Convexity, Extension of Linear Operators, Approximation and Applications

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By

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To My Family

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INTRODUCTION

OCTAV OLTEANU

The aim of the present book is to emphasize recent results in the following actual research fields, presented in the eight chapters of the book. The main topics are: 1) Hahn-Banach type theorems and some of their motivations; the moment problem and related problems; giving a direct sharp proof for a main generalization of Hahn-Banach theorem; proving and applying polynomial approximation on unbounded subsets, that leads to existence and uniqueness of the solution for some Markov moment problems; characterizing the existence and uniqueness of the linear solution T, such that $T_1 \leq T \leq T_2$ on the positive cone of the domain space. Here T_1 and T_2 are given bounded linear operators; reviewing a construction of a polynomial solution for multidimensional moment problem; applying the Mazur-Orlicz theorem to concrete spaces and operators. 2) Elements of representation theory. The notion of a barycenter for a probability measure. 3) Pointing out properties, evaluating and optimizing convex functions and convex operators. 4) Constructing invariant subspaces for a large class of bounded linear operators and related results. 5) Proving results from linear analysis for convex operators; emphasizing relationship between linear and sublinear continuous operators; extending inequalities via Krein-Milman theorem. 6) Proving topological versions of sandwich theorems of type $f \leq h \leq g$, where $f, -g$ are convex and h is affine, on bounded and unbounded special convex subsets, called finite-simplicial sets. Generally, a finite-simplicial subset of a real vector space can be unbounded in any locally convex topology on the entire space. 7) Pointing out a global Newton like method for convex functions and operators and its connection to contraction principle. 8) Proving results on special functional equations over the real and over the complex fields. All our theorems are accompanied by (or represent) examples, solving concrete problems related to basic spaces of functions and respectively equations. The interested reader can find detailed proofs of recent results, as well as of our earlier basic results. A common point of most of these themes is the notion of convex function (or operator). In this respect, the linear solution of a Markov moment problem is dominated by a convex operator and is minorated by the null operator on the convex cone of the domain space. This last property is called positivity of the solution. The dominating convex continuous operator controls the norm of the linear solution. Recall that a Markov moment problem is an interpolation problem with two constraints on the linear solution. Here the dominated operator is not necessarily null. For details, see the introductory part of the first chapter. The existence of such a solution is proved by means of Hahn-Banach type theorems and their generalizations. All necessary such earlier results are recalled without their proofs. A direct sharp proof of a basic such result is given in Chapter 1. Another important aspect of the Markov moment problem is the uniqueness of the solution. Sometimes the uniqueness follows from the proof of the existence for the solution, via polynomial approximation results. Since the values of the linear continuous solution on polynomials are prescribed, the uniqueness of the solution follows whenever the polynomials are dense in the domain function space. The most interesting case is that of function spaces in several real variables, for which polynomial approximation on arbitrary (unbounded) closed subsets holds in some L^1 spaces. The most interesting results hold for Cartesian product of closed intervals. In particular, it works for spaces such as $L^1_\mu(\mathbb{R}^n), L^1_\mu([0,\infty)^n)$, etc., where μ is a product measure: $\mu = \mu_1 \times \cdots \times \mu_n$, μ_j being a positive regular Borel $M-$determinate (moment determinate) measure on $\mathbb{R}$, with finite moments $\int_{\mathbb{R}} t^k d\mu_j$ of all orders $k \in \mathbb{N}, j = 1, \dots, n,$ respectively μ_j is an $M-$ determinate measure on $[0,\infty), j = 1, \dots, n.$ Recall that a measure ν on $\mathbb{R}$ is called $M-$ determinate if it is uniquely determinate by its classical moments $\int_{\mathbb{R}} t^k d\nu,\ k \in \mathbb{N}$, or, equivalently, by its values on polynomials. Our polynomial approximation results partially solve the difficulty arising from the fact that there exist nonnegative polynomials on $\mathbb{R}^n, n \geq 2,$ which are not expressible as sums of squares. This way, some of the conditions in our statements are formulated in terms of quadratic forms. On the other hand, these approximation methods lead to a characterization of positivity of some bounded linear operators only in terms of quadratic forms. Notably, in Chapter 1 a special attention is focused on the Mazur-Orlicz theorem in concrete spaces. Many of the references of this book refer to applications related to convexity, geometric aspects in analysis and functional analysis and/or Hahn-Banach theorem. Various aspects of the moment problem are discussed in [1], [2], [4-7], [12], [13], [19], [23-25], [27-29], [31], [32], [35], [41], [45], [47-49], [51-53], [55], [58-63], [67], [70], [71], [75], [76], [78]. For some recent results and applications related to the present work, see [11], [20], [21], [27], [33], [39], [40], [59-64], [71], [74], [76]. As we will see in the introduction of the first chapter, solving many moment problems requires both Hahn-Banach type theorems and polynomial approximation on Cartesian products of closed (unbounded) intervals. This is the reason for placing the next chapter as the first one. Chapter 2 deals with elements of representation theory, while chapter 3 is devoted to evaluating and optimizing convex functions and operators, under certain convex or even linear constraints. For Pareto optimization and related applications see [9], [10]. For further results on convex functions and operators see [30], [36], [39], [43]. Chapter 4 is devoted to construction of invariant (closed) subspaces for a class of bounded linear operators. A related differential equation is studied and discussed in this respect. Invariance of the unit ball of some L^1 spaces is also under attention. Here polynomial approximation is applied again. It allows passing from inequalities verified on special nonnegative polynomials, to the same inequalities on arbitrary nonnegative functions in the domain space. In Chapter 5, a uniform boundedness property for classes of convex operators is proved and its relationship with previous related results in the literature is briefly discussed. As an important particular case, we point out applications of this result to classes of sublinear operators. Extending inequalities via Krein-Milman theorem is considered in the end of this chapter. The first aim of Chapter 6 is to prove topological versions for sandwich results $f \leq h \leq g$, where $f, -g$ are convex and h is affine on K. The case when K is a Choquet simplex, as well as that of K being a finite- simplicial set [2] is under attention. Second, applications of Krein-Milman and Carathéodory's theorem are emphasized. Chapter 7 deals with a global Newton's method for convex monotone operators. The Newton iterations are well-known. The problem is that the obtained sequence is not always convergent. Newton's method works sometimes only locally. When the involved function (or operator) is convex and isotone (monotone increasing) (or anti-isotone), with continuous derivative of first order, the method leads to a rapidly convergent sequence. Its limit is the zero of the given function (respectively operator), which can be approximated with the control of the norm of the error. The strength of this method consists in the fact that is global, and the weakness is its limitation to convex (or concave) decreasing or increasing operators. The connection with the contraction principle is emphasized and proved in detail. In some cases, approximation in terms of a contraction constant is preferable to that from the classical Newton method. Chapter 8 is devoted to a special kind of functional equations. Both real and complex cases are under attention. Matrix equations are also briefly discussed. To conclude, old results have been recently applied to obtain new theorems, published in [39], [52] [53], [55], [58-63]. Some other results on polynomial approximation on unbounded subsets were motivated by solving Markov moment problems.

