Solutions to the N-Body Problem

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PREFACE

The n-body problem of 6n-12 degrees of freedom with the twelve inherent constraints creates a difficult situation in working the equations of motion for three or more masses. This befalls the mathematical physicist to remedy the situation by determining proper constraints to get past this dilemma by requiring a specialized set of conditions to define a unique problem. Procedurally, this results in working n-body problems in a case by case environment. On the assumption that deterministic solutions are of interest, then the population of problems solved under these conditions appear to be limited. The question arises, if a general approach to solving certain classes of n-body problems exists and how they could be formulated to meet the deterministic constraint. Existing in this work is a large body of n-body problems that can be formulated and solved using methods whereby the given initial conditions defining these structured mass configurations allows deterministic solutions. In the following chapters specialized n-body configurations will be structured systematically and analyzed revealing characteristics of their behavior.

The n-body problem is usually approached in a deductive manner as implied above, that is, searching for a general solution of the equations of motion over a finite or infinite interval. This method can be quite daunting due to the lack of constraints. However, approaching the n-body problem in an inductive manner, such as the inverse problem of dynamics, can reduce the degrees of freedom as a result of the forces being determined by the given properties of their motion. Using the infinitesimal interval method for tiered n-body systems with type one geometry and the sidereal synodic relations at time zero relative to the barycenter in the configuration plane of motion is a compromise that exploits both deductive and inductive approaches. Combined deductive and inductive processes simplify the formulation complexity of the tiered n-body problem resulting in a more intuitive understanding of particle interactions in terms of configuration subsystems. Binary, trinary, quadruple etc. subsystem finite stability within the n-body configuration is one such example. Subsystem perturbations, particle velocities, state vectors, period ratios and sphere of influence are other examples. The first two chapters address the tiered structure of the n-body problem using the methodology just described. These configuration solutions are a specialized subset of all possible solutions contained in Newton's equations of motion.

Concentric regular n-gons are covered in the third chapter using methods for n-body deterministic solutions similar to that of the collinear n-body infinitesimal interval problem. Systematically, starting with solving the single polygon structure and then proceeding by evolving to more complex systems results in analysis that allows solving the multiple and infinite concentric polygon problem. Central to the concentric n-gon formulation are determinations of structured particle distributions necessary to find unique solutions of regular multi-polygon configurations. Required distributions necessary to solve this problem involve determining intra/inter perturbations, multi-polygon separation distance and mass scaling. These distributions are not unique therefore requiring further analysis using numerical optimization to find the minimum (extrema) potential energy solution. Depending on how multi-polygons are structured can force the intra/inter perturbations to have singularities. Resolving this situation leads to greater structural complexity requiring non-regular as well as regular n-gons to make the concentric polygon structure singularity free.

The Orthogonal collinear configuration presented in chapter four is a variation of the collinear infinitesimal interval problem worked in chapters one and two. This class of structured configurations use collinear masses along the inertial x and y axes possessing symmetry where the geometric center and the barycenter coincide. Masses on the x-axis as well as on the y-axis are balanced relative to the barycenter in a restricted problem to maintain this symmetry. Several of these configurations are solved beginning with the most fundamental restricted double binary, which is composed of one mass on either side of the x axis relative to the barycenter and one mass on either side of the y-axis relative to the barycenter and then proceeding to a quadruple binary with two binaries on each axis symmetric to the barycenter. Stability analysis for the double binary configuration has been worked in the sense of Lagrange using the first order approximation from the eigen value problem to compare to numerically integrated state vectors trajectories with the purpose to show that although a trajectory is not stable for all time, it can easily be stable for a finite time.

Placing collinear configurations at the L4 and L5 equilibrium locations with the purpose of verifying that finite stable orbital systems can exist there is studied in the appendix. Two such configurations are investigated. Approximate state vectors have been derived for binary and double binary configurations orbiting the equilibrium locations at the Lagrange L4 and L5 points assuming that a pseudo mass resides there. Numerically integrating these state vectors has verified that they can be finite stable. It is suspected this analysis could be extended to show that large numbered collinear mass systems can be finite stable at the L4 and L5 locations for extended periods of time.

In presenting a specialized work on the collinear infinitesimal interval and other related n-body problems, it has been assumed that a back ground in the basic mathematical and physical concepts forming the foundations of celestial mechanics are already known. There exists a plethora of work on celestial and classical dynamics, non-linear differential equations, figures of equilibrium and stability of motion published over the last several centuries that substantially embody this field. To this end a collection of these publications have been listed pertaining to celestial mechanics, classical mechanics, stability theory and linear/nonlinear differential equations. These references can be found posted in the reference section at the end of this book.

Chapter One

Collinear N-Body Problem

Within its elegant mathematical formulation, the n-body equations of motion comprise every possible classical trajectory per any given set of masses (dense set). Infinite problems of great complexity reside within these coupled sets of non-linear differential equations. It would be of interest to formulate complex systems by structuring the n-body configurations in such a manner where they could be solved under the proper constraints resulting in systematic deterministic solutions consistent with the given conservation laws of physics. Therefore, it would be within capability to reduce the 'every possible' particle trajectories to just the possible structured subset that is properly constrained to get a deterministic solution. Although these configurations are a subset of the 'every possible' n-body solutions and do not encompass the whole classical 'real universe' of solutions, they are still viable and provide the mathematical physicist with insight into the nature of particle motion.

No general agreed on, or accepted finite/infinite time interval deterministic solution to the n-body equations of motion exists, including any particular general solution of three, four or more bodies. Trying to solve the n-body problem starting directly from the equations of motion for a particular n-body configuration can be a difficult way to proceed, whereas, determining the forces from the given properties of motion as in the inverse problem of dynamics (Besant, 1914, pp 140-144, Ramsey, 1929, pp 253-254, Galiullin, 1984, pp 25-28, Santilli, 1978, pp 219-223) may be more instructive. Constraints on particle position reduce 'every possible position' to just one possible collinear particle distribution, thereby reducing the degrees of freedom. Constraints on particle velocity and acceleration will reduce 'every possible velocity and acceleration' to just one possible velocity and acceleration distribution thus further reducing the degrees of freedom. Finally, using sidereal synodic relations, type one geometry and constraining time to the infinitesimal interval will reduce the degrees of freedom to zero. Properly constraining the structured n-body position, velocity and acceleration at time zero, results in a solution domain only existing at one point, that is, a state vector solution domain.

Formulating the n-body problem into a tiered structured collinear system of particles constrained by type one geometry, sidereal synodic relations that are rotating within an infinitesimal interval at time zero, will allow deterministic state vector solutions. Infinite families comprise the tiered n-body configurations where every family is populated by infinite members and those infinite members have infinite variations. Deterministic solutions can be found for every one of these collinear configurations using Jacoby coordinates within the infinitesimal interval relative to the barycenter. Particles at time zero are in instantaneous circular orbits verified by showing the collinear solution is consistent with conservation of energy where twice the kinetic energy is equal to the potential energy. Numerical integration of the instantaneous solution state vectors will result in trajectories that are the most circular or least elliptical of all orbits for that particular configuration. This orbital characteristic is due to the type one geometry constraint where all particle velocity vectors are perpendicular to all particle position vectors within the infinitesimal interval relative to the barycenter at time zero in the given plane of motion.

1.1 Collinear N-Body Structure

The structured collinear n-body configurations are defined to exist over an infinitesimal interval at time zero (t = 0). Newtonian equations of motion are infinitesimally rotated in the complex plane within the infinitesimal interval relative to the barycenter and constrained by type one geometry and the sidereal synodic relations. The Newtonian system is defined as a dynamical, discrete, classical non-relativistic Euclidean space and consistent with the laws of conservation (all masses are non-zero).

