the standard procedure for the inverse scattering transform (IST) method is expanded for the case of multiple poles. Using the VPE as an example, we have shown how, in the IST method, to take into account the two-multiple poles, among single poles, in the discrete part of the spectral data. The special line spectrum of continuum states in the IST method, for which the mathematical procedure is similar to that for the discrete spectrum for two-multiple poles, is considered as well. In this case the account of the time-dependence is shown to be essentially different from the standard procedure. This approach can be applied to other integrable nonlinear equations.

# Chapter 2
# Models of relaxing medium

As a rule the behavior of media under the action of high-frequency wave perturbations is not described in the framework of equilibrium models of continuum mechanics. So, to develop physical models for wave propagation through media with complicated inner kinetics, notions based on the relaxational nature of a phenomenon are regarded to be promising. From the non-equilibrium thermodynamics standpoint, models of a relaxing medium are more general than equilibrium models. Thermodynamic equilibrium is disturbed owing to the propagation of fast perturbations. There are processes of interaction that tend to return the equilibrium. The parameters characterizing this interaction are referred to as the inner variables unlike the macro-parameters such as the pressure $p$, mass velocity $u$ and density $\rho$. In essence, the change of macro-parameters caused by the changes of inner parameters is a relaxation process.

## 2.1 Evolution equation for relaxing medium

Starting from a general idea of relaxing phenomena in real media via a hydrodynamic approach, we will derive a nonlinear evolution equation for describing high-frequency waves. We restrict our attention to barotropic media. An equilibrium state equation of a barotropic medium is a one-parameter equation. As a result of relaxation, an additional variable $\xi$ (the inner parameter) appears in the state equation

$$p = p(\rho, \xi) \tag{2.1.1}$$

and defines the completeness of the relaxation process. There are two limiting cases with corresponding sound velocities:

(i) lack of relaxation (inner interaction processes are frozen) for which $\xi = 1$:

$$p = p(\rho, 1) \equiv p_f(\rho); \tag{2.1.2}$$

(ii) relaxation is complete (there is local thermodynamic equilibrium) for which $\xi = 0$:

$$p = p(\rho, 0) \equiv p_e(\rho). \tag{2.1.3}$$

The state equations (2.1.2) and (2.1.3) are considered to be known. These relationships enable us to introduce the sound velocities for fast processes

$$c_f^2 = dp_f/d\rho; \tag{2.1.4}$$

and for slow processes

$$c_e^2 = dp_e/d\rho. \tag{2.1.5}$$

Slow and fast processes are compared using the relaxation time $\tau_p$.

The following dynamic state equation is applied to account for the relaxation effects

$$\tau_p \left( \frac{dp}{dt} - c_f^2 \frac{d\rho}{dt} \right) + (p - p_e) = 0. \tag{2.1.6}$$

The equilibrium equations of state are considered to be known

$$\rho_e - \rho_0 = \int\limits_{p_0}^{p} c_e^{-2} dp. \tag{2.1.7}$$

Clearly, for the fast processes ($\omega\tau_p \gg 1$) we have the relation (2.1.2), and for the slow ones ($\omega\tau_p \ll 1$) we have (2.1.3).

The substantiation of equation (2.1.6) within the framework of the thermodynamics of irreversible processes has been given in [1, 2, 3, 4]. As far as we know the first work in this field was the article by Mandelshtam and Leontovich [5] (see also Section 81 in [2]). We note that the mechanisms of the exchange processes are not defined concretely when deriving the dynamic state equation (2.1.6). In this equation the thermodynamic and kinetic parameters appear only as sound velocities $c_e$, $c_f$ and relaxation time $\tau_p$. These are very common characteristics and they can be found experimentally. Hence it is not necessary to know the inner exchange mechanism in detail. The dynamic state equation (2.1.6) enables us to take into account the exchange processes completely.

The phenomenological approach for describing the relaxation processes in hydrodynamics have been developed in many publications [2, 4, 6, 7]. The dynamic state equation was used to describe the propagation of sound in a relaxing medium [2], to take into account the exchange processes within media (gas-solid particles [4]), and to study wave fields in gas-liquid media [6] and in [7] soils. The well-known Zener phenomenological model of a standard linear solid [8] is generalized to describe the sandstone deformation [9]. Within the context of mixture theory, Biot [10] attempted to account for the non-equilibrium in velocities between components directly in the equations of motion in the form of dissipative terms. In most works, the state equation had been derived from the concept of some concrete mechanism for the inner process.

To analyze the wave motion, we use the following hydrodynamic equations in the Lagrangian coordinates:

$$\frac{\partial V}{\partial t} - \frac{1}{\rho_0}\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0. \tag{2.1.8}$$

Here $V \equiv \rho^{-1}$ is the specific volume, and $x$ is the Lagrangian space coordinate.

The closed system of equations consists of two motion equations (2.1.8) and the dynamic state equation (2.1.6). The motion equations (2.1.8) are written in Lagrangian coordinates since the state equation (2.1.6) is related to the element of mass of the medium.