CHAPTER ONE

ON HAHN-BANACH TYPE THEOREMS, POLYNOMIAL APPROXIMATION ON UNBOUNDED SUBSETS, THE MOMENT PROBLEM AND MAZUR-ORLICZ THEOREM

1. Introduction

We recall the classical formulation of the moment problem, under the terms of T. Stieltjes, given in 1894-1895 (see the basic book of N.I. Akhiezer [1] for details): find the repartition of the positive mass on the nonnegative semi-axis, if the moments of arbitrary orders k $(k = 0,1,2, \ldots)$ are given. Precisely, in the Stieltjes moment problem, a sequence of real numbers $(s_k)_{k\geq 0}$ is given and one looks for a nondecreasing real function $\sigma(t)$ $(t \geq 0)$, which verifies the moment conditions:

$$\int_0^\infty t^k \, d\sigma = s_k, \qquad (k = 0,1,2, \ldots).$$

This is a one-dimensional moment problem, on an unbounded interval. Namely, is an interpolation problem with the constraint on the positivity of the measure dσ. The numbers $s_k, k \in \mathbb{N} = \{0,1,2, \ldots\}$ are called the moments of the measure $d\sigma$. Existence, uniqueness, and construction of the solution σ are studied. The present work concerns firstly the existence problem. However, the uniqueness is studied as well. The moment problem is an inverse problem: we are looking for an unknown measure, starting from its moments. The direct problem might be: being given the measure $d\sigma$, compute its moments $\int_0^\infty t^k d\sigma, k = 0,1,2, \ldots$. The connection with the positive polynomials and extensions of linear positive functional and operators is quite clear. Namely, if one denotes by $\varphi_j, \varphi_j(t) := t^j, j \in \mathbb{N}, t \in [0, \infty)$, $\mathcal{P}$ the vector space of polynomials with real coefficients and

$$T_0:\mathcal{P}\to\mathbb{R}, T_0\left(\sum_{j\in J_0}\alpha_j\varphi_j\right):=\sum_{j\in J_0}\alpha_j s_j,$$

where $J_0\subset\mathbb{N}$ is a finite subset, then the moment conditions $T_0(\varphi_j)=s_j, j\in\mathbb{N}$ are clearly verified. It remains to check whether the linear form T_0 has nonnegative values at all nonnegative polynomials. If the latter condition is also accomplished, then one looks for the existence of a linear positive extension T of T_0 to a larger ordered function space X which contains both $\mathcal{P}$ and the space of continuous compactly supported functions, then representing T by means of a positive regular Borel measure μ on $[0,\infty)$, via Riesz representation theorem. Alternately one can apply directly Haviland theorem [25]. We start reviewing existence of a solution for simplest classical one-dimensional moment problems: the Hamburger moment problem (when the closed subset $F\subseteq\mathbb{R}^n$ is $F=\mathbb{R}$), Stieltjes moment problem (when $F=\mathbb{R}_+$) and Hausdorff moment problem (when $F=[0,1]$). In the sequel, the following notations are used: $\mathbb{N}=\{0,1,2,\dots\}, \mathbb{R}_+=[0,\infty), C_0(F)$ is the vector space of all real valued compactly supported continuous functions defined on F, $(C_0(F))_+$ is the convex cone of all functions in $C_0(F)$ which take nonnegative values at each point of F. $\mathcal{P}_+(F)$ is the convex cone of all polynomial functions with real coefficients, which are nonnegative on F. If an interval (for example $[a,b]$, $\mathbb{R}$, or $[0,\infty)$) is replaced by a closed subset F of $\mathbb{R}^n, n\geq 2,$ we have a multidimensional moment problem. Moments appear naturally in physics and probability theory, as discussed in the Introduction of [71]. Passing to an example of the multidimensional real classical moment problem, let us denote

$$\varphi_j(t)=t^j=t_1^{j_1}\cdots t_n^{j_n}, j=(j_1,\dots,j_n)\in\mathbb{N}^n,$$

$$t=(t_1,\dots,t_n)\in\mathbb{R}_+^n, n\in\mathbb{N}, n\geq 2. \tag{1.1}$$

If a sequence $(y_j)_{j\in\mathbb{N}^n}$ is given, one studies the existence, uniqueness, and construction of a positive linear form T defined on a function space containing polynomials and continuous real valued compactly supported functions, such that the moment conditions

$$T(\varphi_j)=y_j, \qquad j\in\mathbb{N}^n \tag{1.2}$$

are satisfied. Usually, the positive linear form T can be represented by means of a positive regular Borel measure on $\mathbb{R}_+^n$. When un upper constraint on the solution T is required too, we have a Markov moment problem. This requirement is formulated as T being dominated by a convex functional, which might be a norm, and its goal is to control the continuity and the norm of the solution. Clearly, the classical (Stieltjes) moment problem is an extension problem for a linear functional, from the subspace of polynomials to a function space which contains both polynomials as well as the continuous compactly supported real functions on $\mathbb{R}_+^n$. From solutions linear functionals, many authors considered solutions linear operators. Of course, in this case the moments $y_j, j \in \mathbb{N}^n$ are elements of an ordered vector space Y (usually Y is an order complete Banach lattice). The order completeness is necessary for applying Hahn-Banach type results for operators defined on polynomials and having Y as codomain. Various aspects of the classical moment problem have been studied (see the References). The case of multidimensional moment problem on compact semi-algebraic subsets in $\mathbb{R}^n$ was intensively studied. Clearly, the classical moment problem is related to the form of positive polynomials on the involved closed subsets of $\mathbb{R}^n$. As it is well-known, there exist nonnegative polynomials on the entire space $\mathbb{R}^n, n \geq 2,$ which are not sums of squares of polynomials (see [4], [71]), contrary to the case $n = 1,$ which is reviewed in [1]. The analytic form of positive polynomials on special closed unbounded finite dimensional subsets is crucial in solving classical moment problems on such subsets (see [34] for the expression of nonnegative polynomials on a strip, in terms of sums of squares). Such results are useful in characterizing the existence of a positive solution by means of signatures of quadratic forms. In case of Markov moment problem, approximation of nonnegative compactly supported continuous functions (with their support contained in a closed subset) by special dominating polynomials on that subset, having known analytic form is very important. Based on the form of nonnegative polynomials on unbounded intervals and the above-mentioned approximation results, Markov moment problem on Cartesian products of unbounded intervals can be solved in terms of products of quadratic forms. In most of the cases, the uniqueness of the solution of the Markov moment problem on spaces $L_\nu^1(F)$ follows too, thanks to the density of polynomials in such spaces; here F is a closed unbounded subset of $\mathbb{R}^n$ and ν is a positive regular M-determinate Borel measure on F. Recall that a measure ν is M −determinate (moment determinate), if it has finite moments of all orders and is uniquely determined by its moments