Collinear n-body configurations are conservative dynamical systems centered at the barycenter (0,0,0) fixed in three-dimensional Euclidean space with an inertial right-handed coordinate system x, y, z. The n-body equations of motion are formulated in the complex plane over the infinitesimal interval (Lass, 1957, pp 314-319, Pollard, 1966, p-49) as specified below

$$\ddot{z}_k + 2wi\dot{z}_k - w^2 z_k = G \sum_{\substack{j=1 \\ j\neq k}}^{N} \frac{m_j}{r_{jk}^3} z_{jk} \quad k = 1, 2, 3, \ldots, N \qquad \text{Equation 1.1.1}$$

where $i = \sqrt{-1}$, $z_k = x'_k + iy'_k$ and $r_{jk} = |z_j - z_k|$

The sidereal synodic relations (Kurth, 1959, pp 1-8, Bauer, 2001, pp 1-2) that specify the general geometric conditions under which the collinear n-body configurations rotate

$$n_N P_N = (n_N + n_{N-1})P_{N-1} = \cdots = (n_N + n_{N-1} + \cdots + n_1)P_1$$

Equation 1.1.2

where P_i is the period of the given motion at time zero and n_i the corresponding coefficients.

Collinear configurations in the n-body complex formulation are transformed into Jacoby coordinates reducing the number of equations per configuration to n – 1 coupled equations of motion. By systematically coupling these n – 1 Jacoby configurations, tiered structures can be created and solved by using equations 1.1.1 and 1.1.2 with type one geometry. Any tiered collinear configuration can be illustrated in terms of its basic component subsystems as defined by their Jacoby coordinates. For example, the three-body collinear family is composed of one member (symbolically written as $\underline{12}3$) with five additional variations. These six total configurations represent an ordered set where mass three is systematically placed in all possible gravitational subsystems relative to mass one and mass two. The six total configurations can be written out in the following manner

$3\underline{12}$ $\underline{31}2$ $\underline{13}2$ $1\underline{32}$ $1\underline{23}$ $\underline{12}3$ Equation 1.1.3

where underscoring represents Jacoby coordinate coupling. A mirror image solution exists for each of the configurations however, they are considered redundant. General form of a collinear three-body solution can be written as a quintic polynomial which results in the above six configurations expressed as six quintic equations. The first two quintic equations (configurations) in equation 1.1.3 intersect, with that intersection resulting in an additional quintic equation. This is also true of the third and fourth configurations and of the fifth and sixth configurations. There now exists three intersection quintics resulting from the paired six configurations, and it can be shown that the three intersection quintic equations are the same quintic equations derived by Euler (Roy, 1988, p 119, Pollard, 1966, p 51, Wintner, 1964, p 430, Szebehely, 1967, p 297) for his solution of the collinear three-body problem.

Adding another mass, the collinear four-body problem can be described in a similar manner as the collinear three-body problem where the four-body family is composed of two members written as $\underline{12}34$ and $\underline{12}\ \underline{34}$. The second configuration is a double binary system with masses one and two representing the first binary subsystem and masses three and four representing the second binary subsystem with each subsystem in orbit about the other at time zero. Two collinear four body family members plus the member variations consist of approximately fifty unique configurations. Complexities arising from four body ordering account for the large increase in configuration number. Two four body ordered subset systems analogous to the three-body ordered system can be written as follows

$$3(\underline{12}4)\quad (3\underline{12})4\quad (\underline{31}2)4\quad (\underline{13}2)4\quad (1\underline{32})4\quad (1\underline{23})4\quad (\underline{12}3)4\quad \underline{1234}\quad \underline{1243}\quad (\underline{12}4)3$$

$$3(\underline{21}4)\quad (3\underline{21})4\quad (\underline{32}1)4\quad (\underline{23}1)4\quad (2\underline{31})4\quad (2\underline{13})4\quad (\underline{21}3)4\quad \underline{2134}\quad \underline{2143}\quad (\underline{21}4)3$$

Equation 1.1.4

where ordering is defined by systematically arranging particle configurations in such a manner that the resultant systems are unique configurations.

Interchanging mass one with mass two differentiates the second mass ordered system from the first. The first ordered set is by convention the reference set. Three-body ordered configurations in equation 1.1.3 can be seen sequentially embedded in the second through seventh four-body configurations of the first four-body ordered set. Parenthesis are used to delineate the three-body subsets. By systematically interchanging the masses one and three, two and three, one and four, two and four and three and four in the reference set, the remaining combinations can be generated. Approximately thirty unique combinations out of these remaining configurations will be left after removing redundant mirror and duplicate configurations. It will be noted that all mass interchanging has been done relative to the first ordered reference set.

This collinear n-body process can be continued by the method described ad infinitum. With each additional mass added there results a significant increase in configurations due to complexity in generating all possible unique combinations. Every collinear n-body configuration is solvable producing a state vector at time zero which can be numerically integrated to determine trajectory behavior for stability analysis (Lehnigh, 1966, pp 25-71, Merkin, 1996, pp 103-111). The n-body collinear configuration hierarchy sequence can be represented as a mathematical logical system.

Found below is a listing of the first six-tiered n-body family configurations not including the family variations.

1 Equation 1.1.5

$\underline{12}$

$\underline{12}$ 3

$\underline{12}$ 34 $\underline{12}$ $\underline{34}$

$\underline{12}$ 345 $\underline{12}$ $\underline{34}$ 5 $\underline{12}$ 3 $\underline{45}$

$\underline{12}$ 3456 $\underline{12}$ $\underline{34}$ 56 $\underline{12}$ 3 $\underline{45}$ 6 $\underline{12}$ 34 $\underline{56}$ $\underline{12}$ $\underline{34}$ $\underline{56}$ $\underline{12}$ 3, $\underline{45}$ 6

$\underline{12}$ 34567…

This mathematical logical system can be generated by either sequencing mass from the left or right of the unary mass. The above listing is generated from mass addition to the right and is a mirror image of a left mass generated mathematical logical system. With each additional mass the listing becomes more intricate as the mass number approaches infinity. For example, the last entry for the six-body configuration is a double trinary system represented as two three body configurations separated by a comma. For clarification, as masses are added, trinary subsystems may require a set of parentheses about each trinary to distinguish them systematically from every other subsystem when generating higher order configurations. In general, a system of underscoring, parentheses commas etc. will be needed to illustrate the n-body tiered configurations accurately. There are many possible ways in which this structuring can be symbolically formatted. However, whichever systematic representation employed, the proper perspective is needed to maintain the correct collinear rotating subsystems that compose the n-body configurations.

Each column in the mathematical logical system of equation 1.1.5 is a family of configurations that can be solved as one set of deterministic solutions using equations 1.1.1 and 1.1.2. Analysis of these fundamental configuration structures requires formulating the collinear n-body problem in terms of Jacoby coordinates. Jacoby coordinates can be classified into three distinct categories. They are, mass to mass distance, mass to sub-configuration center of gravity distance and sub-configuration center of gravity to sub-configuration center of gravity distance. Using the infinitesimally rotating Jacoby coordinates with type one geometry, sidereal synodic relations in the infinitesimal interval at time zero results in a deterministic solution set. This solution set includes inertial velocities, state vectors, period ratios, perturbation coefficients and sphere of influence. The solution for the first column of the mathematical logical system is presented in the next subsection.