Let us consider a small perturbation $p' \ll p_0$. The equations of state for fast (2.1.2) and slow (2.1.3) processes are considered to be known. They can be expanded as the power series with accuracy $O(p'^2)$

$$\begin{aligned} V_f(p_0 + p') &= V_0 - V_0^2 c_f^{-2} p' + \frac{1}{2}\left.\frac{d^2 V_f}{dp^2}\right|_{p=p_0} p'^2 + \dots\ , \\ V_e(p_0 + p') &= V_0 - V_0^2 c_e^{-2} p' + \frac{1}{2}\left.\frac{d^2 V_e}{dp^2}\right|_{p=p_0} p'^2 + \dots\ . \end{aligned} \tag{2.1.9}$$

Hereafter, the velocities $c_e$, $c_f$ are related to the initial pressure $p_0$. Combining these two relationships with the equations of motion (2.1.8), we obtain the equation in one unknown quantity (the dash in $p'$ is omitted) [11, 12, 13, 14]:

$$\tau_p \frac{\partial}{\partial t} \left( \frac{\partial^2 p}{\partial x^2} - c_f^{-1} \frac{\partial^2 p}{\partial t^2} + \frac{1}{2V_0^2} \left. \frac{d^2 V_f}{dp^2} \right|_{p=p_0} \frac{\partial^2 p^2}{\partial t^2} \right) + \left( \frac{\partial^2 p}{\partial x^2} - c_e^{-1} \frac{\partial^2 p}{\partial t^2} + \frac{1}{2V_0^2} \left. \frac{d^2 V_e}{dp^2} \right|_{p=p_0} \frac{\partial^2 p^2}{\partial t^2} \right) = 0. \tag{2.1.10}$$

A similar equation has been obtained in Ref. [1], but without nonlinear terms.

The hydrodynamic nonlinearity $p\partial p/\partial x$ and the complicated dispersive law are inherent in a medium which is described by the evolution equation (2.1.10). Now we consider the dispersive relation which follows from equation (2.1.10) after a substitution of the slow perturbation in a form $p' \sim \exp[\mathrm{i}(kx - \omega t)]$,

$$-\mathrm{i}\omega\tau_p \frac{c_e^2}{c_f^2}(\omega^2 - c_f^2 k^2) + (\omega^2 - c_e^2 k^2) = 0. \tag{2.1.11}$$

From this relationship we obtain the functional dependence $k = k(\omega)$

$$k^2 = \frac{\omega^2}{c_f^2} \cdot \frac{\tau_p^2 \omega^2}{1 + \tau_p^2 \omega^2} \cdot \left( 1 + \frac{\mathrm{i}}{\tau_p \omega} \cdot \frac{c_f^2 - c_e^2}{c_e^2} + \frac{1}{\tau_p^2 \omega^2} \cdot \frac{c_e^2}{c_f^2} \right). \tag{2.1.12}$$

Taking the roots we write the result in the form $k = k' + \mathrm{i}k''$. It is clear that $k''$ is associated with the speed of wave attenuation as a function of the distance [2], while a value $c = \omega/k'$ can be considered as the velocity of the perturbation propagation. The expressions for $k'$ and $k''$ take the form

$$k' = a_1 \sqrt{\sqrt{a_2^2 + a_3^2} + a_2}, \qquad k'' = a_1 \sqrt{\sqrt{a_2^2 + a_3^2} - a_2},$$

$$a_1 = \frac{\tau_p^2 \omega^2}{\sqrt{2} c_f \sqrt{1 + \tau_p^2 \omega^2}}, \qquad a_2 = 1 + \frac{c_f^2}{\tau_p^2 \omega^2 c_e^2}, \qquad a_3 = \frac{c_f^2 - c_e^2}{\tau_p \omega c_e^2}.$$

In Fig. 2.1, for example, we show the dependencies $c$ and $k''$ on $\tau_p \omega$ for water-saturation soil with a concentration of air 0.1. For this

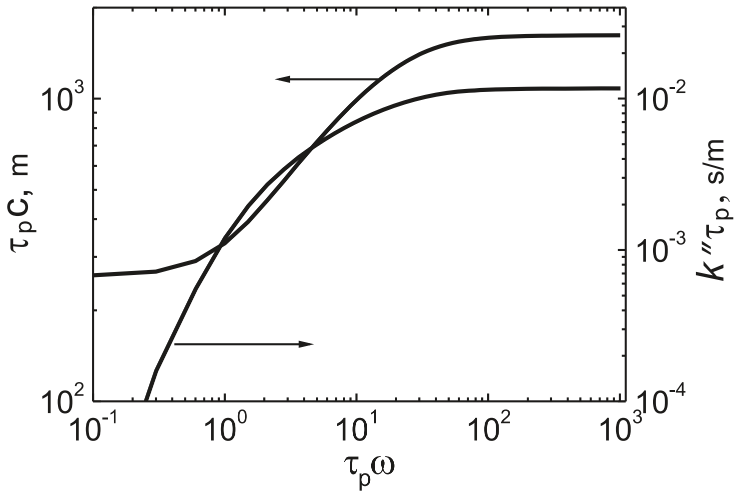


Figure 2.1: The dependencies of the velocity $c$ and the attenuation factor $k''$ on frequency $\tau_p\omega$.