$$\int_F t^j \, d\nu, j \in \mathbb{N}^n,$$

(or, equivalently, by its values on polynomials; see (1.1) for the notation $t^j, j \in \mathbb{N}^n$). Connections of the moment problem with other fields, such as operator theory, fixed point theory, algebra, polynomial approximation, optimization, complex functions, are emphasized by means of the corresponding references. Applications of various moments in physics and probabilities theory are briefly reviewed in the Introduction of the monograph [71]. To conclude, for characterizing the existence of a solution of a classical moment problem in terms of moments, extension Hahn-Banach results and their generalizations accompanied by knowing the analytic form of positive polynomial on the set under discussion are the basic tools. Sometimes, especially in Markov moment problem, the uniqueness of the (continuous linear) solution follows too, by the proof of its existence, thanks to the density of polynomials in some function spaces (even on an unbounded closed subset). In this respect, for both existence and uniqueness of the solution, polynomial approximation on unbounded subsets is essential. Otherwise, the uniqueness problem requires specific methods (see [6], [71], [75], [76]). The rest of this chapter is organized as follows. Section 2 is devoted to recalling various results and methods on Hahn-Banach type extension for linear functionals and operators, preserving positivity or sandwich conditions, which are applied in the sequel. We start this discussion with geometric form of Hahn-Banach theorem, which leads rapidly to Krein-Milman theorem. In Section 3, earlier and recent results on existence and uniqueness of the solution for the classical moment problem are reviewed. Section 4 deals with moment problems and Mazur-Orlicz theorems in concrete spaces. Some of the solutions are Markov operators. Generally, the norms of the solutions can be determined by means of the norms of the bounded sublinear upper constraints. Section 5 is devoted to polynomial approximation on unbounded subsets and its applications to the Markov moment problem. As an application, the positivity of some continuous linear operators is characterized only in terms of quadratic forms (see Theorem 5. 13). From the point of view of this result, the difficulty created by the fact that on $\mathbb{R}^n, n \geq 2$, there exist polynomials which are not sums of squares is solved.

2. Hahn-Banach type theorems, abstract Markov moment problems and Mazur-Orlicz theorem in ordered vector spaces setting

Almost all the results of this section will be applied in the sequel. Some of them have been published first in [45], using earlier results on extension of linear operators preserving two constraints published in [42], [44]. Detailed proofs can be found in [44]. Before going on with the above-mentioned results, which are valid for operators, we review the geometric form of Hahn-Banach theorem. This variant is useful for some of the results stated in the next chapters. The following lemma is the key result for the direct proof of the geometric version of Hahn-Banach theorem (see [69], p. 45-46).

Lemma 2.1. (See [69]). *Let X be a real topological vector space (t.v.s.) of dimension at least 2. If D is an open convex subset not containing $\mathbf{0}$, there exists a one-dimensional subspace of X not intersecting D.*

Lemma 2.1 and a standard application of Zorn's lemma yield the next result.

Theorem 2.2. (See [69]). *Let X be a real t.v.s., let M be a linear manifold in X, and let D be a non-empty open convex subset of X, not intersecting M. Then there exists a closed hyperplane H in X, containing M and not intersecting D.*

Corollary 2.3. *Let E be a t.v.s., C an open convex subset of E, E_1 a vector subspace of E such that $E_1 \cap C \neq \emptyset, T_1 \in L(E_1, \mathbb{R})$ a continuous linear functional, $P: C \to \mathbb{R}$ a convex upper semi-continuous functional such that $T_1(x) \leq P(x)$ for all $x \in E_1 \cap C$. Then there exists a continuous linear functional $T \in L(E, \mathbb{R})$ which extends T_1, such that $T(x) \leq P(x)$ for all $x \in C$.*

To deduce Corollary 2.3 from Theorem 2.2 one applies the latter statement, where X stands for $E \times \mathbb{R}$, M stands for the graph of T_1 $\left(M = \{(x, T_1(x)); x \in E_1\}\right), D$ stands for $\{(x,t) \in C \times \mathbb{R}; P(x) < t\}$. According to Theorem 2.2, there exists a closed hyperplane H in $E \times \mathbb{R}$ which contains M, such that $H \cap D = \emptyset$. Due to condition $E_1 \cap C \neq \emptyset, H$ cannot be vertical, hence is the graph of a linear functional $T \in L(E, \mathbb{R})$. From the details of this sketch of the proof, it is easy to observe that T extends $T_1, T(x) \leq P(x), x \in C$ and T is continuous (and linear) from E to $\mathbb{R}$ (see also [69, Exercise 6 , p. 69]).

The next result holds in locally convex spaces. All such spaces are assumed to be Hausdorff.

Theorem 2.4. (See [69, Theorem 4.2, p. 49]). *Let X be a t.v.s., whose topology is locally convex. If T_1 is a linear form, defined and continuous on a subspace M of X, then T_1 has a continuous extension T to the entire space X.*

Corollary 2.5. *Given $n \in \{1,2,\dots\}$ and n linearly independent elements x_ν of a l.c.s. X, there exist n continuous linear forms T_μ on X such that $T_\mu(x_\nu) = \delta_{\mu\nu}, (\mu, \nu = 1, \dots, n)$.*

The next result is basic in finite dimensional convex analysis due to its applications, including the maximum principle for convex functions.

Theorem 2.6. (Carathéodory; see [38] and/or [65]). *Let $K \subset \mathbb{R}^n$ $(n \in \{1,2,\dots\})$ be a convex compact subset. Then any $x \in K$ can be written as convex combination of at most $n+1$ extreme points of K.*

A simple proof of Theorem 2.6 (by induction on the dimension n) is given in [65, p. 7-8], essentially using Theorem 2.2 stated above. Here is a main application of Theorem 2.6 to convex optimization (in particular, to linear optimization).

Corollary 2.7. (See [69, Exercise 26, p. 71]). *Let $K \subset \mathbb{R}^n$ be a nonempty compact subset. Then its convex hull $co(K)$ is compact.*

Theorem 2.8. (See [38, p. 171). *If f is a continuous convex real function on a convex compact subset $K \subset \mathbb{R}^n$ $(n \in \{1,2,\dots\})$, then f attains a global maximum at an extreme point of K.*

Theorem 2.9. (The maximum principle [38]). *Let C be a convex subset of $\mathbb{R}^n$. If a convex function $f: C \to \mathbb{R}$ attains its maximum on C at a point from the relative interior of C, then f is constant on C.*

Next, we recall the following basic results, derived from Theorem 2.2.

Theorem 2.10. (First Separation Theorem [69]). *Let A be a convex subset of a t.v.s. X, such that $int(A) \neq \emptyset$ and let B be a nonempty convex subset of X, not intersecting the interior $int(A)$ of A. There exists a closed hyperplane H separating A and B; if A and B are both open, H separates A and B strictly.*

Theorem 2.11. (Second Separation Theorem [69]). *Let A, B be nonempty, disjoint convex subsets of a locally convex Hausdorff space (l.c.s.) X, such that A is closed and B is compact. There exists a closed hyperplane in X strictly separating A and B.*

Corollary 2.12. *Let X be a l.c.s. and $x_1, x_2 \in X, x_1 \neq x_2$. Then there exists a continuous linear functional $x^\star \in X^\star$ such that $x^\star(x_1) \neq x^\star(x_2)$.*

The preceding corollary says that the topological dual $X^\star$ of a l.c.s. X separates the points of X. On the other hand, by the definition of weak topology on a l.c.s. X, any weak closed subset of X is closed in the initial topology on X. For convex closed subsets, the reverse implication holds as well. Namely, we recall the following well-known consequence of Theorem 2.11:

Corollary 2.13. (See [69]) *Let X be a locally convex space and $C \subset X$ a convex closed subset. Then C is the intersection of all closed half-spaces containing it. In particular, C is closed with respect to the weak topology $w(X, X^\star)$ on X.*

The next key lemma is used in the proof of the main Theorem 2.15 (Krein-Milman).

Lemma 2.14. (See [69]). *If C is a compact, convex subset of a locally convex space, every closed hyperplane supporting C contains at least one extreme point of C.*

We recall that, by definition a closed hyperplane H in the locally convex space X under attention is supporting C if $C \cap H \neq \emptyset$ and C is contained in one of the two half-spaces defined by H. A point $e \in C$ is called an extreme point of C if from $x_1, x_2 \in C, t \in (0,1)$, the equality $e = (1-t)x_1 + tx_2$ implies $x_1 = x_2 = e$. In other words, e cannot be an interior element of any line segment of ends elements of C.

Theorem 2.15. (Krein-Milman; see [69]). *Every compact convex subset of a locally convex space is the closed convex hull of its extreme points.*

Krein-Milman theorem says that in any compact convex subset C of a l.c.s. there are many extreme points, which generate C (any element of C is the limit of a net whose elements are convex combinations of extreme points of C).

Theorem 2.16. (See [69, Theorem 10.5, p. 68]). *If K is a compact subset of a locally convex space such that the closed convex hull C of K is compact, then each extreme point of C is an element of K.*

From Theorem 2.6 (Carathéodory), Corollary 2.7 and Theorem 2.16, the following consequence follows:

Corollary 2.17. *If $K \subset \mathbb{R}^n$ is a compact nonempty subset, then its convex hull $co(K)$ is compact and $co(K) = co(Extr(K))$. Moreover, each point of $co(K)$ can be written as convex combination of at most $n+1$ extreme points of K.*

The above results are deduced from the geometric form of Hahn-Banach theorem. In most of cases, motivated by further applications, analytic proofs of Hahn-Banach type theorems are more suitable. Here is a first main result, completely proved in [38, p. 339-340].

Theorem 2.18. (The Hahn-Banach theorem). *Let X be a vector space, $P: X \to \mathbb{R}$ a sublinear functional, $M \subset X$ a vector subspace $L: M \to \mathbb{R}$ a linear functional, such that $L(x) \leq P(x)$ for all $x \in M$. Then L has a linear extension $T: X \to \mathbb{R}$, such that T is dominated by P on the entire space X.*

Corollary 2.19. (See [38, p. 340]) *If P is a sublinear functional on a real vector space X, then for every element $x_0 \in X$ there exists a linear functional T such that*

$$T(x_0) = P(x_0) \text{ and } T(x) \leq P(x) \text{ for all } x \in X.$$

Theorem 2.20. (The Hahn-Banach theorem on normed vector spaces; see [38]). *Let X_0 be a vector subspace of the real normed vector space X and $T_0: X_0 \to \mathbb{R}$ a continuous linear functional. Then T_0 has a continuous linear extension $T: X \to \mathbb{R}$, with $\|T\| = \|T_0\|$.*

Corollary 2.21. (See [38]). *If X is normed vector space, then for each $x_0 \in X, x_0 \neq \mathbf{0}$, there exists a linear functional T on X, such that $T(x_0) = \|x_0\|$, and $\|T\| = 1$.*

One of the reasons of using analytic proofs of Hahn-Banach type theorems is that they work not only for extending linear functionals, but also for operators. As in the case of functional, the proofs of such type results are quite simple, by means of Zorn's Lemma and extension of linear operators from a subspace S of the involved domain space X, to a space $S \oplus$

$Span\{x_0\}$, where $x_0 \in X \backslash S$, preserving some constraints on the extension. The codomains of the operators for which Hahn-Banach type theorems hold must be order complete vector spaces, or even order complete vector lattices. We recall that an ordered vector space is a vector space Y endowed with an order relation which is compatible with the algebraic structure of a vector space. Namely, the following two properties are satisfied:

$$y_1 \leq y_2, y \in Y \Rightarrow y_1 + y \leq y_2 + y,\ y_1 \leq y_2, \alpha \in \mathbb{R}_+ \Rightarrow \alpha y_1 \leq \alpha y_2.$$

We say that such an order relation is linear. If Y is an ordered vector space, then $Y_+ = \{y \in Y; y \geq \mathbf{0}\}$ is a convex cone, called the positive cone of Y. We always assume that the positive cone is generating ($Y = Y_+ - Y_+$). An ordered vector space Y is called order complete (Dedekind complete) if for any upper bounded subset $B \subset Y$, there exists a least upper bound for B in Y, denoted by $sup(B)$. A vector lattice is an ordered vector space Y with the property that for any $y_1, y_2 \in Y$, there exists $sup\{y_1, y_2\} \in Y$. In a vector lattice Y, for any element $y \in Y$ one denotes $|y| = sup\{y, -y\}$. An ordered Banach space is a Banach space Y which is also an ordered vector space, such that the positive cone Y_+ is closed and the norm is monotone on Y_+:

$$0 \leq y_1 \leq y_2 \Rightarrow \|y_1\| \leq \|y_2\|.$$

A Banach lattice Y is a Banach space, which is also a vector lattice, such that

$$y_1, y_2 \in Y, |y_1| \leq |y_2| \Rightarrow \|y_1\| \leq \|y_2\|.$$

Obviously, any Banach lattice is an ordered Banach space. In an ordered Banach space, there exists also a compatibility of the topology defined by the norm with the order relation. There exist ordered Banach spaces which are not lattices. For example, the space Y of all $n \times n$ symmetric matrixes with real entries, endowed with the norm

$$\|V\| = \max_{\|x\| \leq 1} |< Vx, x >|$$

and the order relation $V \preccurlyeq W \Leftrightarrow < Vx, x > \leq < Wx, x >$ for all $x \in \mathbb{R}^n, V, W \in Y$, is an ordered Banach space which is not a lattice for $n \geq 2$. Here the norm $\|x\|$ is the Euclidean norm of the vector $x \in \mathbb{R}^n$. In the same way, if H is a real or complex Hilbert space, the real vector space $Y = \mathcal{A}(H)$ of all self-adjoint operators acting on H, with the norm and order relation defined similarly to the case of symmetric matrixes, is an ordered Banach space which is not a lattice (here $\mathbb{R}^n$ is replaced by H). Almost all usual function spaces and sequence spaces have natural structures of Banach lattices. On a vector space $\mathcal{F}(S)$ of real valued functions defined on a set S, the usual order relation is: $f \le g \Leftrightarrow f(t) \le g(t)$ for all $t \in S$. For example, if S is a compact Hausdorff topological space, the space $C(S)$ of all real valued continuous functions over S, is a Banach lattice with respect to the above defined order relation and usual norm. If we additionally assume that S is connected and contains at least two different points, then $C(S)$ is not order complete. A particular such a Banach lattice is $C([0,1])$. The Lebesgue spaces $L^p(F), 1 \le p \le \infty, F \subseteq \mathbb{R}^n$, and the sequence spaces $l^p, 1 \le p \le \infty$, are order complete Banach lattices.

Here is one of the old results on this subject, with many applications to the vector valued moment problem. Let X_1 be an ordered vector space whose positive cone $X_{1,+}$ is generating ($X_1 = X_{1,+} - X_{1,+}$). Recall that in such an ordered vector space X_1 a vector subspace S is called a majorizing subspace if for any $x \in X_1$ there exists $s \in S$ such that $x \le s$. The following theorem holds true.

Theorem 2.22. (See [30, Theorem 1.2.1]). *Let X_1 be an ordered vector space whose positive cone is generating, $X_0 \subset X_1$ a majorizing vector subspace, Y an order complete vector space, $T_0: X_0 \to Y$ a positive linear operator. Then T_0 admits a positive linear extension $T: X_1 \to Y$.*

We go on with Hahn-Banach type theorems. Now a condition on the operator solution of being dominated by a convex operator defined on a convex subset of the domain space is required. In other words, a generalized Hahn-Banach theorem will be reviewed. The relationship between the next result and its corollary (existence of subgradients of convex operators) will appear clearly. A point x_0 of the subset A of a vector space X is called an (algebraic) interior point of A if for each $x \in X$ there is a positive λ_0 such that $\lambda x + (1-\lambda)x_0 \in A$ for $|\lambda| \le \lambda_0$. The point x_0 is said to be an (algebraic) relative interior point of A if for each x of the affine variety generated by A (affine hull of A) there is a positive λ_0 such that $\lambda x + (1-\lambda)x_0 \in A$ for $|\lambda| \le \lambda_0$. The set of all interior points of A is denoted by A^{int} and the set of all relative interior points by A^{ri}. For the next result see [81, Theorem 2.1, p. 284-286].

Theorem 2.23. (A generalized Hahn-Banach theorem; see [81]). *Let X be a vector space, $M \subset X$ a vector subspace, Y an order complete vector space, $A \subseteq X$ a convex subset, $P: A \to Y$ a convex operator, $T_M: M \to Y$ a linear operator such that*

$$T_M(x) \le P(x) \textit{ for all } x \in M \cap A.$$

If $A^{int} \cap M \neq \emptyset$*, then there exists a linear operator* $T: X \to Y$ *such that*

$$T(x) = T_M(x) \textit{ for all } x \in M \textit{ and } T(x) \le P(x) \textit{ for all } x \in A.$$

Corollary 2.24. (See [81, Corollary 2.7, p. 286]). *Let* X *be a vector space,* Y *an order complete vector space,* $A \subseteq X$ *a convex subset,* $P: A \to Y$ *a convex operator. If* $x_0 \in A^{ri}$*, then there exists a linear operator* $T: X \to Y$ *such that*

$$T(x) - T(x_0) \le P(x) - P(x_0) \textit{ for all } x \in A. \quad (1.3)$$

A linear operator T satisfying (1.3) is called a subgradient of P at x_0. Corollary 2.24 says that a convex operator having as codomain an order complete vector space possesses a subgradient at every relative interior point of its domain. This result (with a somewhat different proof) goes back to [77]. The set of all subgradients of P at x_0 is called the subdifferential of P at x_0 and is denoted by $\partial_{x_0}P$. This is a convex set, and, for convex operators P satisfying the hypothesis of Corollary 2.24, is nonempty.

In the results stated above, the order relation who naturally exists on concrete spaces does not appear on the domain space X in any way. The next results take into consideration linear order structures on X as well. This way, from now on, we have three conditions on the linear operator solution T. Namely T must extend a given linear operator defined on a subspace of X, it is dominated by a given convex operator P and dominates a given concave operator Q. If $Q|_{X_+} \ge \mathbf{0}$, then the linear extension T is positive: $x \in X_+ \Longrightarrow T(x) \in Y_+$. Recall that an ordered vector space X which is also a topological vector space is called an ordered topological vector space if the positive cone X_+ is topologically closed. The next result is due to H. Bauer and independently to I. Namioka (for citation of the original sources see [69, p. 227]).

Theorem 2.25. (See [69, Theorem 5.4, p. 227]). *Let* X *be an ordered t.v.s. with positive cone* X_+ *and* M *a vector subspace of* X*. For a linear form* T_0 *on* M *to have a linear continuous positive extension* $T: X \to \mathbb{R}$ *it is necessary and sufficient that* T_0 *be bounded above on* $M \cap (U - X_+)$*, where* U *is a suitable convex* $\mathbf{0}$ *−neighborhood in* X*.*

The next result is motivated by Theorem 2.25 and the discussion preceding it. In the sequel, all theorems are valid for operators. In particular, the corresponding cases of real valued functionals follow as consequences. In the next theorem, X will be a real vector space, Y an order-complete vector lattice, $A, B \subseteq X$ convex subsets, $Q: A \to Y$ a concave operator, $P: B \to Y$ a convex operator, $M \subset X$ a vector subspace, $T_0: M \to Y$ a linear operator. All vector spaces and linear operators are considered over the real field.

Theorem 2.26. (See [42, Theorem 1]). *Assume that* $T_0(x) \geq Q(x)\ \forall x \in M \cap A, T_0(x) \leq P(x)\ \forall x \in M \cap B$. *The following two statements are equivalent:*

(a) *there exists a linear extension* $T: X \to Y$ *of the operator* T_0 *such that*

$$T|_A \geq Q, T|_B \leq P;$$

(b) *there exists* $P_1: A \to Y$, *convex, and* $Q_1: B \to Y$ *concave operator such that for all*

$$(\rho, t, \lambda, a_1, a, b_1, b, v) \in [0,1]^2 \times (0, \infty) \times A^2 \times B^2 \times M,$$

the following implication holds:

$$(1-t)a_1 - tb_1 = v + \lambda\big((1-\rho)a - \rho b\big) \Rightarrow$$

$$(1-t)P_1(a_1) - tQ_1(b_1) \geq T_0(v) + \lambda\big((1-\rho)Q(a) - \rho P(b)\big).$$

Notable, the extension T of Theorem 2.26 satisfies the following conditions: is an extension of T_0, it is dominated by P on B and it dominates Q on A. Here the convex subsets A, B are arbitrary, with no restriction on the existence of relative interior points or on their position with respect to the subspace M. The following theorems follow more or less directly as corollaries of Theorem 2.26. For details and applications to the abstract Markov moment problem see [42], [44], [45]. For applications to characterizing the isotonicity of a convex operator over a convex cone see [39]. The same article [39] contains a large class of examples of concrete spaces and operators for which the developed theory works. From Theorem 2.26 we obtain.

Theorem 2.27. (See [42], [44]). *Let* X *be an ordered vector space,* Y *an order complete vector space,* $M \subset X$ *a vector subspace,* $T_1: M \to Y$ *a linear operator,* $P: X \to Y$ *a convex operator. The following statements are equivalent:*

(a) *there exists a positive linear extension* $T: X \to Y$ *of* T_1 *such that* $T \leq P$ *on* X;
(b) *we have* $T_1(h) \leq P(x)$ *for all* $(h, x) \in M \times X$ *such that* $h \leq x$.

One observes that in the very particular case $E_+ = \{\mathbf{0}\}$, when the order relation on E is the equality, from Theorem 2.27 one obtains the Hahn-Banach extension theorem for linear operators dominated by convex operators stated below.

Corollary 2.28. *Let X be a vector space, Y an order complete vector space,* $M \subset X$ *a vector subspace,* $T_1: M \to Y$ *a linear operator,* $P: X \to Y$ *a convex operator. Assume that* $T_1(x) \leq P(x)$ *for all* $x \in M$. *Then there exists a linear extension* $T: X \to Y$ *of* T_1 *such that* $T(x) \leq P(x)$ *for all* $x \in X$.

Theorem 2.27 is equivalent to the following result, formulated in the abstract Markov moment setting.

Theorem 2.29. (See [45, Theorem 1]). *Let X be a preordered vector space, Y an order complete vector lattice,* $P: X \to Y$ *a convex operator,* $\{x_j\}_{j \in J} \subset X, \{y_j\}_{j \in J} \subset Y$ *given families. The following statements are equivalent:*

(a) *there exists a linear positive operator* $T: X \to Y$ *such that*

$$T(x_j) = y_j, j \in J, T(x) \leq P(x), x \in X;$$

(b) *for any finite subset* $J_0 \subseteq J$, *and any* $\{\lambda_j; j \in J_0\} \subseteq \mathbb{R}$, , *we have*

$$\sum_{j \in J_0} \lambda_j x_j \leq x \in X \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq P(x).$$

If we additionally assume that P is isotone $\left(u \leq v \Rightarrow P(u) \leq P(v)\right)$, *the assertions (a) and (b) are equivalent to (c), where:*

(c) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}$, *the following inequality holds:*

$$\sum_{j \in J_0} \lambda_j y_j \leq P\left(\sum_{j \in J_0} \lambda_j x_j\right).$$

When the convex operator P is defined only on the positive cone of X, one obtains the following variant of Theorem 2.27 (see [44]):

Theorem 2.30. (See [44], [59]). *Let X be an ordered vector space, Y an order complete vector space, $M \subset X$ a vector subspace, $T_1\colon M \to Y$ a linear operator, $P\colon X_+ \to Y$ a convex operator. The following statements are equivalent:*

(a) *there exists a positive linear extension $T\colon X \to Y$ of T_1 such that $T|_{X_+} \leq P$;*
(b) *we have $T_1(h) \leq P(x)$ for all $(h, x) \in M \times X_+$ such that $h \leq x$.*