1.2 First Infinite Family Configurations

Although theoretically, the n-body mathematical logical system symbolically represented by equation 1.1.5 could be solved as one formulation of infinite configurations, only two subsets will be considered in this chapter due to the complexity of the general problem. The first infinite family is column one, as listed in equation 1.1.5 and defined in equation 1.1.1, have sidereal synodic relations under which the individual masses within the n-body configurations rotate consistent with equation 1.1.2. First family Jacoby coordinates in the infinitesimal interval at time zero are $\vec{r}_{21}, \vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3, \dots, \vec{\rho}_{N-2}$ where $\vec{r}_{21}$ represents the Jacoby coordinate for the mass one mass two binary subsystem, $\vec{\rho}_1$ represents the Jacoby coordinate from the center of gravity of the binary subsystem to mass three, $\vec{\rho}_2$ represents the Jacoby coordinate from the center of gravity of the trinary subsystem to mass four etc.

First infinite family collinear equations of motion in the complex plane using equation 1.1.1 describing infinitesimal rotations in the infinite interval at time zero, are as follows

$$\ddot{z}_{21} + 2wi\acute{z}_{21} - w^2 z_{21} = -\frac{G\mu_{r1}}{r_{21}^3} z_{21} + \frac{Gm_3}{r_{32}^3} z_{32} - \frac{Gm_3}{r_{31}^3} z_{31} + \cdots + \frac{Gm_N}{r_{N2}^3} z_{N2} - \frac{Gm_N}{r_{N1}^3} z_{N1}$$

$$\ddot{z}_3 + 2wi\acute{z}_3 - w^2 z_3 = \frac{Gm_1}{r_{13}^3} z_{13} + \frac{Gm_2}{r_{23}^3} z_{23} + \frac{Gm_4}{r_{43}^3} z_{43} + \cdots + \frac{Gm_N}{r_{N3}^3} z_{N3}$$

$$\ddot{z}_4 + 2wi\acute{z}_4 - w^2 z_4 = \frac{Gm_1}{r_{14}^3} z_{14} + \frac{Gm_2}{r_{24}^3} z_{24} + \frac{Gm_3}{r_{34}^3} z_{34} + \cdots + \frac{Gm_N}{r_{N4}^3} z_{N4}$$

$$-w^2 z_N = \frac{Gm_1}{r_{1N}^3} z_{1N} + \frac{Gm_2}{r_{2N}^3} z_{2N} + \frac{Gm_3}{r_{3N}^3} z_{3N} + \cdots + \frac{Gm_N}{r_{(N-1),N}^3} z_{(N-1),N}$$

First equation is a result of coupling $\ddot{z}_1$ and $\ddot{z}_2$ where $\mu_{r1} = m_1 + m_2$, and N is mass number. The last relation is the equation of motion from the barycenter to mass N. This vector is along the x' axis rotating relative to the x axis at inertial angular velocity w in the infinitesimal interval (with $\ddot{z}_N = \acute{z}_N = 0$), the x' and x axis are coincident at time zero. Modifying sidereal syndic relations equation 1.1.2 for the first family gives

$$n_N P_N = (n_N + n_{N-1})P_{N-1} = \cdots = (n_N + n_{N-1} + \cdots + n_{21})P_{21}$$

Starting from the three-body problem through to higher tiered first family collinear n-body configurations, solutions can be found by use of vector and scalar geometry. Using the sidereal synodic relations in conjunction with the vector and scalar geometry allows determination of the infinitesimal rotation of the Jacoby coordinates in the infinitesimal interval relative to the x' axis. For example, rotating Jacoby coordinate z_{21} can be formulated

$$z_{21} = r_{21} e^{i\frac{n_{N-1}+\cdots+n_3+n_{21}}{n_N}wt}$$

Substituting the second derivative of the above equation into the $\ddot{z}_{21}$ equation and using the vector configuration geometry will give after some analysis

$$\frac{G\mu_{r1}}{w^2 r_{21}^3} - \frac{P_N^2}{P_{21}^2} = \frac{G}{w^2 r_{21}^3}\left(\left(\frac{r_{21}^2}{r_{32}^2} - \frac{r_{21}^2}{r_{31}^2}\right)m_3 + \left(\frac{r_{21}^2}{r_{42}^2} - \frac{r_{21}^2}{r_{41}^2}\right)m_4 + \ldots + \left(\frac{r_{21}^2}{r_{N2}^2} - \frac{r_{21}^2}{r_{N1}^2}\right)m_N\right) = \frac{G}{w^2 r_{21}^3}\phi_1 \qquad \text{Equation 1.2.1}$$

where $\phi_1 = \left(\frac{r_{21}^2}{r_{32}^2} - \frac{r_{21}^2}{r_{31}^2}\right)m_3 + \left(\frac{r_{21}^2}{r_{42}^2} - \frac{r_{21}^2}{r_{41}^2}\right)m_4 + \cdots + \left(\frac{r_{21}^2}{r_{N2}^2} - \frac{r_{21}^2}{r_{N1}^2}\right)m_N$

Continuing this derivation with the acceleration vectors $\ddot{z}_3$, $\ddot{z}_4$,… z_N will yield the complete set of $\phi_i's$ needed to find the Jacoby velocities for the first family state vectors

$$\phi_2 = \frac{r_{21}^2}{r_{31}^2}m_1 + \frac{r_{21}^2}{r_{32}^2}m_2 - \frac{r_{21}^2}{r_{34}^2}m_4 - \cdots - \frac{r_{21}^2}{r_{3N}^2}m_N$$

$$\phi_3 = \frac{r_{21}^2}{r_{41}^2}m_1 + \frac{r_{21}^2}{r_{42}^2}m_2 + \frac{r_{21}^2}{r_{43}^2}m_3 - \cdots - \frac{r_{21}^2}{r_{4N}^2}m_N$$

$$\phi_{N-1} = \frac{r_{21}^2}{r_{N1}^2}m_1 + \frac{r_{21}^2}{r_{N2}^2}m_2 + \frac{r_{21}^2}{r_{N3}^2}m_3 + \cdots + \frac{r_{21}^2}{r_{N,(N-1)}^2}m_{N-1}$$

where ϕ_1, ϕ_2, ϕ_3,…, ϕ_{N-1} are the perturbation coefficients associated with the respective $\vec{r}_{21}, \vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_{n-2}$ Jacoby coordinates.

Period ratios are another aspect of the first family collinear n-body solution and result as a function of using the sidereal synodic relations in solving the infinitesimal interval problem, with the number of ratio combinations determined by $\frac{(N-1)!}{2(N-3)!}$ for mass number N greater than or equal to three. The period ratio structure for the first family configurations are

$$\frac{P_3}{P_{21}} \quad \frac{P_4}{P_{21}} \quad \frac{P_5}{P_{21}} \cdots\cdots\cdots \frac{P_N}{P_{21}}$$ Equation 1.2.2

$$\frac{P_4}{P_3} \quad \frac{P_5}{P_3} \cdots\cdots\cdots \frac{P_N}{P_3}$$

$$\frac{P_5}{P_4} \cdots\cdots\cdots \frac{P_N}{P_4}$$

$$\cdot \quad \cdot \quad \cdot$$

$$\frac{P_N}{P_{N-1}}$$

Collinear n-body period ratio solutions as structured in equation 1.2.2 and determined from the equations of motion as found described in equation1.2.1 are

$$\frac{P_A^2}{P_{21}^2} = \frac{\mu - \phi_1}{\phi_{A-1} + \frac{1}{\sum_{i=1}^{A} m_i} \sum_{i=1}^{n-A} m_{A+i} \phi_{A+i-1}} \frac{\sum_{i=1}^{A-1} m_i}{\sum_{i=1}^{A} m_i} l_{A-2}$$ Equation 1.2.3

where $A \geq 3 \qquad A = 3, 4, 5, \ldots, N$

Additional period ratios are calculated from $\frac{P_A^2}{P_{21}^2}$ by systematically inverting and multiplying the ratios of each configuration set. For example, the five body first family configuration period ratios determined from equation 1.2.3 are $(\frac{P_3}{P_{21}}, \frac{P_4}{P_{21}}, \frac{P_5}{P_{21}})$ and after inverting and multiplying results $(\frac{P_4}{P_3}, \frac{P_5}{P_3}, \frac{P_5}{P_4})$ consistent with $\frac{(N-1)!}{2(N-3)!}$.

