Introduction to Digital Control of Linear Time Invariant Systems

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By

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PREFACE

New technologies in engineering, automatics and robotics are creating problems in which control plays a major role. Solutions to many of these problems require the use of digital signals. This manuscript attempts to provide the reader with an insight into digital control of time-invariant linear systems.

My objective is to offer an accessible, self-contained research monograph which can also be used as a graduate text. The material presented, in this book, is of interest to a wide population of students, teachers, engineers, and researchers working in engineering, computing, electronic, robotics and automatics. It can also be used as a reference book by control engineers in industry and research students in automation and control.

The first chapter covers fundamental concepts in the sampling and reconstruction of signals. The material presented in the second chapter can serve as an advanced text for courses on z-transform and inverse z-transform. Indeed, the inspection method, the direct division method, the partial-fraction expansion method, the recurrence inversion method and the contour integration method are all detailed. The third chapter introduces the transfer function. In fact, the absence or presence of an input sampler is crucial in determining the transfer function of a system. For this reason, different examples of the position of the sampler are treated to improve its efficiency and its influence. The fourth chapter presents the stability condition of discrete-time systems in the closed loop. The global stability definition, the algebraic stability criterion and the stability in the frequency domain are discussed. The fifth chapter introduces the synthesis of a digital controller for linear time invariant system. The last section, in this book, shows the use of a digital PID controller in the practical speed control of a DC motor using an Arduino card, to encourage readers to explore new applied areas of digital control. In all these chapters simple examples are used to illustrate important concepts.

I hope that the publication of this work will have a positive impact on students' interest in the subject. I have been benefited from my students, through my teaching and other interactions with them; in particular, their questions asking me to explain many of the topics covered in this book with simple examples.

Ayachi ERRACHDI, University of Kairouan, Kairouan, 2021

ABOUT THE AUTHOR

Ayachi ERRACHDI studied at the National Engineering School of Monastir, Tunisia, obtaining a degree in electrical engineering and a master's degree in automatic and industrial maintenance in 2005 and 2007, respectively. He obtained his PhD degree in electrical engineering from the National Engineering School of Tunis El Manar, Tunisia, in 2012. He is currently an associate professor of electrical engineering at the University of Kairouan.

He is currently a member of the Automatic Research Laboratory, at the National Engineering School of Tunis El Manar, Tunisia. His major research interests are in nonlinear, continuous and discrete-time control systems, artificial neural networks, system identification, adaptive control systems, fractional order systems, observers, simulation and target tracking. He has published many papers in journals and conference proceedings, and has supervised many doctoral students.

CHAPTER ONE

SAMPLING AND RECONSTRUCTION

1.1. Introduction

In this chapter we are going to focus on the sampling and reconstruction of an analog signal. Indeed, a continuous-time signal is an infinite and uncountable set of numbers. Between a start and end time, there are infinite possible values for the time t and the instantaneous amplitude. When continuous-time signals are brought into a computer, they must be digitized. In a discrete-time signal, the number of elements in the set, as well as the possible values of each element, is finite, countable, representable by computer bits and can be stored on a digital storage medium.

Digital systems attempt to overcome the analog system's susceptibility to noise by sacrificing the infinite aspect of the time and the amplitude resolution to obtain perfect reproduction of the signal no matter how long it has been stored or how many times it has been duplicated.

The discrete time and discrete amplitude nature of the digital signal provide a buffer to any noise that may enter the system through transmission or otherwise. Digital signals are usually stored and transmitted in the form of ones and zeros. If a digital receiver knows that only zeros or ones are being transmitted and when approximately to expect them, there is a certain acceptable level of noise that the receiver can handle.

Beyond the advantages of noise robustness during reproduction and transmission, digital signals have many other advantages. These include the ability to use computer algorithms to filter the signal, data compression to save storage space and signal processing to extract information that may not be possible through manual human analysis. Thus, there can be a large benefit in converting many signals that are used in cardiology to digital form.

1.2. Digitization of continuous-time signal

An analog signal is a continuous function with respect to the time and amplitude variables.

To find a digital signal, three steps are needed: sampling, quantization and coding as represented by the following figure.

Fig. 1.1 Three processes of digital processing

where $x(t)$ is an analog signal, $x_s(t)$ is a sampled signal, $x_q(k)$ is a quantized signal and $x_d(k)$ is a digital signal.

These blocks are defined as follows:

- sampling (sampler block): using a sampler block, we find a sampled signal that is discrete in time and continuous in amplitude.
- quantization (quantizer block): using a quantizer block, we find a quantized signal that is discrete in time and discrete in amplitude.
- coding (encoder block): each sample quantized to a finite number of bits.