Proof. The implication $(a) \Longrightarrow (b)$ is obvious; indeed, we have

$$T_1(h) = T(h) \leq T(x) \leq P(x),$$

for all $(h, x) \in M \times X_+$ such that $h \leq x$, thanks to the positivity of T, also using the property $T(x) \leq P(x)$ for all $x \in E_+$. To prove the converse, let P be an arbitrary convex operator over X_+, verifying the conditions mentioned at (b). We are going to apply Zorn lemma to the set $\mathcal{L}$ of all pairs (K, T_K), where K is a vector subspace of $X, M \subset K, T_K\colon K \to Y$ is a linear operator such that $T_K|_H = T_1$ and $T_K(h) \leq P(x)$ for all $h \in K$ and $x \in X_+$ such that $h \leq x$. The set $\mathcal{L}$ contains the pair (M, T_1) and is inductively ordered by the order relation

$$(K, T_K) \ll (Z, T_Z) \Leftrightarrow K \subset Z, T_Z|_K = T_K\,.$$

According to Zorn's Lemma, there exists a maximal pair $\left(H_M, T_{H_M}\right) \in \mathcal{L}$. Our aim is to prove that $H_M = X$. Assuming this is done, and taking $T \coloneqq T_{H_M}$, we have $T(h) \leq P(x)$ for all $h \in M$ and $x \in E_+$ such that $h \leq x$. Application of this inequality for $h = -ny, y \in X_+, n \in \mathbb{N}, x = 0$, yields

$$nT(-y) = T(-ny) \leq P(\mathbf{0}), n \in \mathbb{N}.$$

Since any order complete vector space is Archimedean, it results $T(-y) \leq \mathbf{0}, y \in X_+$, that is the positivity of T. Also, taking $h = x \in X_+$, one obtains $T(x) \leq P(x)$ and T will be the expected positive extension of $T_1, T|_{X_+} \leq P$. This will end the proof. If $H_M \neq X$, then we can choose $v_0 \in X \backslash H_M$ and define $H_0 = H_M \oplus \mathbb{R} v_0, T_0\colon H_0 \to Y$,

$$T_0(h + rv_0) = T_{H_M}(h) + ry_0\,, h \in H_M, r \in \mathbb{R},$$

where $y_0 \in Y$ will be chosen such that $\left(H_M, T_{H_M}\right) \ll (H_0, T_0)$ and $(H_0, T_0) \in \mathcal{L}$. This will contradict the maximality of $\left(H_M, T_{H_M}\right)$ in $\mathcal{L}$. Thus $H_M = X$. To prove that $(H_0, T_0) \in \mathcal{L}$ for suitable $y_0 \in Y$, we have to show that

$$h \in H_M, r \in \mathbb{R}, h + rv_0 \leq x \in X_+ \Rightarrow T_{H_M}(h) + ry_0 \leq P(x).$$

For $r = \alpha > 0$, multiplying by α^{-1} the relation $h_1 + \alpha v_0 \leq x_1 \in X_+$, the above implication becomes:

$$h_1 + \alpha v_0 \leq x_1 \in X_+ \Rightarrow y_0 \leq \alpha^{-1}\left(P(x_1) - T_{H_M}(h_1)\right), \alpha > 0,$$

and, similarly,

$$h_2 + \beta v_0 \leq x_2 \in X_+ \Rightarrow y_0 \geq \beta^{-1}\left(P(x_2) - T_{H_M}(h_2)\right), \beta < 0.$$

To have both conditions on y_0 verified, according to order completeness of Y, it is necessary and sufficient to prove that

$$\beta^{-1}\left(P(x_2) - T_{H_M}(h_2)\right) \leq \alpha^{-1}\left(P(x_1) - T_{H_M}(h_1)\right).$$

The last inequality may be written as

$$T_{H_M}(\alpha^{-1}h_1 - \beta^{-1}h_2) \leq \alpha^{-1}P(x_1) - \beta^{-1}P(x_2)$$

To prove this last inequality, we eliminate v_0 by adding the previous inequalities, multiplied by $\alpha^{-1} > 0$, respectively by $-\beta^{-1} > 0$, as follows:

$$(\alpha^{-1}h_1 + v_0 \leq \alpha^{-1}x_1, -\beta^{-1}h_2 - v_0 \leq -\beta^{-1}x_2) \Rightarrow$$

$$\alpha^{-1}h_1 - \beta^{-1}h_2 \leq \alpha^{-1}x_1 - \beta^{-1}x_2 \in X_+,$$

where $h_j \in H_M, x_j \in X_+, j = 1,2, \alpha > 0, \beta < 0$. Since $\left(H_M, T_{H_M}\right) \in \mathcal{L}$ and P is convex, these further yields:

$$\frac{1}{\alpha^{-1} - \beta^{-1}} T_{H_M}(\alpha^{-1}h_1 - \beta^{-1}h_2)$$
$$= T_{H_M}\left(\frac{\alpha^{-1}}{\alpha^{-1} - \beta^{-1}} h_1 + \frac{-\beta^{-1}}{\alpha^{-1} - \beta^{-1}} h_2\right) \leq$$

$$P\left(\frac{\alpha^{-1}}{\alpha^{-1}-\beta^{-1}}x_1+\frac{-\beta^{-1}}{\alpha^{-1}-\beta^{-1}}x_2\right)\le$$

$$\frac{\alpha^{-1}}{\alpha^{-1}-\beta^{-1}}P(x_1)+\frac{-\beta^{-1}}{\alpha^{-1}-\beta^{-1}}P(x_2).$$

Thus, the expected inequality follows, and the proof is complete. □

The next result provides a sufficient condition on the given linear operators for the existence of the linear extensions. When $X=\mathbb{R}^2, Y=\mathbb{R}$, it has an interesting geometric meaning.

Theorem 2.31. (See [44]). *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S\subset X$ a vector subspace. Let $A\subset X$ be a convex subset with the following properties:*

(a) *there exists a neighborhood V of the origin such that* $(S+V)\cap A=\emptyset$; (*that is, by definition, A and S are distanced*).
(b) *A is bounded.*

For any equicontinuous family of linear operators $\{f_j\}_{j\in J}\subset\mathcal{L}(S,Y)$ and for any $\tilde{y}\in Y_+\setminus\{\mathbf{0}\}$, there exists an equicontinuous family $\{T_j\}_{j\in J}\subset\mathcal{L}(X,Y)$ such that

$$T_j(s)=f_j(s), s\in S,\qquad T_j(\psi)\ge\tilde{y}, \psi\in A, j\in J.$$

Moreover, if V is a convex balanced neighbourhood of the origin such that

$$f_j(\mathrm{V}\cap\mathrm{S})\subset[-u_0,u_0],\qquad (S+V)\cap A=\emptyset,$$

and if $\alpha>0$ is such that $P_V(a)\le\alpha\ \forall a\in A$ and $\alpha_1>0$ is large enough such that $\tilde{y}\le\alpha_1 u_0$, then the following relations hold

$$T_j(x)\le(1+\alpha+\alpha_1)P_V(x)u_0,\qquad x\in X, j\in J.$$

We have denoted by P_V the gauge attached to V.