Perturbation coefficients ϕ_i in equation 1.2.3 as determined from first family collinear n-body solutions found in equation 1.2.1 are used to construct the instantaneous state vectors. Summary of first family perturbation coefficients from equation 1.2.1 in matrix format are listed below

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \cdot\\ \phi_\alpha\end{pmatrix} = r_{21}^2 \begin{pmatrix} ..0.. & ..0.. & e13 & e14 & e15 & ..... & e1n\\ e21 & e22 & ..0.. & e24 & e25 & ..... & e2n\\ e31 & e32 & e33 & ..0.. & e35 & ..... & e3n\\ e41 & e42 & e43 & e44 & ..0.. & ..... & e4n\\ ..... & ..... & ..... & ..... & ..... & ..... & .....\\ e\alpha1 & e\alpha2 & e\alpha3 & e\alpha4 & e\alpha5 & ..... & ..0.. \end{pmatrix} \begin{pmatrix} m_1\\ m_2\\ m_3\\ m_4\\ m_5\\ \cdot\\ m_n\end{pmatrix}$$

Equation 1.2.4

where e21 = $\frac{1}{r_{31}^2}$ e22 = $\frac{1}{r_{32}^2}$

e31 = $\frac{1}{r_{41}^2}$ e32 = $\frac{1}{r_{42}^2}$ e33 = $\frac{1}{r_{43}^2}$

e41 = $\frac{1}{r_{51}^2}$ e42 = $\frac{1}{r_{52}^2}$ e43 = $\frac{1}{r_{53}^2}$ e44 = $\frac{1}{r_{54}^2}$

eα1 = $\frac{1}{r_{n1}^2}$ eα2 = $\frac{1}{r_{n2}^2}$ eα3 = $\frac{1}{r_{n3}^2}$ eα4 = $\frac{1}{r_{n4}^2}$ eα5 = $\frac{1}{r_{n5}^2}$

e24 = -e33 e25 = -e43 e2n = -eα3

e35 = -e44 e3n = -eα4

e4n = -eα5

e13 = e22 – e21 e14 = e32 – e31 e15 = e42 – e41 e1n = eα2 – eα1

$\alpha = n - 1$

First row double zeros in the equation 1.2.4 perturbation matrix denote a physical Jacoby binary coupling between mass one and mass two. In general, double zeros in the perturbation matrix denote mass coupling in hierarchical families.

Scalar position differences r_{jk} from equation 1.2.4 determined from the n-body configuration geometry can be expressed

$$\frac{r_{21}}{r_{31}} = \frac{1}{x_1+1}, \ldots\ldots\ldots\ldots, \frac{r_{21}}{r_{n1}} = \frac{1}{x_{n-2}+1}$$ Equation 1.2.5

$$\frac{r_{21}}{r_{32}} = \frac{1}{x_1}, \ldots\ldots\ldots\ldots\ldots, \frac{r_{21}}{r_{n2}} = \frac{1}{x_{n-2}}$$

$$\frac{r_{21}}{r_{43}} = \frac{1}{x_2 - x_1}, \ldots\ldots\ldots\ldots, \frac{r_{21}}{r_{n3}} = \frac{1}{x_{n-2} - x_1}$$

$$\frac{r_{21}}{r_{54}} = \frac{1}{x_3 - x_1}, \ldots\ldots\ldots\ldots, \frac{r_{21}}{r_{n4}} = \frac{1}{x_{n-2} - x_2}$$

$$\frac{r_{21}}{r_{n,n-1}} = \frac{1}{x_{n-2} - x_{n-3}}$$

Distance between masses $m_2 m_3$, $m_2 m_4$, $m_2 m_5$,…,$m_2 m_n$ is defined to be x_1, x_2, x_3,…, x_{n-2} respectively.

Scaling parameters $l_1, \ldots, l_{n-2}$ in equation 1.2.3 determined from the scalar configuration geometry are defined as follows

$$\rho_1 = l_1 r_{21}, \rho_2 = l_2 r_{21}, \ldots, \rho_{n-2} = l_{n-2} r_{21} \qquad \text{Equation 1.2.6}$$

where $$l_1 = x_1 + \frac{m_1}{\mu_1}$$

$$l_2 = x_2 - \frac{m_3}{\mu_2} x_1 + \frac{m_1}{\mu_2}$$

$$l_3 = x_3 - \frac{m_4}{\mu_3} x_2 - \frac{m_3}{\mu_3} x_1 + \frac{m_1}{\mu_3}$$

.

$$l_{n-2} = x_{n-2} + \frac{1}{\sum_{i=1}^{n-1} m_i} \left(m_1 - \sum_{i=1}^{n-3} m_{i+2}\, x_i\right)$$

$$\mu_i = \sum_{i=1}^{n-1} m_i \qquad n \geq 3$$

The three-dimensional first family collinear state vector solution defined in the infinitesimal interval at time zero is presented below and is a line of nodes solution defined in the barycentric inertial coordinate system. Required inputs are configuration masses, perturbation coefficients, inertial velocities, Jacoby coordinates and inclination angles.

$$x_1 = -\frac{m_2}{\mu_1} r_{21} - \frac{m_3}{\mu_2} \rho_1 - X \qquad \dot{x}_1 = 0 \qquad 1.2.7$$

$$y_1 = 0 \qquad \dot{y}_1 = -\frac{m_2}{\mu_1} V_{r_{21}} cosi_{r_{21}} - \frac{m_3}{\mu_2} V_{\rho_1} cosi_{\rho_1} - \dot{Y}$$

$z_1 = 0$ $\dot{z}_1 = -\frac{m_2}{\mu_1} V_{r_{21}} sini_{r_{21}} - \frac{m_3}{\mu_2} V_{\rho_1} sini_{\rho_1} - \dot{Z}$

$x_2 = \frac{m_1}{\mu_1} r_{21} - \frac{m_3}{\mu_2} \rho_1 - X$ $\dot{x}_2 = 0$

$y_2 = 0$ $\dot{y}_2 = \frac{m_1}{\mu_1} V_{r_{21}} cosi_{r_{21}} - \frac{m_3}{\mu_2} V_{\rho_1} cosi_{\rho_1} - \dot{Y}$

$z_2 = 0$ $\dot{z}_2 = \frac{m_1}{\mu_1} V_{r_{21}} sini_{r_{21}} - \frac{m_3}{\mu_2} V_{\rho_1} sini_{\rho_1} - \dot{Z}$

$x_3 = \frac{\mu_1}{\mu_2} \rho_1 - X$ $\dot{x}_3 = 0$

$y_3 = 0$ $\dot{y}_3 = \frac{\mu_1}{\mu_2} V_{\rho_1} cosi_{\rho_1} - \dot{Y}$

$z_3 = 0$ $\dot{z}_3 = \frac{\mu_1}{\mu_2} V_{\rho_1} sini_{\rho_1} - \dot{Z}$

$x_{n-\lambda} = Q\rho_{n-\lambda-2} - R\rho_{n-i+1}$ $\dot{x}_{n-\lambda} = 0$

$y_{n-\lambda} = 0$ $\dot{y}_{n-\lambda} = QV_{\rho_{n-\lambda-2}} cosi_{n-\lambda-2} - RV_{n-i-1} cosi_{n-i+1}$

$z_{n-\lambda} = 0$ $\dot{z}_{n-\lambda} = QV_{\rho_{n-\lambda-2}} sini_{n-\lambda-2} - RV_{n-i-1} sini_{n-i+1}$

where $X = \sum_{i=1}^{n-3} \frac{m_{i+3}}{\sum_{j=1}^{i+3} m_j} \rho_{i+1}$ $\dot{Y} = \sum_{i=1}^{n-3} \frac{m_{i+3}}{\sum_{j=1}^{i+3} m_j} V_{\rho_{i+1}} cosi_{\rho_{i+1}}$

$\dot{Z} = \sum_{i=1}^{n-3} \frac{m_{i+3}}{\sum_{j=1}^{i+3} m_j} V_{\rho_{i+1}} sini_{\rho_{i+1}}$

$Q = \frac{\sum_{i=1}^{n-\lambda-1} m_i}{\sum_{i=1}^{n-\lambda} m_i}$ $R = \sum_{i=1}^{\lambda} \frac{m_{n-i+1}}{\sum_{j=1}^{n-i+1} m_j}$

and $\lambda = 0, 1, 2, \ldots, n-4$ $n \geq 4$

Collinear state vector instantaneous inclination angles $i_{r_{21}}, i_{\rho_1}, \ldots, i_{\rho_{n-2}}$ in equation 1.2.7 are relative to the velocities $V_{r_{21}}, V_{\rho_1}, \ldots, V_{\rho_{n-2}}$ in Jacoby r and

ρ coordinates. First family state vector velocities are calculated from the following equations

$$V_{r_{21}}^2 = \frac{G}{r_{21}}(\mu_{r_{21}} - \phi_1) \quad V_{\rho_j}^2 = \frac{Gl_j^2}{\rho_j \sum_{i=1}^{j+1} m_i}\left(\phi_{j+1} \sum_{i=1}^{j+2} m_i + \sum_{k=j+3}^{n} m_k\, \phi_{k-1}\right)$$

for $\quad j = 1, 2, 3, \ldots, n-2$ Equation 1.2.8

The first family collinear state vector equations 1.2.7 are formulated relative to the three-body configuration. Any configurations greater than three-bodies will need to use the $n - \boldsymbol{\lambda}$ position and velocity coordinate equations. These state vector equations are only valid for the first family and not for the first family variations. Due to constraint conditions placed on this problem, the resultant state vectors when numerically integrated produce trajectories that are the most circular/least elliptical for that mass, position and velocity combination.

Kinetic and potential energy equations formulated for the first family are as follows

$$2T = \frac{m_1 m_2}{\mu} V_{r_{21}}^2 + \sum_{i=1}^{n-2} \frac{m_{i+2} \sum_{j=1}^{i+1} m_j}{\sum_{j=1}^{i+2} m_j} V_{\rho_i}^2 \qquad \text{Equation 1.2.9}$$

$$U = G \sum_{1 \le j < k \le n}^{n} \frac{m_j m_k}{r_{jk}}$$

All collinear configurations are consistent with conservation of energy where twice the kinetic energy is equal to the potential energy within the infinitesimal interval at time zero. This is realized for the first infinite family configurations when the velocity equations 1.2.8 are substituted into the energy equations 1.2.9 to show consistency with the conservation laws. Conservation of energy for the first family can also be verified by use of the Lagrange-Jacoby identity $\ddot{I} = 2T - U$. Where I is the moment of inertia and defined to be

$$I = \frac{1}{2} \sum_k m_k r_k^2$$

Since distance within the infinitesimal interval for the n-body configurations at time zero is constant, the second derivative of the moment of inertia is zero and therefore twice the kinetic energy will be equal to the potential energy, thus, giving the expected result.

1.3 Infinite Binary Configurations

Any family of configurations in the mathematical logical system should be solvable in the manner presented in subsection 1.2 using Jacoby coordinates with the sidereal synodic relations and type one geometry in the infinitesimal interval at time zero. Hierarchical solutions can also be constructed between families by grouping binary, trinary, quadruple etc. configurations in the proper order under the same constraints. In this section the collinear infinite binary solution will be presented, that is, formatting the collinear n-body equations of motion in terms of double binary, triple binary, quadruple binary etc. configurations. Jacoby coordinates will be designated by $r1, r2, \dots rN$ for each binary subsystem with the Jacoby coordinates $\rho 1, \rho 2, \dots \rho(N-1)$ designate the distance between the binary centers of gravity. The vector between the binary center of gravity is also a binary subsystem.

Infinite binary collinear equations of motion in the complex plane describing infinitesimal rotations in the infinite interval at time zero, are listed below. These coupled binary equations of motion will be used to determine the perturbation coefficients, inertial velocities, period ratios and state vectors for the infinite binaries.

$$\ddot{z}_{21} + 2wi\acute{z}_{21} - w^2 z_{21} = -\frac{G\mu_{r1}}{r_{21}^3} z_{21} + \frac{Gm_3}{r_{32}^3} z_{32} - \frac{Gm_3}{r_{31}^3} z_{31} + \cdots + \frac{Gm_n}{r_{n,2}^3} z_{n,2} - \frac{Gm_n}{r_{n,1}^3} z_{n,1}$$

$$\ddot{z}_{43} + 2wi\acute{z}_{43} - w^2 z_{43} = -\frac{G\mu_{r2}}{r_{43}^3} z_{43} + \frac{Gm_1}{r_{14}^3} z_{14} - \frac{Gm_1}{r_{13}^3} z_{13} + \cdots + \frac{Gm_n}{r_{n,4}^3} z_{n,4} - \frac{Gm_n}{r_{n,3}^3} z_{n,3}$$

$$\ddot{z}_{65} + 2wi\acute{z}_{65} - w^2 z_{65} = -\frac{G\mu_{r3}}{r_{65}^3} z_{65} + \frac{Gm_1}{r_{16}^3} z_{16} - \frac{Gm_1}{r_{15}^3} z_{15} + \cdots + \frac{Gm_n}{r_{n,6}^3} z_{n,6} - \frac{Gm_n}{r_{n,5}^3} z_{n,5}$$

$$\ddot{z}_{n,(n-1)} + 2wi\acute{z}_{n,(n-1)} - w^2 z_{n,(n-1)} = -\frac{G\mu_{rN}}{r_{n,(n-1)}^3} z_{n,(n-1)} + \frac{Gm_1}{r_{1,n}^3} z_{1,n} - \frac{Gm_1}{r_{1,(n-1)}^3} z_{1,(n-1)} + \cdots + \frac{Gm_{n-2}}{r_{(n-2),n}^3} z_{(n-2),n} - \frac{Gm_{n-2}}{r_{(n-2),(n-1)}^3} z_{(n-2),(n-1)}$$

The initial equations of motion using equation 1.1.1 have been reduced by a factor of two by converting to Jacoby coordinates where $\ddot{z}_{21}, \ddot{z}_{43}, \ldots, \ddot{z}_{n(n-1)}$ represents this coupling. Total mass number is represented by n (even) and total binary number by N ($n = 2N$) with mass sums $\mu_{rN} = m_{2N-1} + m_{2N}$. The r_{21}, r_{43}, $r_{n,n-1}$ Jacoby coordinates will be referenced as $r1, r2, \ldots rN$ in future equations.