medium $c_f = 1620$ m/s and $c_e = 260$ m/s [7]. The velocity $c$ increases monotonically from $c_e$ to $c_f$ at bottom-up sweep $\tau_p\omega$. The dependence $k'' = k''(\omega)$ indicates that at $\omega \to 0$ the dispersion is absent, while at high frequency the variable $k''$ becomes a constant and does not depend on $\omega$ (see Fig. 2.1) with the limit value

$$\tau_p k'' = \frac{c_f^2 - c_e^2}{2c_f^2 c_e^2}.$$

Hence, the energy in the high-frequency wave dissipates always. For this wave the pressure attenuation is the same as at fixed distance and does not depend on frequency $\omega$.

The equation of state in the form (2.1.9) enables us to describe the effects associated with bulk viscosity of a medium. Let us show that for slow processes (since for these processes the notion of viscosity coefficient is defined, i.e. for processes in which a small deviation from equilibrium is taken into account in linear approximation) a bulk viscosity coefficient relates to the relaxation time $\tau_\rho = \tau_p c_e^2/c_\rho^2$ [1, 2, 5]

$$\zeta = \tau_\rho \rho (c_f^2 - c_e^2). \tag{2.1.13}$$

Let us rewrite (2.1.9) in a form of the power series $p$ in $\tau_\rho d/dt$. To do it, we differentiate equation (2.1.9) with respect to time $t$ and substitute the result into the same equation (2.1.9). Repeating several times this procedure, we obtain with required accuracy the expression

$$dp = c_e^2 d\rho + \tau_\rho (c_f^2 - c_e^2) d\dot{\rho} - \tau_\rho^2 (c_f^2 - c_e^2) d\ddot{\rho} + \dots \ . \tag{2.1.14}$$

Let us consider two terms only in this relation. The value $c_e^2 d\rho$ associates with an increase of a pressure $dp_e$ in infinitely slow process, i.e. $dp_e = c_e^2 d\rho$. It is noted that value $p$ acquires more general sense than merely a pressure. With the accuracy of a sign the value $(-p)$ is nothing other than a stress $\pi_{ii}$. By definition, in the low-frequency approximation the stress is written through the bulk viscosity coefficient [2]

$$\pi_{ii} = -p_e + \zeta \frac{\partial u}{\partial x}.$$

Then it is easy to obtain the expression for the bulk viscosity coefficient in the form (2.1.13).

## 2.2 Low-frequency perturbations and high-frequency perturbations

Now we shall show that for low-frequency perturbations the equation (2.1.10) is reduced to the Korteweg-de Vries-Burgers (KdVB) equation, while for high-frequency waves we shall obtain the equation with hydrodynamic nonlinearity and term that appeared in the Klein-Gordon equation. To analyze the equation (2.1.10) let us apply the multiscale method [15, 16]. The value $\varepsilon \equiv \tau_p \omega$ is chosen to be small (large) parameter where the quantity $\omega$ is the characteristic frequency of wave perturbation. For the sake of convenience we rewrite the equation (2.1.10) as follows:

$$\begin{gathered} \tau_p \omega \frac{\partial}{\partial (t\omega)} \left( \frac{\partial^2 p}{\partial (x\omega)^2} - c_f^{-2} \frac{\partial^2 p}{\partial (t\omega)^2} + \alpha_f \frac{\partial^2 p^2}{\partial (t\omega)^2} \right) + \\ + \left( \frac{\partial^2 p}{\partial (x\omega)^2} - c_e^{-2} \frac{\partial^2 p}{\partial (t\omega)^2} + \alpha_e \frac{\partial^2 p^2}{\partial (t\omega)^2} \right) = 0, \end{gathered} \tag{2.2.1}$$

$$\alpha_f = \frac{1}{2V_0^2} \left. \frac{d^2 V_f}{dp^2} \right|_{p=p_0}, \qquad \alpha_e = \frac{1}{2V_0^2} \left. \frac{d^2 V_e}{dp^2} \right|_{p=p_0},$$

and introduce the new independent variables [11, 12, 13, 14]

$$T_0 = t\omega, \quad X_0 = x\omega, \quad T_{-2} = t\omega/\varepsilon^2, \quad X_{-2} = x\omega/\varepsilon^2. \qquad (2.2.2)$$