The next result was published in the following version in [45, Theorem 4]. It can be regarded as a generalization of a result of M.G. Krein [29] to arbitrary infinite set of moment interpolation conditions and to vector valued mappings. This result is formulated in terms of the abstract Markov moment problem.

Theorem 2.32. (See [45]). *Let X be an ordered vector space, Y an order complete vector lattice, $\{\varphi_j\}_{j\in J} \subset X, \{y_j\}_{j\in J} \subset Y$ given arbitrary families, $T_1, T_2 \in L(X,Y)$ two linear operators. The following statements are equivalent:*

(a) *there is a linear operator $T \in L(X,Y)$ such that*

$$T_1(x) \leq T(x) \leq T_2(x) \;\; \forall x \in X_+, T(\varphi_j) = y_j \;\; \forall j \in J;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, the following implication holds true*

$$\left(\sum_{j\in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1, \psi_1, \psi_2 \in X_+\right) \Rightarrow \sum_{j\in J_0} \lambda_j y_j \leq T_2(\psi_2) - T_1(\psi_1).$$

If X is a vector lattice, then assertions (a) and (b) are equivalent to (c), where:

(c) $T_1(w) \leq T_2(w)$ *for all $w \in X_+$ and for any finite subset $J_0 \subset J$ and $\forall\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, we have*

$$\sum_{j\in J_0} \lambda_j\, y_j \leq T_2\left(\left(\sum_{j\in J_0} \lambda_j \varphi_j\right)^+\right) - T_1\left(\left(\sum_{j\in J_0} \lambda_j \varphi_j\right)^-\right)$$

The following theorem is also a Hahn-Banach- type result (see Theorem 2.29 stated above) but is formulated in terms which are similar to those of the abstract Markov moment problem [45]. However, the condition $T(x_j) = y_j, j \in J$ of the abstract moment problem (Theorem 2.29 stated above) is replaced by $T(x_j) \geq y_j, j \in J$. Consequently it is sufficient that the implication mentioned at point (b) holds only for nonnegative scalars λ_j.

Theorem 2.33. (Mazur-Orlicz: see [45, Theorem 5]). *Let X be a preordered vector space, Y an order complete vector space, $\{x_j\}_{j\in J}, \{y_j\}_{j\in J}$ families of elements in X, respectively in Y, $P: X \to Y$ a sublinear operator. The following statements are equivalent:*

(a) *there exists a linear positive operator* $T: X \to Y$ *such that*

$$T(x_j) \geq y_j, j \in J, T(x) \leq P(x), x \in X;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}_+ = [0, \infty)$, *the following implication holds true*

$$\sum_{j \in J_0} \lambda_j x_j \leq x \in X \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq P(x).$$

If in addition we assume that P is isotone, the assertions (a) and (b) are equivalent to (c), where:

(c) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}_+$, *the following inequality holds*

$$\sum_{j \in J_0} \lambda_j y_j \leq P\left(\sum_{j \in J_0} \lambda_j x_j\right).$$

In the end of this section, we state a general constrained extension result which can be proved as a consequence of Theorem 2.26. Probably Theorems 2.26 and 2.34 are equivalent.

Theorem 2.34. (See [44]). *Let X be a vector space, Y be an order complete vector lattice,* $M \subset X$ *a vector subspace,* $T_0: M \to Y$ *a linear operator,* $A \subseteq X$ *a convex subset,* $Q: A \to Y$ *a concave operator. Assume that* $T_0(x) \geq Q(x)\ \forall x \in M \cap A$. *The following statements are equivalent:*

(a) *There exists a linear operator* $T: X \to Y$ *which extends* T_0, *such that* $T|_A \geq Q$.
(b) *There exists a convex operator* $P: A \to Y$ *such that for all* $(x, r, a) \in M \times (0, \infty) \times A$ *the following implication holds:*

$$x + ra \in A \Rightarrow T_0(x) + rQ(a) \leq P(x + ra).$$

Moreover, if P satisfies the requirements of (b), then the extension T of (a) verifies the relation $T|_A \leq P$.

It seems that the general Theorems 2.26 and 2.34 are equivalent. On the other hand, since all concrete spaces are endowed with a natural linear order relation, we restate Theorem 2.34 in the framework of ordered vector spaces.

Theorem 2.35. *Let X be an ordered vector space, let Y be an order complete vector lattice, $M \subset X$ a vector subspace, $T_0: M \to Y$ a linear operator, $Q: X_+ \to Y$ a supralinear operator, $P: X_+ \to Y$ a convex operator. The following statements are equivalent:*

(a) *There exists a linear operator $T: X \to Y$ which extends T_0, such that $Q \le T|_{X_+} \le P$.*
(b) *For all $(h, \varphi_1, \varphi_2) \in M \times X_+ \times X_+$, the following implication holds*:

$$h = \varphi_2 - \varphi_1 \Longrightarrow T_0(h) \le P(\varphi_2) - Q(\varphi_1).$$

The next consequence of Theorem 2.35 is a sandwich type result. It can be obtained from Theorem 2.35, applied to $M = \{\mathbf{0}\}, T_0 = \mathbf{0}$.

Corollary 2.36. *Let X, Y, P, Q be as in the statement of Theorem 2.35. Assume that $Q \le P$ on X_+. Then there exists a linear operator $T: X \to Y$, such that $Q \le T|_{X_+} \le P$.*

The next two variants of the same controlled regularity property of some linear operators are also consequence of Theorem 2.26. Recall that a linear operator T is called regular if it can be written as a difference of two positive linear operators V, W: $T = V - W$. If V is dominated by a given convex operator P, we say that we have a controlled regularity for T. This terminology is motivated by the fact that in the topological framework, P is assumed to be continuous and $V \le P$ on the entire domain space usually implies the continuity of V. Sometimes, the norm of V can be evaluated or determinate as well.

Theorem 2.37. (See [44]). *Suppose that X is an ordered vector space, Y is an order complete vector lattice and $P: X_+ \to Y$ is a convex operator. Then for any linear operator $T: X \to Y$ the following two statements are equivalent:*

(a) *there exist positive linear operators $V, W: X \to Y$ such that*

$$T = V - W, V|_{X_+} \le P.$$

(b) $T(x_1) \le P(x_2)$ *for all x_1, x_2 in X such that $\mathbf{0} \le x_1 \le x_2$.*