1.2.1. The sampling of continuous-time signal

The sampling process converts a continuous-time signal to a discrete-time signal with a defined time resolution. This is determined by what is known as the sampling rate, and it is usually expressed in Hertz (Hz) or samples per second. The sampling rate needed for a faithful reproduction of the signal depends on the fluctuation sharpness of the signal that is being sampled.

1.2.1.1. Ideal sampler

To sample an analog signal, means to register some values of this signal at given times. An ideal sampler is generally represented by an interrupter. The time closing is considered equal to zero. The ideal sampler is given by Figure 1.2:

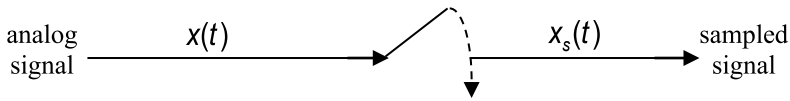

Fig. 1.2. The ideal sampler of an analog signal

We obtain a sampled signal $x_s(t)$ at equally spaced times $t = kT_s$:

$$x_s(t) = x(kT_s) \qquad \forall k \in \left]-\infty, +\infty\right[$$

where T_s is called the sampling period and it is inversely related to the sampling rate F_s, that is

$$F_s = \frac{1}{T_s}$$

The sampled signal $x_s(t)$ is found by multiplying the continuous-time signal $x(t)$ by a series of unit impulses, which are called the Dirac comb, given by

$$x_s(t) = x(t)p(t)$$

where $p(t)$ is

$$p(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$$

and $\delta(t)$ is called the Dirac delta function and is defined by:

$$\delta(t) = \begin{cases} 1 & if \quad t = 0 \\ 0 & if \quad t \neq 0 \end{cases}$$

The Dirac delta function $\delta(t)$ is a unit pulse in which the duration approaches zero but the area of the pulse is equal to one. This means as the width of the pulse τ approaches to zero, the amplitude of the pulse $\frac{1}{\tau}$ must approach infinity to maintain a unity area. The Dirac delta function is presented by this figure:

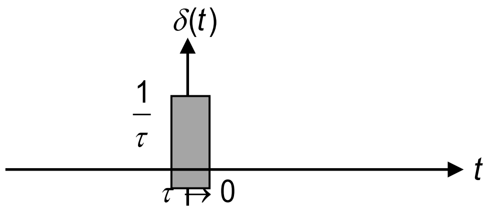

Fig. 1.3. The Dirac delta function

The periodic train impulse $p(t)$ is presented as follows

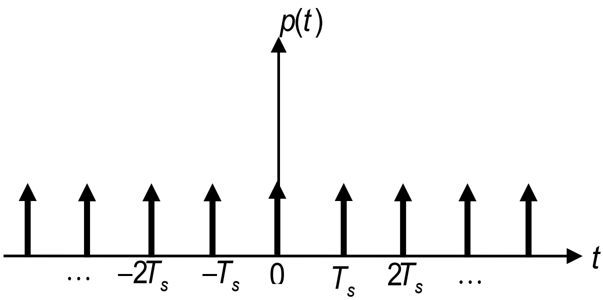

Fig. 1.4. Periodic impulse train

Sampling $x(t)$ is equivalent to multiplying it by a train of impulses as given here:

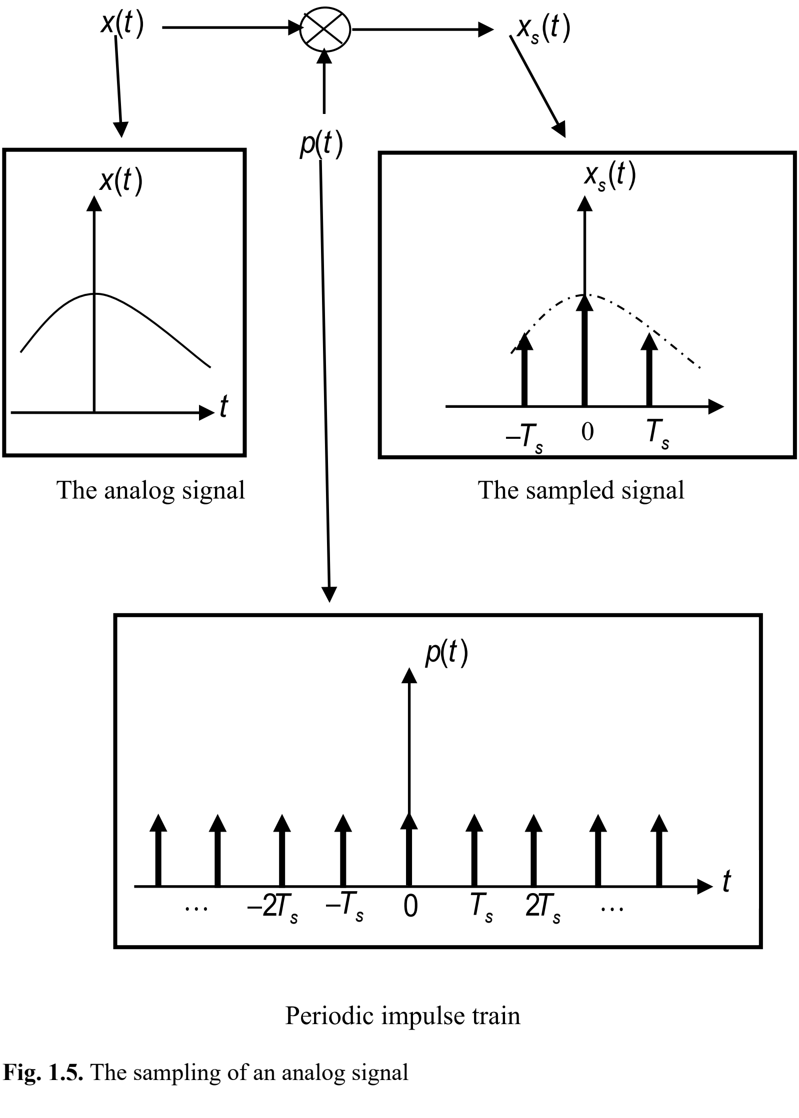

Fig. 1.5. The sampling of an analog signal

The found sampled signal is given as:

$$x_s(t) = \sum_{k=-\infty}^{+\infty} x(kT_s)\delta(t - kT_s)$$

whereas the continuous-time signal $x(t)$ is supposedly causal, that is to say

$$x(t) = 0 \qquad \forall \ t < 0$$

Therefore, the sampled signal is given as:

$$x_s(t) = x(0)\delta(t) + x(T_s)\delta(t - T_s) + x(2T_s)\delta(t - 2T_s) +, \dots, + x(kT_s)\delta(t - kT_s)$$

$$x_s(t) = \sum_{k=0}^{+\infty} x(kT_s)\delta(t - kT_s)$$

Some properties are described as follows:

$$\int_{-\infty}^{+\infty} \delta(t - kT_s)dt = 1 \quad , \qquad \forall k \in \mathbf{R}$$

The Laplace transform of the Dirac function is

$$L\left[\delta(t - kT_s)\right] = e^{-kT_s s} \quad , \qquad \forall k \in \mathbf{R}$$

and the Laplace transform of the sampled signal is

$$L\left[x_s(t)\right] = L\left[\sum_{k=0}^{+\infty} x(kT_s)\delta(t - kT_s)\right]$$

$$= \sum_{k=0}^{+\infty} L\left[x(kT_s)\delta(t - kT_s)\right]$$

$$= \sum_{k=0}^{+\infty} x(kT_s) L\left[\delta(t - kT_s)\right]$$

$$= \sum_{k=0}^{+\infty} x(kT_s)e^{-kT_s s}$$

1.2.1.2. Practical sampler

In the practical case $\forall k \in \mathbf{R}$, the interrupter has an important closing time which leads to samples depending on the duration of the sampling λ.

$$x_s(t) = \sum_{k=0}^{+\infty} x(kT_s)\lambda\delta(t - kT_s)$$

where λ is the duration of sampling caused by the interrupter.

1.2.1.3. Commonly used functions

a. Unit impulse

Consider a continuous-time impulse $\delta(t)$, defined by:

$$\delta(t) = \begin{cases} 1 & if \quad t = 0 \\ 0 & if \quad t \neq 0 \end{cases}$$

The sampled impulse is:

$$\delta_s(t) = \sum_{k=0}^{+\infty} \delta(kT_s)\delta(t - kT_s) = \begin{cases} 1 & if \quad k = 0 \\ 0 & if \quad k \neq 0 \end{cases}$$

and is given by the following figure:

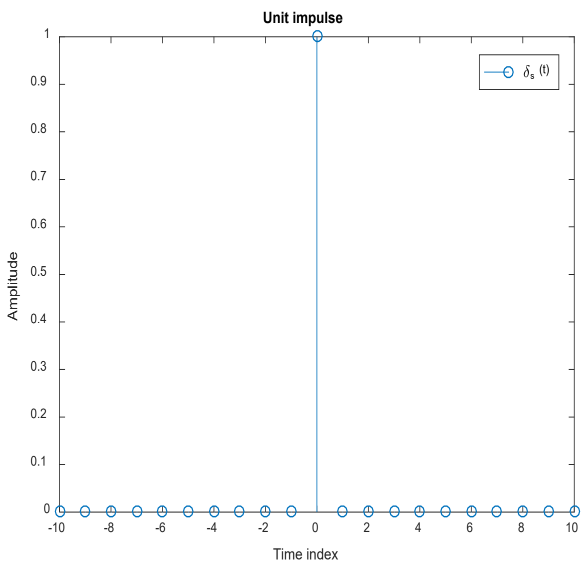

Fig. 1.6. The unit impulse

b. Unit step

Consider a continuous-time signal $u(t)$, defined by:

$$u(t) = \begin{cases} 1 & if \quad t \geq 0 \\ 0 & if \quad t < 0 \end{cases}$$

The sampled unitary step is:

$$u_s(t) = \sum_{k=0}^{+\infty} u(kT_s)\delta(t - kT_s) = \begin{cases} 1 & if \quad k \geq 0 \\ 0 & if \quad k < 0 \end{cases}$$

and is represented by the following figure:

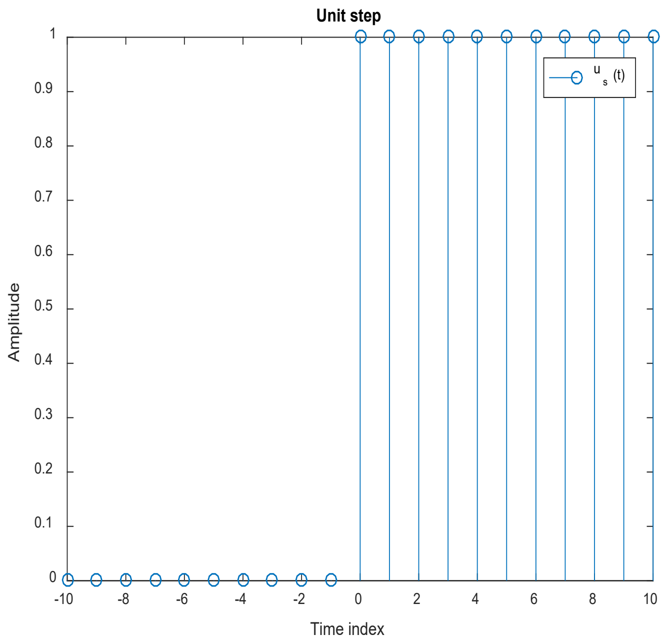

Fig. 1.7. The sampled unitary step

c. Unit ramp

Consider a continuous-time signal $r(t)$, defined by:

$$r(t) = \begin{cases} t & if \quad t \geq 0 \\ 0 & if \quad t < 0 \end{cases}$$

The sampled ramp is:

$$r_s(t) = \sum_{k=0}^{+\infty} r(kT_s)\delta(t - kT_s) = \begin{cases} k & if \quad k \geq 0 \\ 0 & if \quad k < 0 \end{cases}$$

and it is given by the following figure:

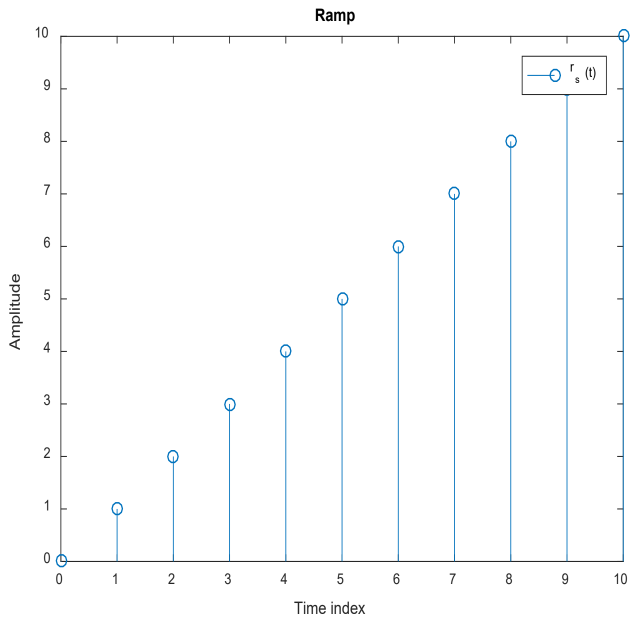

Fig. 1.8. The sampled ramp

d. Sinusoidal signal

Consider an analog signal $s(t)$, defined by:

$$s(t) = A\sin(wt + \varphi)$$

where A is an amplitude, w is an angular frequency (radians/s) and φ is a phase (radians). The expression of the sampled sinusoidal signal is:

$$s_s(t) = \sum_{k=0}^{+\infty} s(kT_s)\delta(t - kT_s) = A\sin(wkT_s + \varphi)$$

and it is given by the following figure:

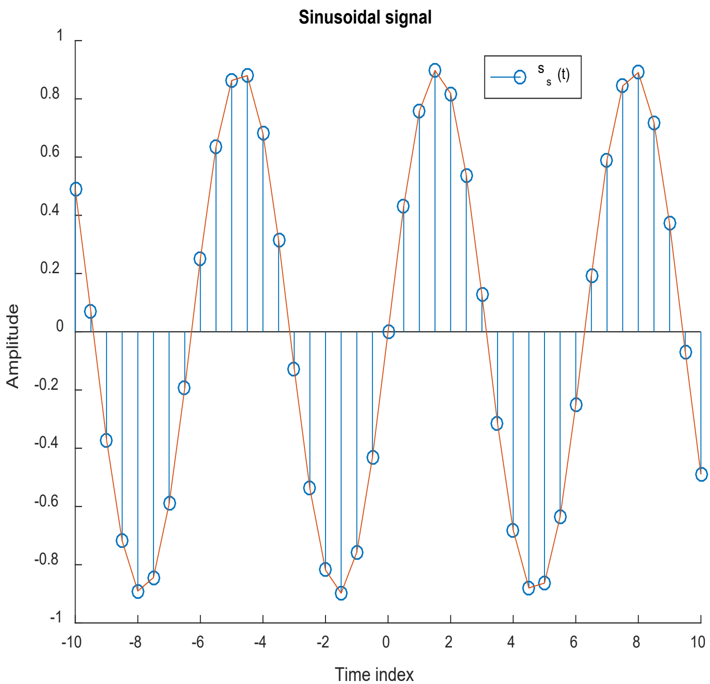

Fig. 1.9. The sampled sinusoidal signal

1.2.1.4. Sampling period's choice

A small amount of information is lost by sampling an analog signal at a very short time but all the information contained in the signal can be lost when the sampling times are too spaced from each other.

Example: Consider the analog signal defined by the following function:

$$x(t) = \sin(2\pi ft)$$

This function is sampled at a frequency $F_s = 2f$, that is, at a sampling period $T_s = \frac{T}{2}$.

$$x_s(t) = \sum_{k=0}^{+\infty} x(kT_s)\delta(t - kT_s) = \sum_{k=0}^{+\infty} \sin(\pi k)\delta(t - kT_s) = 0$$

An analog signal $x(t)$ having a low-pass type spectrum of width $w_{\max}$ is fully described by the further completeness of its instantaneous values $x_s(t)$ if they are elevated to a sampling pulse w_s such that $w_s > 2.w_{\max}$.

a. Theorem of spectrum concept

The Fourier transform of a bandlimited sampled signal $x_s(t)$ is the frequency spectrum of the signal:

$$\begin{aligned} X_s(f) &= F\left(x_s(t)\right) \\ &= \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X\left(f - kF_s\right) \\ &= F_s \sum_{k=-\infty}^{+\infty} X(f - kF_s) \end{aligned}$$

Proof:

The impulse function is given by the following equation

$$p(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$$

The impulse function has a periodic distribution of period T_s from which it can be decomposed into a Fourier series:

$$p(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_s) = \sum_{n=-\infty}^{+\infty} c_n e^{\frac{jn\pi}{T_s}t}$$

with

$$c_n = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} p(t) e^{\frac{jn\pi}{T_s}t}\, dt$$

Using the property of the Dirac distribution:

$$\int_a^b f(t)\delta(t-kT_s) = \begin{cases} f(kT_s) & if \quad a < kT_s < b \\ 0 & if \quad no \end{cases}$$

it can be easily shown that $c_n = \frac{1}{T_s}$, from where, we deduce that:

$$p(t) = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} e^{\frac{jn\pi}{T_s}t}$$

Let us apply the Fourier transform to the expression of the sampled signal $x_s(t)$. This would be transformed to:

$$X_s[f] = F[x_s(t)] = F[x(t)p(t)]$$

Using the property of the impulse function and the linearity of the Fourier transform, we obtain:

$$X_s[f] = F[x_s(t)] = \frac{1}{T_s}F\left[x(t)\sum_{k=-\infty}^{+\infty} e^{\frac{jk\pi}{T_s}t}\right] = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} F\left[x(t)e^{\frac{jk\pi}{T_s}t}\right]$$

By the use of the spectral translation property of the Fourier transform, a shift in the spectral domain corresponds to a multiplication by a complex exponential in the time domain, according to the formula:

$$F\left[e^{j\pi F_s t}x(t)\right] = X(f-F_s)$$

The Fourier transform of the sampled signal becomes

$$\boxed{X_s(f) = F(x_s(t)) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(f-kF_s)}$$

We note that $X_s(f)$ is periodic with period T_s and it is obtained by the sum of an infinity of complex functions, each of them being the Fourier transform $X(f)$ of the continuous-time signal $x(t)$, shifted by kF_s where F_s is the frequency of sampling and $k \in \mathbf{Z}$.