The dependent variable $p$ is a function of $T_0, X_0, T_{-2}, X_{-2}$, i.e. $p = p(T_0, X_0, T_{-2}, X_{-2})$. The existing derivatives in (2.2.1) are to be rewritten in the new independent variables

$$\begin{aligned}
\frac{\partial}{\partial(x\omega)} &= \frac{\partial}{\partial X_0} + \varepsilon^{-2}\frac{\partial}{\partial X_{-2}},\\
\frac{\partial}{\partial(t\omega)} &= \frac{\partial}{\partial T_0} + \varepsilon^{-2}\frac{\partial}{\partial T_{-2}},\\
\frac{\partial^2}{\partial(x\omega)^2} &= \frac{\partial^2}{\partial X_0^2} + 2\varepsilon^{-2}\frac{\partial^2}{\partial X_0 \partial X_{-2}} + \varepsilon^{-4}\frac{\partial^2}{\partial X_{-2}^2},\\
\frac{\partial^2}{\partial(t\omega)^2} &= \frac{\partial^2}{\partial T_0^2} + 2\varepsilon^{-2}\frac{\partial^2}{\partial T_0 \partial T_{-2}} + \varepsilon^{-4}\frac{\partial^2}{\partial T_{-2}^2},\\
\frac{\partial^3}{\partial(t\omega)^3} &= \frac{\partial^3}{\partial T_0^3} + 3\varepsilon^{-2}\frac{\partial^3}{\partial T_0^2 \partial T_{-2}} + 3\varepsilon^{-4}\frac{\partial^3}{\partial T_0 \partial T_{-2}^2} + \varepsilon^{-6}\frac{\partial^3}{\partial T_{-2}^3},\\
\frac{\partial^3}{\partial(t\omega)\partial(x\omega)^2} &= \frac{\partial^3}{\partial X_0^2 \partial T_0} + \varepsilon^{-2}\left(\frac{\partial^3}{\partial X_0^2 \partial T_{-2}} + 2\frac{\partial^3}{\partial T_0 \partial X_0 \partial X_{-2}}\right)\\
&\quad + \varepsilon^{-4}\left(\frac{\partial^3}{\partial T_0 \partial X_{-2}^2} + 2\frac{\partial^3}{\partial X_0 \partial X_{-2} \partial T_{-2}}\right) + \varepsilon^{-6}\frac{\partial^3}{\partial X_{-2}^2 \partial T_{-2}}.
\end{aligned} \qquad (2.2.3)$$

It is precisely these variables that cause the equations obtained within the framework of the multiscale method [15, 16]

$$
\begin{aligned}
O(\varepsilon^{+1}) &: \quad \frac{\partial}{\partial T_0}\left(\frac{\partial^2 p}{\partial X_0^2} - c_f^{-2}\frac{\partial^2 p}{\partial T_0^2} + \alpha_f \frac{\partial^2 p^2}{\partial T_0^2}\right) = 0, \\
O(\varepsilon^{0}) &: \quad \frac{\partial^2 p}{\partial X_0^2} - c_e^{-2}\frac{\partial^2 p}{\partial T_0^2} + \alpha_e \frac{\partial^2 p^2}{\partial T_0^2} = 0, \\
O(\varepsilon^{-1}) &: \quad \left(\frac{\partial^3}{\partial X_0^2 \partial T_{-2}} + 2\frac{\partial^3}{\partial T_0 \partial X_0 \partial X_{-2}}\right) p \\
&\qquad -3c_f^{-2}\frac{\partial^3 p}{\partial T_0^2 \partial T_{-2}} + 3\alpha_f \frac{\partial^3 p^2}{\partial T_0^2 \partial T_{-2}} = 0, \\
O(\varepsilon^{-2}) &: \quad \frac{\partial^2 p}{\partial X_0 \partial X_{-2}} - c_e^{-2}\frac{\partial^2 p}{\partial T_0 \partial T_{-2}} + \alpha_e \frac{\partial^2 p^2}{\partial T_0 \partial T_{-2}} = 0, \qquad (2.2.4) \\
O(\varepsilon^{-3}) &: \quad \left(\frac{\partial^3}{\partial T_0 \partial X_{-2}^2} + 2\frac{\partial^3}{\partial X_0 \partial X_{-2} \partial T_{-2}}\right) p \\
&\qquad -3c_f^{-2}\frac{\partial^3 p}{\partial T_0 \partial T_{-2}^2} + 3\alpha_f \frac{\partial^3 p^2}{\partial T_0 \partial T_{-2}^2} = 0, \\
O(\varepsilon^{-4}) &: \quad \frac{\partial^2 p}{\partial X_{-2}^2} - c_e^{-2}\frac{\partial^2 p}{\partial T_{-2}^2} + \alpha_e \frac{\partial^2 p^2}{\partial T_{-2}^2} = 0, \\
O(\varepsilon^{-5}) &: \quad \frac{\partial}{\partial T_{-2}}\left(\frac{\partial^2 p}{\partial X_{-2}^2} - c_f^{-2}\frac{\partial^2 p}{\partial T_{-2}^2} + \alpha_f \frac{\partial^2 p^2}{\partial T_{-2}^2}\right) = 0,
\end{aligned}
$$

to be partially uncoupled [17, 18, 19, 20, 21, 22]. The two leading equations depend on $T_0$ and $X_0$ only, while the last two equations include the independent variables $T_{-2}$ and $X_{-2}$ only. Thus, the low-frequency perturbations are described by the two leading equations, and the high-frequency perturbations by the last two equations. An interaction between these perturbations is described by the three center equations. A similar approach was applied to obtain the evolution equation with cubic nonlinearity [23, 24].

Let us write out the equations of motion for low-frequency and high-frequency perturbations in the initial variables $x$ and $t$. For low-frequency perturbations the main terms $\partial^2 p/\partial X_0^2$ and $c_e^{-2}\partial^2 p/\partial T_0^2$ (and only they) appear in the first and second equations of the system (2.2.4), while for high-frequency perturbations the main terms

$\partial^2 p/\partial X_{-2}^2$ and $c_f^{-2}\partial^2 p/\partial T_{-2}^2$ (and only they) appear in the sixth and seventh equations of the system (2.2.4).

For low-frequency perturbations ($\tau_p \omega \ll 1$) propagating in one direction ($\partial/\partial x - c_e^{-1}\partial/\partial t \simeq 2\partial/\partial x$), we obtain an evolution equation

$$\frac{\partial p}{\partial t} + c_e\frac{\partial p}{\partial x} + \alpha_e c_e^3 p\frac{\partial p}{\partial x} - \beta_e\frac{\partial^2 p}{\partial x^2} + \gamma_e\frac{\partial^3 p}{\partial x^3} = 0, \qquad (2.2.5)$$

$$\alpha_e = \frac{1}{2V_0^2}\left.\frac{d^2 V_e}{dp^2}\right|_{p=p_0}, \qquad \beta_e = \frac{c_e^2\tau_p}{2c_f^2}(c_f^2 - c_e^2),$$

$$\gamma_e = \frac{c_e^3\tau_p^2}{8c_f^4}(c_f^2 - c_e^2)(c_f^2 - 5c_e^2).$$

This equation can be derived in the following way. A dispersion relation for the linearized equation (2.1.10) can be written down with an accuracy $O(k^3)$ in the form $\omega = c_e k + \mathrm{i}\beta_e k^2 - \gamma_e k^3$, if the terms $\partial p/\partial x$ and $c_e^{-1}\partial p/\partial t$ are the main ones. For this dispersion relation we write a linear equation in which a nonlinear term is reconstructed in agreement with the initial equation.

The equation (2.2.5) is the well-known KdVB equation. It is encountered in many areas of physics to describe nonlinear wave processes [25, 26, 27, 28, 29]. In [30] it was shown how hydrodynamic equations reduce to either the KdV or Burgers equation according to the choices for the state equation and the generalized force when analyzing the gasdynamical waves, waves in shallow water [30], hydrodynamic waves in cold plasma [31], and ion-acoustic waves in cold plasma [32].

As is known, the investigation of the KdV equation ($\beta_e = 0$) in conjunction with the nonlinear Schrödinger (NLS) and sine-Gordon equations gives rise to the theory of solitons [25, 27, 28, 29, 30, 33, 34, 35, 36, 37]. As well as having soliton solutions, these equations have other inherent striking properties, in particular integrability. The equations can be integrated, for example, by the inverse scattering method. Details on the study of the aforementioned equations can be found in the monographs [25, 27, 28]. In general, the existence of soliton solutions to a nonlinear evolution equation points to distinctive features for the equation such as integrability, the applicability of the inverse scattering method, the Hirota method and Bäcklund transformation, and the existence of conservation laws. Consequently, the finding of soliton solutions for a new evolution equation is of considerable interest.

For high-frequency perturbations ($\tau_p\omega \gg 1$), using the last two equations of the system (2.2.4), we get the following evolution equation:

$$\frac{\partial^2 p}{\partial x^2} - c_f^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_f c_f^2 \frac{\partial^2 p^2}{\partial x^2} + \beta_f \frac{\partial p}{\partial x} + \gamma_f p = 0. \tag{2.2.6}$$

$$\alpha_f = \frac{1}{2V_0^2}\left.\frac{d^2 V_f}{dp^2}\right|_{p=p_0}, \quad \beta_f = \frac{c_f^2 - c_e^2}{\tau_p c_e^2 c_f}, \quad \gamma_f = \frac{c_f^4 - c_e^4}{2\tau_p^2 c_e^4 c_f^2}.$$

In addition to the nonlinear term with coefficient $\alpha_f$, the equation has dissipative $\beta_f \partial p/\partial x$ and dispersive $\gamma_f p$ terms. If $\alpha_f = \beta_f = 0$, this is a linear Klein-Gordon equation. There is a Green function for this equation [38] that enables us to find the solution in quadrature, at least. The numerical solutions of the Klein-Gordon equation modeling the propagation of high-frequency perturbations in gas-liquid media have been presented in [39]. A similar evolution equation for high-frequency perturbations was described in a monograph by Whitham [40]. However, it coincides with Eq. (2.2.6) only when $\alpha_f = 0$ and $\gamma_f = 0$.

Landau and Lifshitz showed that for high frequencies the dissipative term under high transport of heat agrees with the corresponding term in the equation (2.2.6) (see section 79 and 81 in [2]). Thus, the dynamic state equation (2.1.9) enables us to take into account the dissipative processes completely. But the form of the dissipative terms describing the inner exchange processes (transport of heat and momentum) are different for the high and low frequencies.

We call attention to the fact that the dispersion relations $\omega = \omega(k)$ for the linearized equations (2.2.5) and (2.2.6) have been restricted by the finite power series in $k$ and in $k^{-1}$, respectively:

$$\omega = c_e k + \mathrm{i}\beta_e k^2 - \gamma_e k^3, \qquad \tau_p\omega \ll 1,$$

$$\omega^2 = c_f^2 k^2 (1 + \mathrm{i}\beta_f k^{-1} - \gamma_f k^{-2}), \qquad \tau_p\omega \gg 1.$$

At the time we were carrying out our research, it turned out that equation (2.2.6) had not been investigated much. This is likely connected with the fact, noted by Whitham in Ref. [40], that high-frequency perturbations attenuate very quickly. However in Whitham's monograph [40], the evolution equation (2.2.6) without nonlinear and dispersive terms was considered. Certainly, the lack of such terms restricts the class of solutions. At least, there is no solution in the form of a solitary wave which is caused by nonlinearity and dispersion.

## 2.3 Evolution equation for high-frequency perturbations

The equation (2.2.6), which we are interested in,

$$\frac{\partial^2 p}{\partial x^2} - c_f^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_f c_f^2 \frac{\partial^2 p^2}{\partial x^2} + \beta_f \frac{\partial p}{\partial x} + \gamma_f p = 0$$

is written down in a dimensionless form. Let us restrict our consideration to the propagation of high-frequency waves in positive direction $x$, then with necessary accuracy we can write the operator

$$\frac{\partial^2}{\partial x^2} - c_f^{-2}\frac{\partial^2}{\partial t^2} =$$

$$= \left(\frac{\partial}{\partial x} - c_f^{-1}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + c_f^{-1}\frac{\partial}{\partial t}\right) \to 2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + c_f^{-1}\frac{\partial}{\partial t}\right).$$

In the moving coordinates system with velocity $c_f$, the equation has the form in dimensionless variables

$$\tilde{x} = \sqrt{\frac{\gamma_f}{2}}(x - c_f t), \quad \tilde{t} = \sqrt{\frac{\gamma_f}{2}} c_f t, \quad \tilde{u} = \alpha_f c_f^2 p$$

(tilde over variables $\tilde{x}, \tilde{t}, \tilde{u}$ is omitted) [11, 12, 13, 41]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + \alpha\frac{\partial u}{\partial x} + u = 0. \tag{2.3.1}$$

The constant $\alpha = \beta_f/\sqrt{2\gamma_f}$ is always positive. The equation (2.3.1) without the dissipative term has the form of the nonlinear equation [41, 42]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0. \tag{2.3.2}$$

Historically, the equation (2.3.2) has been called the Vakhnenko equation (VE) and we shall use this name subsequently.

It is interesting to note that equation (2.3.2) follows as a particular limit of the following generalized Korteweg-de Vries equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \beta\frac{\partial^3 u}{\partial x^3}\right) = \gamma u \tag{2.3.3}$$

derived by Ostrovsky [43] to model small-amplitude long waves in a rotating fluid ($\gamma u$ is induced by the Coriolis force) of finite depth. Subsequently, equation (2.3.2) was known by different names in the literature, such as the Ostrovsky-Hunter equation, the short-wave equation, the reduced Ostrovsky equation and the Ostrovsky-Vakhnenko equation depending on the physical context in which it is studied.

The consideration here of equation (2.3.2) has an interest not only from the viewpoint of the investigation of the propagation of high-frequency perturbations, but more specifically from the viewpoint of the study of methods and approaches that may be applied in the theory of nonlinear evolution equations.

# Chapter 3
# The travelling-wave solutions

By investigating equation (2.3.2), we will trace a way in which an uninitiated reader could investigate a new nonlinear equation. As a first step for a new equation, it is necessary to consider the linear analogue and its dispersion relation (these steps for equations (2.2.5) and (2.2.6) are described already in Chapter 2). The next step is, where possible, to link the equation with a known nonlinear equation.

## 3.1 The connection of the VE with the Whitham equation

Now we show how an evolution equation with hydrodynamic nonlinearity can be rewritten in the form of the Whitham equation. The general form of the Whitham equation is as follows [40]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \int_{-\infty}^{\infty} K(x-s)\frac{\partial u}{\partial s}ds = 0. \tag{3.1.1}$$

On the one hand, this equation (3.1.1) has the nonlinearity of hydrodynamic type; on the other hand, it is known (see, Section 13.14 in [40]) that the kernel $K(x)$ can be selected to give the dispersion required. Indeed, the dispersion relation $c(k) = \omega(k)/k$ and the kernel $K(x)$ are connected by means of the Fourier transformation

$$c(k) = F[K(x)], \qquad K(x) = F^{-1}[c(k)]. \tag{3.1.2}$$

Consequently, for the dispersion relation $\omega = -1/k$ corresponding to the linearized version of (2.3.2), the kernel is as follows

$$K(x) = F^{-1}[-1/k^2] = \tfrac{1}{2}|x|. \tag{3.1.3}$$

Thus, the VE (2.3.2) is related to the particular Whitham equation [40]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{2}\int_{-\infty}^{\infty} |x-s|\frac{\partial u}{\partial s}ds = 0. \tag{3.1.4}$$

Since we can reduce the VE to the Whitham equation, we can assert that the VE shares interesting properties with the Whitham equation; in particular, it describes solitary wave-type formations, have periodic solutions and explains the existence of the limiting amplitude [40]. An important property is the presence of conservation laws for waves decreasing rapidly at infinity, namely

$$\frac{d}{dt}\int_{-\infty}^{\infty} u dx = 0, \quad \frac{d}{dt}\int_{-\infty}^{\infty} u^2 dx = 0, \quad \frac{d}{dt}\int_{-\infty}^{\infty} \left(\tfrac{1}{3}u^3 + \widehat{K}u\right) dx = 0, \tag{3.1.5}$$

where by definition $\widehat{K}u = \int_{-\infty}^{\infty} K(x-s)u(s,t)ds$.

For equation (2.3.1) the kernel is $K(x) = \frac{1}{2}[\alpha(2\Theta(x)-1) + |x|]$, where $\Theta(x)$ is the Heaviside function. Hence, (2.3.1) can be written down as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha u + \frac{1}{2}\int_{-\infty}^{\infty} |x-s|\frac{\partial u}{\partial s}ds = 0. \tag{3.1.6}$$

It is important that there is no derivative in the dissipative term $\alpha u$ of Eq. (3.1.6).

## 3.2 Loop-like stationary solutions of the VE

An important step in the investigation of nonlinear evolution equations is to find travelling-wave solutions. These are solutions that are stationary with respect to a moving frame of reference. In this case, the evolution equation (a partial differential equation) becomes an ordinary differential equation (ODE) which is considerably easier to solve.

For the VE (2.3.2) it is convenient to introduce a new dependent variable $z$ and new independent variables $\eta$ and $\tau$ defined by

$$z = (u - v)/|v|, \quad \eta = (x - vt - x_0)/|v|^{1/2}, \quad \tau = t|v|^{1/2}, \tag{3.2.1}$$

where $v$ and $x_0$ are arbitrary constants, and $v \neq 0$. Then the VE becomes

$$z_{\eta\tau} + (zz_\eta)_\eta + z + c = 0, \text{ where } c:= \frac{v}{|v|}. \tag{3.2.2}$$

$c = \pm 1$ corresponding to whether $v \gtrless 0$. We now seek stationary solutions of (3.2.2) for which $z$ is a function of $\eta$ only so that $z_\tau = 0$ and $z$ satisfies the ODE

$$(zz_\eta)_\eta + z + c = 0, \tag{3.2.3}$$

After one integration (3.2.3) gives

$$\begin{aligned} &\tfrac{1}{2}(zz_\eta)^2 = f(z), \\ &f(z) = -\tfrac{1}{3}z^3 - \tfrac{1}{2}cz^2 + \tfrac{1}{6}c^3A = -\tfrac{1}{3}(z - z_1)(z - z_2)(z - z_3), \end{aligned} \tag{3.2.4}$$

where $A$ is a constant. It is easy to verify that if there are complex roots, the value $z$ tends to minus infinity, and this contradicts the physical statement of the problem. Indeed, if we have only one real root, the graph of the function $f(z)$ (see fig. 3.1) crosses the $Oz$ axis once. Thus as $z \to +\infty$ we have $f \to -\infty$ and as $z \to -\infty$ we have $f \to +\infty$. But since the trinomial in (3.2.4) should always be positive in the integration region, as follows from the l.h.s. of (3.2.4), this region extends in $z$ from minus infinity to the value of the single real root. This means the perturbation amplitude $u = (z + c)v$ also tends to minus infinity, which does not correspond to the physical statement of the problem. So, all roots of the trinomial should be real. This requires that $0 \leq A \leq 1$. Note that there are turning points at $z = 0$ and $z = -c$. For periodic or solitary-wave solutions, $z_1$, $z_2$ and $z_3$ are real constants. For definiteness we shall assume that $z_1 \leq z_2 \leq z_3$. Three ways of calculating the roots are given in the Appendix in Section 3.4. From (A.7), we can deduce that for $v > 0$ always the root $z_3 \in [0, 0.5]$ as indicated by curve 2 in Fig. 3.1(a); curve 1 corresponds to $A = 1$ and curve 3 corresponds to $A = 0$. Similarly, for $v < 0$ always $z_3 \in [1, 1.5]$ as indicated by curve 2 in

Fig. 3.1(b); curve 1 corresponds to $A = 0$ and curve 3 corresponds to $A = 1$. It also follows from (A.7) that always $z_1 < 0$, but $z_2 < 0$ for $v > 0$ and $z_2 > 0$ for $v < 0$. Thus the nature of the solutions depends on the sign of $v$. However, the integration of (3.2.4) is not affected by the sign of $v$.

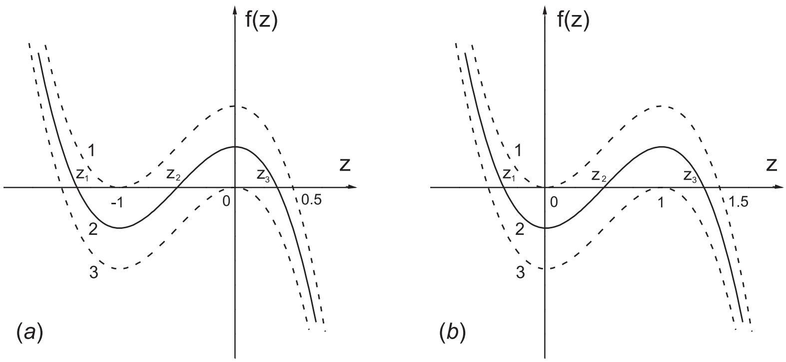


Figure 3.1: The graph of the trinomial $f(z)$: (a) $v > 0$, (b) $v < 0$. The integration region is the interval $(z_2, z_3)$.

The integration region of (3.2.4) is the interval $(z_2, z_3)$ where $f(z) > 0$ (see fig. 3.1). At the points $z = z_2$ and $z = z_3$ the derivatives $z_\eta$ are zero. Hence, we have the relation

$$\pm\sqrt{\tfrac{2}{3}}\eta = \int_z^{z_3} \frac{zdz}{\sqrt{(z - z_1)(z - z_2)(z_3 - z)}} \,. \tag{3.2.5}$$

On using results 236.00 and 236.01 of [44], we may integrate (3.2.5) to obtain

$$\eta = \frac{1}{p}[z_1 F(\varphi, k) + (z_3 - z_1)E(\varphi, k)], \tag{3.2.6}$$

where

$$\sin^2\varphi = \frac{z_3 - z}{z_3 - z_2}, \quad k^2 = \frac{z_3 - z_2}{z_3 - z_1}, \quad p^2 = \frac{(z_3 - z_1)}{6}. \tag{3.2.7}$$

In the notation of [44], $F(\varphi, k)$ and $E(\varphi, k)$ are incomplete elliptic integrals of the first and second kind respectively. We have chosen the constant of integration in (3.2.6) to be zero so that $z = z_3$ at $\eta = 0$. The relations (3.2.6) and (3.2.7) give the required solution in parametric form, with $z$ and $\eta$ as functions of the parameter $\varphi$.

An alternative route to the solution is to follow the procedure described in [45]. We introduce a new independent variable $\zeta$ defined by

$$\frac{d\eta}{d\zeta} = z \tag{3.2.8}$$

so that (3.2.4) becomes

$$\tfrac{1}{2} z_\zeta^2 = f(z). \tag{3.2.9}$$

By means of result 236.00 of [44], (3.2.9) may be integrated to give $w = F\ (\varphi|m)$, where $m\colon = k^2$ and $w = p\zeta$. Here we have used the notation of Chapter 17 in [110]. Thus, on noting that $\sin\varphi = \mathrm{sn}(w|m)$, where sn is a Jacobian elliptic function, we have

$$z = z_3 - (z_3 - z_2)\,\mathrm{sn}^2(w|m). \tag{3.2.10}$$

With result 310.02 of [44], (3.2.8) and (3.2.10) give

$$\eta = \frac{1}{p}[z_1 w + (z_3 - z_1)E(w|m)], \tag{3.2.11}$$

where $E(w|m)$ is the incomplete elliptic integral of the second kind (in the notaion of [110]). Relations (3.2.10) and (3.2.11) are equivalent to (3.2.7) and (3.2.6) respectively and give the solution in parametric form with $z$ and $\eta$ in terms of the parameter $w$.

We define the wavelength $\lambda$ of the solution as the amount by which $\eta$ increases when $\varphi$ increases by $\pi$, or equivalently when $w$ increases by $2K(m)$, where $K(m)$ is the complete elliptic integral of the first kind. It follows from (3.2.11) that

$$\lambda = \frac{2}{p}|z_1 K(m) + (z_3 - z_1)E(m)|, \tag{3.2.12}$$

where $E(m)$ is the complete elliptic integral of the second kind.

As mentioned previously, the VE has two families of solutions corresponding to $v > 0$ and $v < 0$, respectively. We now describe these two cases in detail.