Sidereal synodic relations in equation 1.1.2 have been reformatted to apply to the infinite binary configurations. They are more complex than first family n-body configurations due to the intricate nature of the binary geometry. Differentiation is made between r binaries and ρ binaries when writing out the sidereal synodic relations, where P_{ri} represents the period of the ri binary subsystem and $P_{\rho i}$ represents the period of the vector between the ri binary centers of gravity. Infinite binary sidereal synodic relations can be written as designated below where N indicates the number of binaries in a given configuration.

$$n_{\rho(N-1)}P_{\rho(N-1)} = (n_{\rho(N-1)} + n_{rN})P_{rN} \qquad \text{Equation 1.3.1}$$

$$n_{\rho(N-1)}P_{\rho(N-1)} = (n_{\rho(N-1)} + n_{\rho(N-2)})P_{\rho(N-2)}$$

$$n_{\rho(N-1)}P_{\rho(N-1)} = (n_{\rho(N-1)} + n_{\rho(N-2)} + n_{r(N-1)})P_{r(N-1)}$$

$$n_{\rho(N-1)}P_{\rho(N-1)} = (n_{\rho(N-1)} + n_{\rho(N-2)} + n_{\rho(N-3)})P_{\rho(N-3)}$$

$$n_{\rho(N-1)}P_{\rho(N-1)} = (n_{\rho(N-1)} + n_{\rho(N-2)} + \cdots + n_{\rho 1} + n_{r(N-2)})P_{r(N-2)}$$

$$n_{\rho(N-1)}P_{\rho(N-1)} = (n_{\rho(N-1)} + n_{\rho(N-2)} + \cdots + n_{\rho 1})P_{\rho 1)}$$

.

$$.n_{\rho(N-1)}P_{\rho(N-1)} = (n_{\rho(N-1)} + n_{\rho(N-2)} + \cdots + n_{\rho 1} + n_{r2})P_{r2}$$

$$n_{\rho(N-1)}P_{\rho(N-1)} = (n_{\rho(N-1)} + n_{\rho(N-2)} + \cdots + n_{\rho 1} + n_{r1})P_{r1}$$

The double binary subsystem has two r and one ρ Jacoby coordinates, triple binary subsystem has three r and two ρ Jacoby coordinates, quadruple binary subsystem has four r and three ρ Jacoby coordinates etc., where this geometry is incorporated in equation 1.3.1. Terms used in the sidereal synodic period relations and their coefficients are defined

P_{r1} period of first binary subsystem (mass one mass two)

P_{r2} period of second binary subsystem (mass three mass four)

P_{rN} period of N^{th} binary subsystem

$P_{\rho 1}$ period of double binary subsystem ($m_1 m_2$ *and* $m_3 m_4$ center of gravity distance)

$P_{\rho 2}$ period of triple binary subsystem ($m_1 m_2$ *and* $m_3 m_4$ center of gravity distance to $m_5 m_6$)

$P_{\rho(N-1)}$ period of $\rho(N-1)^{th}$ binary subsystem

It is necessary to diagram the vector binary geometry structure to identify rotating vectors in the complex plane. Formulating these rotating vectors as exponentials to substitute into the coupled equations of motion with the sidereal synodic relations (equation 1.3.1) and type one geometry will result in a collinear solution. To start this process for example, the rotating Jacoby coordinate z_{21} relative to the x' axis can be formulated

$$z_{21} = r1 e^{i\frac{n_{\rho(N-2)}+\cdots+n_{\rho 1}+n_{r1}}{n_{\rho(N-1)}}wt} \qquad \text{Equation 1.3.2}$$

Substituting the first and second derivatives of equation 1.3.2 with respect to time into the $\ddot{z}_{21}$ equation of motion and using the vector configuration geometry will give after analysis

$$\frac{G\mu_{r1}}{w^2 r_{21}^3} - \frac{P_{N-1}^2}{P_{21}^2} = \frac{G}{w^2 r_{21}^3}\left(\left(\frac{r_{21}^2}{r_{32}^2} - \frac{r_{21}^2}{r_{31}^2}\right) m_3 + \left(\frac{r_{21}^2}{r_{42}^2} - \frac{r_{21}^2}{r_{41}^2}\right) m_4 + \cdots + \left(\frac{r_{21}^2}{r_{n2}^2} - \frac{r_{21}^2}{r_{n1}^2}\right) m_n\right) = \frac{G}{w^2 r_{21}^3}\phi_{r1} \qquad \text{Equation 1.3.3}$$

where $$\phi_{r1} = \left(\frac{r_{21}^2}{r_{32}^2} - \frac{r_{21}^2}{r_{31}^2}\right) m_3 + \left(\frac{r_{21}^2}{r_{42}^2} - \frac{r_{21}^2}{r_{41}^2}\right) m_4 + \cdots + \left(\frac{r_{21}^2}{r_{n2}^2} - \frac{r_{21}^2}{r_{n1}^2}\right) m_n$$

(no $r1$ binary terms)

Continuing this derivation with the acceleration vectors $\ddot{z}_{43}$, $\ddot{z}_{65}$,... $\ddot{z}_{n(n-1)}$ will yield the complete set of $\phi_{ri}'s$ using the z_{43}, z_{65}, , $z_{n,n-1}$ rotating Jacoby coordinates with the sidereal synodic relations (equation 1.3.1) to find the velocities for the infinite binary state vectors.

$$\phi_{r2} = \left(\frac{r_{21}^2}{r_{41}^2} - \frac{r_{21}^2}{r_{31}^2}\right) m_1 + \left(\frac{r_{21}^2}{r_{42}^2} - \frac{r_{21}^2}{r_{32}^2}\right) m_2 + \cdots + \left(\frac{r_{21}^2}{r_{n3}^2} - \frac{r_{21}^2}{r_{n4}^2}\right) m_n.$$

$$\phi_{r3} = \left(\frac{r_{21}^2}{r_{61}^2} - \frac{r_{21}^2}{r_{51}^2}\right) m_1 + \left(\frac{r_{21}^2}{r_{62}^2} - \frac{r_{21}^2}{r_{52}^2}\right) m_2 + \cdots + \left(\frac{r_{21}^2}{r_{n5}^2} - \frac{r_{21}^2}{r_{n6}^2}\right) m_n$$

$$\phi_{rN} = \left(\frac{r_{21}^2}{r_{61}^2} - \frac{r_{21}^2}{r_{51}^2}\right) m_1 \left(\frac{r_{21}^2}{r_{62}^2} - \frac{r_{21}^2}{r_{52}^2}\right) m_2 + \ldots + \left(\frac{r_{21}^2}{r_{n,(n-2)}^2} - \frac{r_{21}^2}{r_{(n-1),(n-2)}^2}\right) m_{n-2}$$

. Equation 1.3.4

where ϕ_{r1}, ϕ_{r2}, ϕ_{r3},…, ϕ_{rN} are the perturbation coefficients associated with the respective $r1, r2, r3, \ldots, rN$ Jacoby coordinates. Perturbation coefficient ϕ_{r2} has no $, r2$ binary terms, ϕ_{r3} has no $r3$ binary terms and ϕ_{rN} has no rN binary terms.

To find the perturbation coefficients of the infinite binary subsystems $\rho1, \rho2, \ldots \rho(N-1)$ that are between the binary centers of gravity, the equations of motion will be in terms of the vector from the barycenter to the given binary subsystem center. The equation of motion for the Jacoby coordinate $\rho1$ between the first binary $r1$ and the second binary $r2$ can be derived by using $z_{43c} = \frac{m_3 z_3 + m_4 z_4}{\mu_{r2}}$, which is the vector from the barycenter to the center of gravity of the mass three and mass four binary subsystem. This $\rho1$ equation of motion can also be obtained by following a different vector path, that is, by using $z_{21c} = \frac{m_1 z_1 + m_2 z_2}{\mu_{r1}}$, which is the vector from the barycenter to the center of gravity of the mass one and mass two binary subsystem. Either vector pathway calculation will give the correct answer for the perturbation coefficient. There is no Gm_3 or Gm_4 multiplier in the following equation

$$\ddot{z}_{43c} + 2wi\acute{z}_{43c} - w^2 z_{43c} = \frac{Gm_1}{\mu_{r2}}\left(\frac{m_3}{r_{13}^3} z_{13} + \frac{m_4}{r_{14}^3} z_{14}\right) + \frac{Gm_2}{\mu_{r2}}\left(\frac{m_3}{r_{23}^3} z_{23} + \frac{m_4}{r_{24}^3} z_{24}\right) + \cdots + \frac{Gm_n}{\mu_{r2}}\left(\frac{m_3}{r_{n3}^3} z_{n3} + \frac{m_4}{r_{n4}^3} z_{n4}\right)$$

Equation 1.3.5

The equation of motion for Jacoby coordinate $\rho2$, the vector between the center of gravity of the double binary subsystem $r1$, $r2$ and the center of gravity of $r3$ can be derived by using $z_{65c} = \frac{m_5 z_5 + m_6 z_6}{\mu_{r3}}$ which is the vector from the barycenter to the center of gravity of the mass five and mass six binary subsystem. There is no Gm_5 or Gm_6 multiplier in the following equation

$$\ddot{z}_{65c} + 2wi\acute{z}_{65c} - w^2 z_{65c} = \frac{Gm_1}{\mu_{r3}}\left(\frac{m_5}{r_{15}^3} z_{15} + \frac{m_6}{r_{16}^3} z_{16}\right) + \frac{Gm_2}{\mu_{r3}}\left(\frac{m_5}{r_{25}^3} z_{25} + \frac{m_6}{r_{26}^3} z_{26}\right) + \cdots + \frac{Gm_n}{\mu_{r3}}\left(\frac{m_5}{r_{n5}^3} z_{n5} + \frac{m_6}{r_{n6}^3} z_{n6}\right)$$

Equation 1.3.6

Equation of motion for Jacoby coordinate $\rho(N-1)$, the vector between center of gravity of the $N-1$ binary sub-configuration and the center of gravity of the rN binary can be derived by using equation $z_{n,(n-1)c} = \frac{m_{n-1} z_{n-1} + m_n z_n}{\mu_{rN}}$ which is the vector from the barycenter to the center of gravity of the rN binary subsystem. There is no Gm_{n-1} or Gm_n multiplier in the following equation

$$\ddot{z}_{n,(n-1)c} + 2wi\acute{z}_{n,(n-1)c} - w^2 z_{n,(n-1)c} =$$ Equation 1.3.7

$$\frac{Gm_1}{\mu_{rN}}\left(\frac{m_{n-1}}{r_{1,(n-1)}^3} z_{1,(n-1)} + \frac{m_n}{r_{1,n}^3} z_{1,n}\right) + \frac{Gm_2}{\mu_{rN}}\left(\frac{m_{n-1}}{r_{2,(n-1)}^3} z_{2,(n-1)} + \frac{m_n}{r_{2,n}^3} z_{2,n}\right) + \cdots + \frac{Gm_{n-2}}{\mu_{rN}}\left(\frac{m_{n-1}}{r_{(n-2),(n-1)}^3} z_{(n-2),(n-1)} + \frac{m_n}{r_{(n-2),n}^3} z_{(n-2),n}\right)$$

Infinitesimal rotating vectors z_{43c}, z_{65c}, , $z_{n(n-1)c}$ derived from the binary geometry (equation 1.3.2) when substituted into the acceleration vectors $\ddot{z}_{43c}$, $\ddot{z}_{65c}, \ldots, \ddot{z}_{n(n-1)c}$ will give the $\phi_{\rho i}{}'s$ necessary to find the Jacoby velocities for the infinite binary state vectors. Summarizing the perturbation coefficient results from equations 1.3.5, 1.3.6 and 1.3.7 yields

$$\phi_{\rho 1} = \frac{r_{21}^2}{\mu_{r2}} \sum_{i=1}^{2} \left(\frac{m_4}{r_{4i}^2} + \frac{m_3}{r_{3i}^2}\right) m_i - \frac{r_{21}^2}{\mu_{r2}} \sum_{i=5}^{n} \left(\frac{m_4}{r_{4i}^2} + \frac{m_3}{r_{3i}^2}\right) m_i$$ Equation 1.3.8

$$\phi_{\rho 2} = \frac{r_{21}^2}{\mu_{r3}} \sum_{i=1}^{4} \left(\frac{m_6}{r_{6i}^2} + \frac{m_5}{r_{5i}^2}\right) m_i - \frac{r_{21}^2}{\mu_{r3}} \sum_{i=7}^{n} \left(\frac{m_6}{r_{6i}^2} + \frac{m_5}{r_{5i}^2}\right) m_i$$

$$\phi_{\rho 3} = \frac{r_{21}^2}{\mu_{r4}} \sum_{i=1}^{6} \left(\frac{m_8}{r_{8i}^2} + \frac{m_7}{r_{7i}^2}\right) m_i - \frac{r_{21}^2}{\mu_{r4}} \sum_{i=9}^{n} \left(\frac{m_8}{r_{8i}^2} + \frac{m_7}{r_{7i}^2}\right) m_i$$

$$\phi_{\rho(N-1)} = \frac{r_{21}^2}{\mu_{r4}} \sum_{i=1}^{n-2} \left(\frac{m_n}{r_{ni}^2} + \frac{m_{n-1}}{r_{(n-1)i}^2}\right) m_i$$

where $\phi_{\rho 1}, \phi_{\rho 2}, \phi_{\rho 3}, \ldots, \phi_{\rho(N-1)}$ are the perturbation coefficients associated with the respective $\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_2, \ldots, \vec{\rho}_{n-1}$ Jacoby coordinates. Parameter n is the total number of masses and N is the total number of binaries with $N = \frac{n}{2}$ for $n \geq 4$ and N even. Binary masses are summed $\mu_{r1} = m_1 + m_2$, $\mu_{r2} = m_3 + m_{4,}, \ldots, \mu_{rN} = m_{2N-1} + m_{2N,}$. The x' axis is defined to be from the configuration barycenter to the center of gravity of the N^{th} binary rotating relative to the x axis at inertial velocity w in the infinitesimal interval.

Equation 1.3.1 sidereal synodic relations in conjunction with knowledge of equations 1.3.3, 1.3.4 and 1.3.8 perturbation coefficients result in determination of the binary period ratios. These period ratios numbering $\frac{(n-1)!}{2(n-3)!}$ where n is the number of masses, can be structured in the following manner as presented below

$$\frac{P_{r2}}{P_{r1}} \quad \frac{P_{\rho 1}}{P_{r1}} \quad \frac{P_{\rho 2}}{P_{r1}} \ldots\ldots\ldots\ldots \ldots\ldots \frac{P_{rN}}{P_{r1}} \qquad \text{Equation 1.3.9}$$

$$\frac{P_{\rho 1}}{P_{r2}} \quad \frac{P_{\rho 2}}{P_{r2}} \ldots\ldots\ldots\ldots \frac{P_{rN}}{P_{r2}}$$

$$\frac{P_{\rho 2}}{P_{\rho 1}} \ldots\ldots\ldots\ldots\ldots \frac{P_{rN}}{P_{\rho 1}}$$

$$. \quad . \quad .$$

$$\frac{P_{rN}}{P_{\rho(N-1)}}$$

The individual binary periods for the $r1, r2, \ldots rN$ Jacoby coordinates as a function of the perturbation coefficients from equation 1.3.4 are

$$P_{r1}^2 = 4\pi^2 \left(\frac{r_{r1}^3}{G(\mu_{r1} - \Phi_{r1})}\right) \qquad \text{Equation 1.3.10}$$

$$P_{r2}^2 = 4\pi^2 \left(\frac{r_{r2}^3}{G(\mu_{r2} - \Phi_{r2})}\right)$$

$$P_{rN}^2 = 4\pi^2 \left(\frac{r_{rN}^3}{G(\mu_{rN} - \Phi_{rN})}\right) \qquad \Phi_{rN} = x_{N-1}^2 \phi_{rN}$$

$N \geq 2$ N is even

$$x_1 = \frac{r2}{r1} \qquad x_2 = \frac{r3}{r1} \qquad x_3 = \frac{r4}{r1} \qquad x_n = \frac{r(n+1)}{r1}$$

where $\mu_{r1} = m_1 + m_2$, $\mu_{r2} = m_3 + m_{4,}, \ldots \mu_{rN} = m_{2N-1} + m_{2N}$. Parameter ϕ is the perturbation coefficient and scaling factor x is the $r1, r2, \ldots rN$ binary separation distance ratioed to $r1$.

The corresponding periods for the $\rho 1, \rho 2, \ldots \rho(N-1)$ Jacoby coordinates as a function of the perturbation coefficients from equation 1.3.8 are

$$P_{\rho 1}^2 = 4\pi^2 \left(\frac{\rho_1^3}{G\frac{l_1^2}{\mu_{r1}}(\mu_{\rho 1}\phi_{\rho 1} + \mu_{r3}\phi_{\rho 2} + \cdots + \mu_{rN}\phi_{\rho(N-1)})}\right)$$

$$P_{\rho 2}^2 = 4\pi^2 \left(\frac{\rho_2^3}{G\frac{l_2^2}{\mu_{\rho 1}}(\mu_{\rho 2}\phi_{\rho 2} + \cdots + \mu_{rN}\phi_{\rho(N-1)})}\right)$$

$$P_{\rho(N-1)}^2 = 4\pi^2 \left(\frac{\rho_{N-1}^3}{G\frac{\mu_{\rho(N-1)}}{\mu_{\rho(N-2)}} l_{N-1}^2 \phi_{\rho(N-1)}}\right) \qquad \text{Equation 1.3.11}$$

where $\mu_{\rho 1} = \mu_{r1} + \mu_{r2}$, $\mu_{\rho 2} = \mu_{r1} + \mu_{r2} + \mu_{r3}$, and $\mu_{\rho N} = \sum_{i=1}^{N+1} \mu_{ri}$

Summarizing $r1, r2, \ldots rN$ Jacoby perturbation coefficients from equations 1.3.3 and 1.3.4 in matrix format gives a visual interpretation of the binary structure. Double zeros in each row of the matrix demonstrate the location of every binary in this infinite N binary configuration.

$$\begin{pmatrix} \phi_{r1} \\ \phi_{r2} \\ \phi_{r3} \\ \phi_{r4} \\ \cdot \\ \phi_{rN} \end{pmatrix} = r_{21}^2 B \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ \cdot \\ m_n \end{pmatrix}$$

$$B = \begin{pmatrix} ..0.. & ..0.. & e13 & e14 & e15 & e16 & e17 & e18 & & e1n \\ e21 & e22 & ..0.. & ..0.. & e25 & e26 & e27 & e28 & & e2n \\ e31 & e32 & e33 & e34 & ..0.. & ..0.. & e37 & e38 & & e3n \\ e41 & e42 & e43 & e44 & e45 & e46 & ..0.. & ..0.. & & e4n \\ & \cdot & & & & & \cdot & & & \\ eN1 & eN2 & eN3 & eN4 & e65 & e66 & e67 & e68 & ..0.. & ..0.. \end{pmatrix}$$

where $$\phi_{r1} = r_{21}^2 \sum_{i=3}^{n} (\frac{1}{r_{2i}^2} - \frac{1}{r_{1i}^2}) m_i$$

$$\phi_{r2} = -r_{21}^2 \sum_{i=1}^{2} (\frac{1}{r_{4i}^2} - \frac{1}{r_{3i}^2}) m_i + r_{21}^2 \sum_{i=5}^{n} (\frac{1}{r_{4i}^2} - \frac{1}{r_{3i}^2}) m_i$$

$$\phi_{r3} = -r_{21}^2 \sum_{i=1}^{4} (\frac{1}{r_{6i}^2} - \frac{1}{r_{5i}^2}) m_i + r_{21}^2 \sum_{i=7}^{n} (\frac{1}{r_{6i}^2} - \frac{1}{r_{5i}^2}) m_i$$

$$\phi_{rN} = -r_{21}^2 \sum_{i=1}^{n-2} (\frac{1}{r_{ni}^2} - \frac{1}{r_{(n-1)i}^2}) m_i$$

Each term in the above ϕ_{rN} series corresponds to an element in the rN row. Summarized perturbation coefficients from equation 1.3.8 for the $\rho 1, \rho 2, \dots \rho(N-1)$ Jacoby N binaries are presented below in matrix format. Double zeros show binary coupling existing in this matrix structure as well.

$$\begin{pmatrix} \phi_{\rho 1} \\ \phi_{\rho 2} \\ \phi_{\rho 3} \\ \cdot \\ \phi_{\rho\beta} \end{pmatrix} = r_{21}^2 C \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ \cdot \\ m_n \end{pmatrix}$$

$$C = \begin{pmatrix} e11 & e12 & ..0.. & ..0.. & e15 & e16 & e17 & e18 & & e1n \\ e21 & e22 & e23 & e24 & ..0.. & ..0.. & e27 & e28 & & e2n \\ e31 & e32 & e33 & e34 & e35 & e36 & ..0.. & ..0.. & & e3n \\ & \cdot & & & & & \cdot & & & \\ e\beta 1 & e\beta 2 & e\beta 3 & e\beta 4 & e\beta 5 & e\beta 6 & e\beta 7 & e\beta 8 & ..0.. & ..0.. \end{pmatrix}$$

where $$\phi_{\rho 1} = \frac{r_{21}^2}{\mu_{r2}} \sum_{i=1}^{2} (\frac{m_4}{r_{4i}^2} + \frac{m_3}{r_{3i}^2}) m_i - \frac{r_{21}^2}{\mu_{r2}} \sum_{i=5}^{n} (\frac{m_4}{r_{4i}^2} + \frac{m_3}{r_{3i}^2}) m_i$$

$$\phi_{\rho 2} = \frac{r_{21}^2}{\mu_{r3}} \sum_{i=1}^{4} (\frac{m_6}{r_{6i}^2} + \frac{m_5}{r_{5i}^2}) m_i - \frac{r_{21}^2}{\mu_{r3}} \sum_{i=7}^{n} (\frac{m_6}{r_{6i}^2} + \frac{m_5}{r_{5i}^2}) m_i$$

$$\phi_{\rho 3} = \frac{r_{21}^2}{\mu_{r4}} \sum_{i=1}^{6} (\frac{m_8}{r_{8i}^2} + \frac{m_7}{r_{7i}^2}) m_i - \frac{r_{21}^2}{\mu_{r4}} \sum_{i=9}^{n} (\frac{m_8}{r_{8i}^2} + \frac{m_7}{r_{7i}^2}) m_i$$