The following figures illustrate the frequency-domain representation of sampling in the time domain. The spectrum of the original signal is presented in Fig. 1.10. Three cases are then to be considered according to the value taken by the sampling frequency compared to the highest frequency F_{max} contained in the spectrum $X(f)$ of $x(t)$. The Fourier transform of the sampled signal with $F_s > 2F_{max}$ is shown in Fig. 1.11. The Fourier transform of the sampled signal with $F_s = 2F_{max}$ is shown in Fig. 1.12. The graphical representation using the Fourier transform of the sampled signal where $F_s < 2F_{max}$ is given by Fig. 1.13.

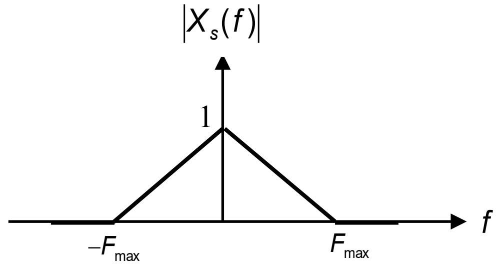

Fig.1.10. The spectrum of the original signal

Case 1: If $F_s > 2F_{max}$, the Fourier transform of the sampled signal is shown in Fig. 1.11.

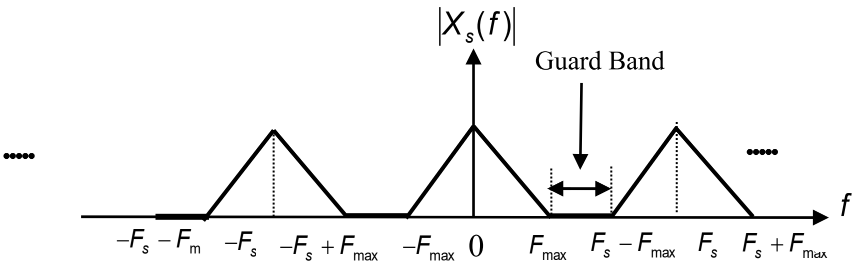

Fig. 1.11. The Fourier transform of the sampled signal with $F_s > 2F_{max}$ where the Guard Band is:

$$G.B = F_{High} - F_{Low}$$
$$= F_s - 2F_{max}$$

In this case, called oversampling, when we sample at a rate which is greater than $2F_{max}$, we say that we are oversampling as shown in Fig.

1.11. All the information contained in the continuous-time signal is found in each of the bands and, in particular, in the band $\left[-F_s, F_s\right]$.

It is, therefore, possible, here, to recover from $X_s(f)$ the continuous signal $X(f)$ isolating in the spectrum of $X_s(f)$ by using a practical Low-Pass Filter (LPF) that removes the sidebands as given in Fig. 1.12.

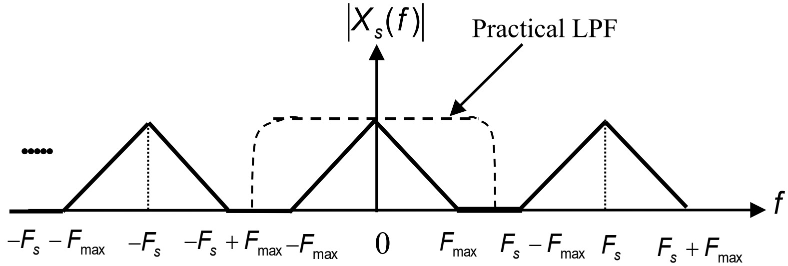

Fig. 1.12. The Fourier transform of the sampled signal with $F_s > 2F_{max}$

Case 2: If $F_s = 2F_{max}$, the Fourier transform of the sampled signal is shown in Fig. 1.13.

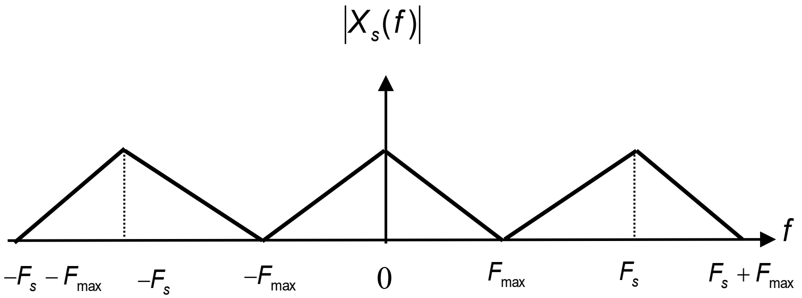

Fig. 1.13. The Fourier transform of the sampled signal with $F_s = 2F_{max}$

In the situation $G.B = 0$, it is possible to reconstruct the base pattern $X(f)$ corresponding to the spectrum of the continuous-time signal $x(t)$ using an ideal low-pass filter with $F_c = F_{max}$, where F_c is the cutoff frequency of the low-pass filter as given in Fig. 1.14.

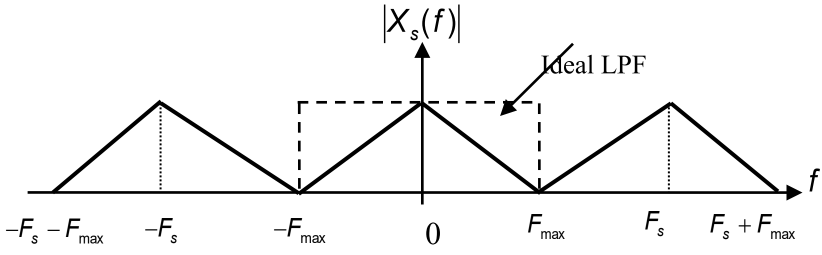

Fig. 1.14. The use of an ideal low-pass filter when $F_s = 2F_{max}$

In this case, it is not recommended to use the practical low-pass filter because the phenomenon of the aliasing effect may be continued as given in the following figure:

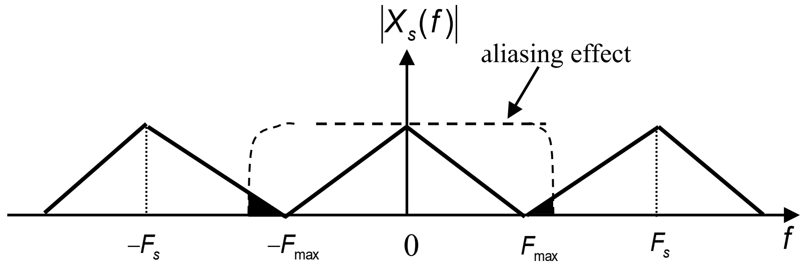

Fig. 1.15. The use of practical low-pass filter when $F_s = 2F_{max}$

Case 3: If $F_s < 2F_{max}$, the Fourier transform of the sampled signal is shown in Fig. 1.16.

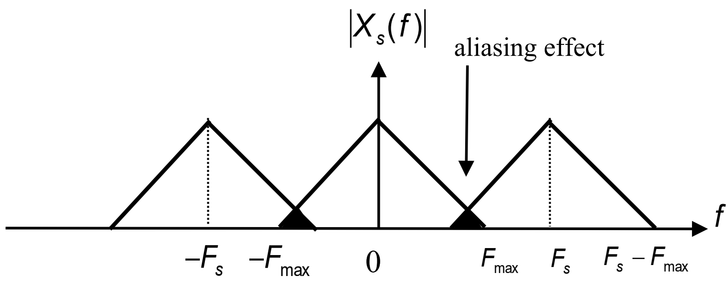

Fig. 1.16. The Fourier transform of the sampled signal with $F_s < 2F_{max}$

In this situation, when we sample at a rate which is less than $2F_{max}$, we say we are undersampling and aliasing will yield misleading results as shown in Fig. 1.16. Distortions occur in the spectrum $\left[-F_s, F_s\right]$, as a result of recombination of its various components. In this case, it is impossible to reconstruct the base pattern $X(f)$ corresponding to the spectrum of the continuous-time signal $x(t)$ even if the practical or ideal low-pass filters are used.

Example:
Let us consider a continuous-time signal $x(t)$ with a maximal frequency $F_{max} = 159.15Hz$, given as:

$$x(t) = 2\sin(1000t) + 3\cos(1000t)$$

The continuous-time signal $x(t)$ is presented by the following figure:

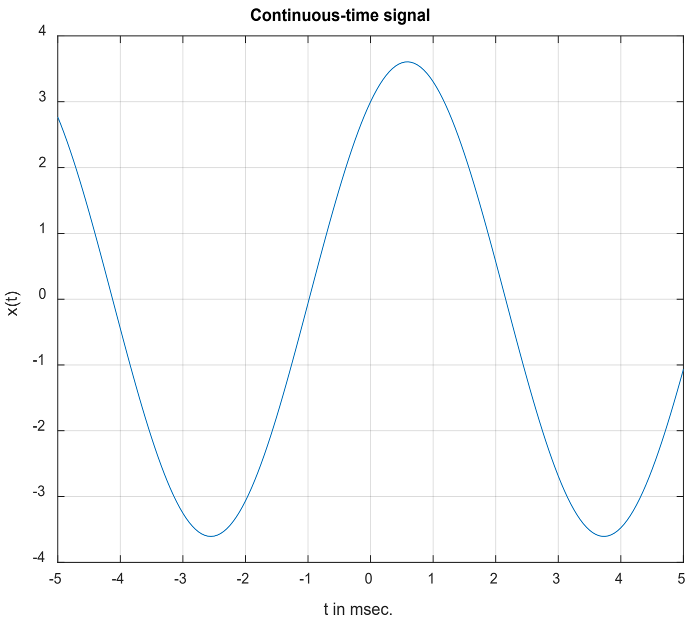

Fig. 1.17. The evolution of $x(t)$

When $F_s > 2F_{max}$, $F_s = 5000Hz$, the discrete-time Fourier Transform $X(f)$ is presented by Fig. 1.18.

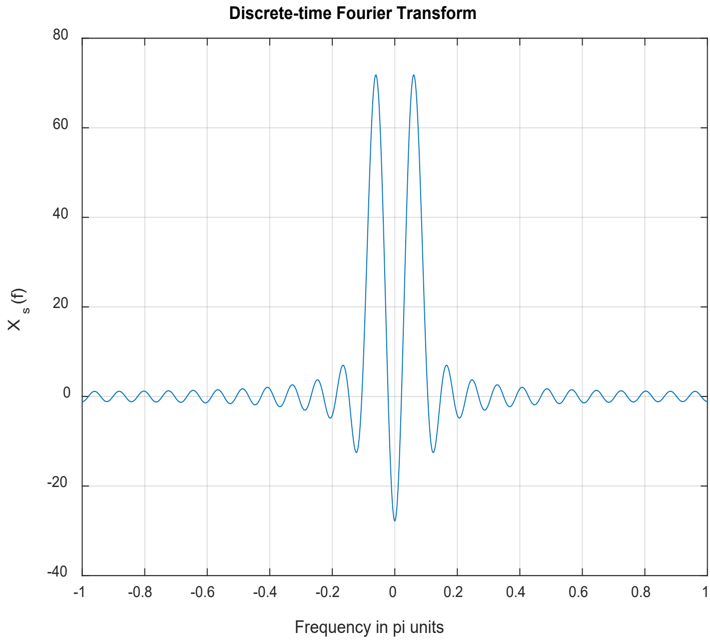

Fig. 1.18. The Fourier transform of the sampled signal with $F_s > 2F_{max}$

If $F_s = 2F_{max} = 318.3098Hz$, the discrete-time Fourier Transform $X_s(f)$ is presented by Fig. 1.19.

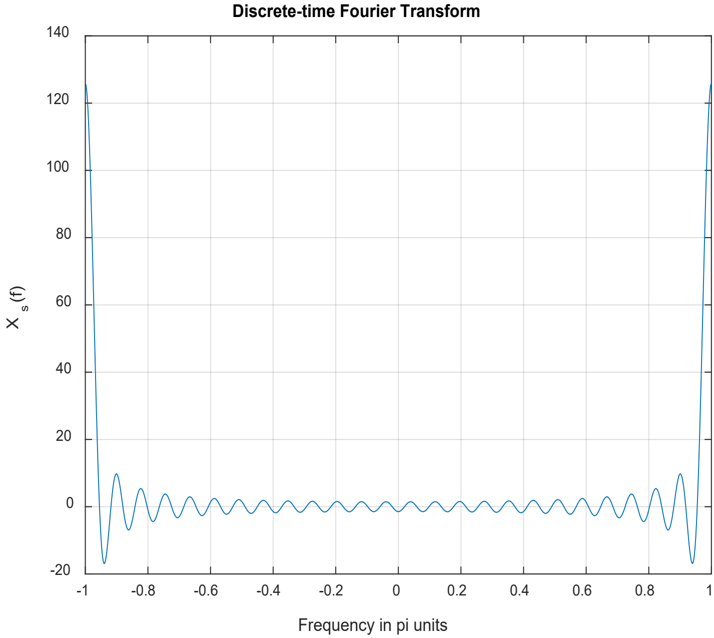

Fig. 1.19. The Fourier transform of the sampled signal with $F_s = 2F_{\max}$

When $F_s < 2F_{\max}$, $F_s = 1000Hz$, the discrete-time Fourier Transform $X_s(f)$ is presented by the following figure: