# Multiplicative Euclidean and Non-Euclidean Geometry

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By

Svetlin G. Georgiev

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# Preface

Differential and integral calculus, the most applicable mathematical theory, was created independently by Isaac Newton and Gottfried Wilhelm Leibniz in the second half of the 17th century. Two operations, differentiation and integration, are basic in calculus and analysis. In fact, they are the infinitesimal versions of the subtraction and addition operations on numbers, respectively. In the period from 1967 till 1970 Michael Grossman and Robert Katz gave definitions of a new kind of derivative and integral, moving the roles of subtraction and addition to division and multiplication, and thus established a new calculus, called multiplicative calculus. Sometimes, it is called an alternative or non-Newtonian calculus as well. Multiplicative calculus can especially be useful as a mathematical tool for economics and finance.

This book is devoted to the multiplicative Euclidean and non-Euclidean geometry. It summarizes the most recent contributions in this area. The book is intended for senior undergraduate students and beginning graduate students of engineering and science courses. The book contains eight chapters. The chapters in the book are pedagogically organized. Each chapter concludes with a section with practical problems.

Chapter 1 introduces the field $\mathbb{R}_\star$ and defines the basic multiplicative arithmetic operations: multiplicative addition, multiplicative subtraction, multiplicative multiplication and multiplicative division and some of their properties are deduced. The basic elementary multiplicative functions are defined and studied. Chapter 2 investigates the multiplicative vector space $\mathbb{R}^2_\star$, the multiplicative inner product space and the multiplicative Euclidean space $E^2_\star$. Multiplicative lines are defined and their equations are deduced. Perpendicular, parallel and intersecting multiplicative lines are introduced. Multiplicative isometries, multiplicative translations, multiplicative rotations and multiplicative glide reflections are investigated. Chapter 3 deals with multiplicative affine transformations, multiplicative affine reflections, multiplicative affine symmetries, multiplicative shears, multiplicative dilatations, and multiplicative similarities and some of their properties are investigated. Multiplicative segments, multiplicative angles and multiplicative rectilinear figures are defined. Some criteria for existence of multiplicative affine transformations that leave multiplicative lines and multiplicative points fixed are given. A multiplicative barycentric coordinate system is introduced and some of its applications are given. Some criteria for congruence of multiplicative angles and triangles are deduced. Chapter 4 is devoted to cyclic and dihedral subgroups of $O_\star(e^2)$ and some of their properties are investigated. Conjugate subgroups, orbits and stabilizers are introduced. In the chapter, regular multiplicative polygons are defined. Chapter 5 introduces multiplicative spheres and multiplicative lines on multiplicative spheres, multiplicative reflections on multiplicative spheres and multiplicative rectilinear figures on multiplicative spheres and some of their properties are deduced. Chapter 6 investigates the projective multiplicative plane $P_\star^2$ and multiplicative perpendicular points, multiplicative lines, multiplicative perpendicular multiplicative lines, multiplicative poles, multiplicative polarities, multiplicative conics, multiplicative tangents, multiplicative secants, and a multiplicative cross product are defined and some of their properties are investigated. Multiplicative analogues of the Desargues and Pappus theorems are proved, as well as, the fundamental theorem of the projective multiplicative geometry is given. Chapter 7 is devoted to the multiplicative distance geometry on $P_\star^2$. Multiplicative orthogonal transformations, multiplicative reflections, multiplicative rotations and multiplicative translations are introduced. Chapter 8 deals with the multiplicative hyperbolic plane $H_\star^2$. Multiplicative lines, multiplicative segments, multiplicative triangles, multiplicative quadrilateral figures, multiplicative circles, multiplicative horocycles and multiplicative equidistant curves are defined and investigated. Multiplicative isometries, multiplicative reflections, multiplicative rotations and multiplicative translations in the multiplicative hyperbolic plane are studied.

This book is addressed to a wide audience of specialists such as mathematicians, physicists, engineers and biologists. It can be used as a textbook at the graduate level and as a reference book for several disciplines.

Paris, January 2022 *Svetlin G. Georgiev*

# Contents

# Chapter 1

# The Field $\mathbb{R}_\star$

In this chapter, the field $\mathbb{R}_\star$ is introduced and the basic multiplicative arithmetic operations: multiplicative addition, multiplicative subtraction, multiplicative multiplication and multiplicative division are defined and some of their properties are deduced. The basic elementary multiplicative functions are introduced and studied.

## 1.1 Definition

Let $\mathbb{R}_\star = (0, \infty)$.

**Definition 1.1.** *In the set $\mathbb{R}_\star$ we define the multiplicative addition or $\star$ addition $+_\star$ in the following manner*

$$a +_\star b = ab, \quad a, b \in \mathbb{R}_\star.$$

**Example 1.2.** *Let $a = 1$, $b = 3$. Then*

$$\begin{aligned} a +_\star b &= 1 +_\star 3 \\ &= 1 \cdot 3 \\ &= 3. \end{aligned}$$

**Definition 1.3.** *In the set $\mathbb{R}_\star$ we define the multiplicative multiplication or $\star$ multiplication $\cdot_\star$ as follows]*

$$a \cdot_\star b = e^{\log a \log b}.$$

**Example 1.4.** *Let $a = 1$ and $b = e$. Then*

$$\begin{aligned} a \cdot_\star b &= e^{\log 1 \log e} \\ &= 1. \end{aligned}$$

**Example 1.5.** *Let $a = 2$, $b = \frac{1}{3}$, $c = 4$. We will find*

$$A = (a +_\star b) \cdot_\star c.$$

*We have*

$$\begin{aligned} a +_\star b &= 2 \cdot \frac{1}{3} \\ &= \frac{2}{3}. \end{aligned}$$

*Then*

$$\begin{aligned} A &= e^{\log(a+_\star b) \log c} \\ &= e^{\log \frac{2}{3} \log 4} \\ &= e^{2 \log 2 \log \frac{2}{3}}. \end{aligned}$$

**Exercise 1.6.** *Let $a = e^3$, $b = e^4$, $c = e^{10}$. Find*

$$A = (a +_\star b) \cdot_\star c.$$

**Answer 1.7.** $A = e^{70}$.

**Definition 1.8.** *In the set $\mathbb{R}_\star$ we define the multiplicative zero($\star$ zero) and multiplicative unit($\star$ unit) as follows*

$$0_\star = 1 \quad \textit{and} \quad 1_\star = e.$$

Below, we will list some of the properties of the multiplicative addition and multiplicative multiplication.

1. **Commutativity of $\star$ Addition.** Let $x, y \in \mathbb{R}_\star$ be arbitrarily chosen. Then

$$\begin{aligned} x +_\star y &= xy \\ &= yx \\ &= y +_\star x. \end{aligned}$$

2. **Associativity of $\star$ Addition.** Let $x, y, z \in \mathbb{R}_\star$ be arbitrarily chosen. Then

$$\begin{aligned} x +_\star (y +_\star z) &= x +_\star (yz) \\ &= xyz \\ &= (xy)z \\ &= (x +_\star y)z \\ &= (x +_\star y) +_\star z. \end{aligned}$$

3. $\star$ **Identity Element of $\star$ Addition.** Let $x \in \mathbb{R}_\star$ be arbitrarily chosen. Then

$$\begin{aligned} x +_\star 0_\star &= x +_\star 1 \\ &= x \cdot 1 \\ &= x. \end{aligned}$$

4. $\star$ **Inverse Elements of $\star$ Addition.** Let $x \in \mathbb{R}_\star$ be arbitrarily chosen. Define

$$-_\star x = \frac{1}{x}.$$

Then

$$\begin{aligned} x +_\star (-_\star x) &= x +_\star \left(\frac{1}{x}\right) \\ &= x \cdot \frac{1}{x} \\ &= 1 \\ &= 0_\star. \end{aligned}$$

5. $\star$ **Identity Element of $\star$ Multiplication.** Let $x \in \mathbb{R}_\star$ be arbitrarily chosen. Then

$$\begin{aligned} x \cdot_\star 1_\star &= x \cdot_\star e \\ &= e^{\log x \log e} \\ &= e^{\log x} \\ &= x. \end{aligned}$$

6. $\star$ **Inverse Elements of $\star$ Multiplication.** Let $x \in \mathbb{R}_\star$ be arbitrarily chosen. Take

$$x^{-1_\star} = e^{\frac{1}{\log x}}.$$

Then

$$\begin{aligned} x \cdot_\star x^{-1_\star} &= x \cdot_\star \left(e^{\frac{1}{\log x}}\right) \\ &= e^{\log x \log e^{\frac{1}{\log x}}} \\ &= e^{\frac{\log x}{\log x}} \\ &= e \\ &= 1_\star. \end{aligned}$$

7. **Distributivity.** Let $x, y, z \in \mathbb{R}_\star$ be arbitrarily chosen. Then

$$\begin{aligned}(x +_\star y) \cdot_\star z &= (xy) \cdot_\star z \\ &= e^{\log(xy)\log z} \\ &= e^{(\log x + \log y)\log z} \\ &= e^{\log x \log z + \log y \log z} \\ &= e^{\log x \log z} e^{\log y \log z} \\ &= (x \cdot_\star z) \cdot (y \cdot_\star z) \\ &= (x \cdot_\star z) +_\star (y \cdot_\star z).\end{aligned}$$

**Definition 1.9.** *For any $x \in \mathbb{R}_\star$, the number*

$$-_\star x = \frac{1}{x}$$

*will be called the multiplicative opposite number or $\star$ opposite number of the number $x$.*

We have

$$\begin{aligned}-_\star(-_\star x) &= -_\star\left(\frac{1}{x}\right) \\ &= \frac{1}{\frac{1}{x}} \\ &= x\end{aligned}$$

for any $x \in \mathbb{R}_\star$.

**Definition 1.10.** *For $x, y \in \mathbb{R}_\star$, define multiplicative subtraction or $\star$ subtraction $-_\star$ as follows*

$$x -_\star y = x +_\star (-_\star y)$$

$$
\begin{aligned}
&= x(-_\star y)\\
&= x\cdot\frac{1}{y}\\
&= \frac{x}{y}.
\end{aligned}
$$

**Definition 1.11.** *For $x \in \mathbb{R}_\star$, $x \neq 0_\star$, the number*

$$x^{-1_\star} = e^{\frac{1}{\log x}}$$

*will be called the multiplicative reciprocal or $\star$ reciprocal of the number $x$.*

We have

$$
\begin{aligned}
\left(x^{-1_\star}\right)^{-1_\star} &= \left(e^{\frac{1}{\log x}}\right)^{-1_\star}\\
&= e^{\frac{1}{\log e^{\frac{1}{\log x}}}}\\
&= e^{\log x}\\
&= x
\end{aligned}
$$

for any $x \in \mathbb{R}_\star$, $x \neq 0_\star$.

**Definition 1.12.** *For $x, y \in \mathbb{R}_\star$, define multiplicative division or $\star$ division $/_\star$ as follows*

$$
\begin{aligned}
x/_\star y &= x\cdot_\star\left(y^{-1_\star}\right)\\
&= x\cdot_\star\left(e^{\frac{1}{\log y}}\right)\\
&= e^{\log x \log e^{\frac{1}{\log y}}}
\end{aligned}
$$

$$= \quad e^{\frac{\log x}{\log y}}.$$

**Example 1.13.** *We will find*

$$A = (2 +_\star 3) \cdot_\star 4 -_\star (3 +_\star 1) /_\star 5.$$

*We have*

$$\begin{aligned} A &= (2 \cdot 3) \cdot_\star 4 -_\star (3 \cdot 1) /_\star 5 \\ &= 6 \cdot_\star 4 -_\star 3 /_\star 5 \\ &= e^{\log 6 \log 4} -_\star e^{\frac{\log 3}{\log 5}} \\ &= e^{2 \log 6 \log 2 - \frac{\log 3}{\log 5}}. \end{aligned}$$

**Exercise 1.14.** *Find*

1. $A = (3 -_\star 5) /_\star 2 +_\star (4 +_\star 2) \cdot_\star e.$
2. $A = 3 +_\star 2 -_\star 3 \cdot_\star (2 +_\star 4).$
3. $A = 1 -_\star 3 +_\star 4 \cdot_\star (1 +_\star 5).$

**Answer 1.15.**

1. $e^{\frac{\log \frac{3}{5}}{\log 2} + 3 \log 2}.$
2. $e^{\frac{\log 6}{3 \log 3 \log 2}}.$
3. $\frac{1}{3} e^{2 \log 2 \log 5}.$

**Theorem 1.16.** *For any $a, b \in \mathbb{R}_\star$, the equation*

$$a +_\star x = b \tag{1.1}$$

*has at least one solution.*

*Proof.* Let

$$x = b -_\star a.$$

Then

$$\begin{aligned} a +_\star (b -_\star a) &= a +_\star \left(\frac{b}{a}\right) \\ &= a \cdot \frac{b}{a} \\ &= b. \end{aligned}$$

This completes the proof. □

**Corollary 1.17.** *Any solution $x \in \mathbb{R}_\star$ of the equation*

$$a +_\star x = a, \quad a \in \mathbb{R}_\star, \tag{1.2}$$

*is a solution of the equation*

$$b +_\star x = b, \quad b \in \mathbb{R}_\star. \tag{1.3}$$

*Proof.* By Theorem 1.16, it follows that the equation (1.2) and the equation

$$a +_\star y = b$$

have at least one solution $x$ and $y$, respectively. Then

$$\begin{aligned} b +_\star x &= (a +_\star y) +_\star x \\ &= a +_\star (y +_\star x) \\ &= a +_\star (x +_\star y) \\ &= (a +_\star x) +_\star y \\ &= a +_\star y \\ &= b, \end{aligned}$$

i.e., $x$ is a solution of the equation (1.3). This completes the proof. □

**Corollary 1.18.** *The equation (1.2) has a unique solution.*

*Proof.* By Theorem 1.16, it follows that the equation (1.2) has at least one solution. Assume that the equation (1.2) has two solutions $x$ and $y$. Then

$$a +_\star x = a$$

and

$$a +_\star y = a.$$

By Corollary 1.17, it follows

$$y +_\star x = y$$

and

$$x +_\star y = x.$$

Hence,

$$\begin{aligned} y &= y +_\star x \\ &= x +_\star y \\ &= x. \end{aligned}$$

This completes the proof. □

**Remark 1.19.** *By Corollary 1.18, it follows that the multiplicative zero* $0_\star$ *is unique.*

**Corollary 1.20.** *For any* $a, b \in \mathbb{R}_\star$*, the equation (1.1) has a unique solution.*

*Proof.* By Theorem 1.16, it follows that the equation (1.1) has at least one solution. Assume that the equation (1.1) has two solutions $x$ and $y$. Then

$$a +_\star x = b$$

and

$$a +_\star y = b.$$

By Theorem 1.16, it follows that the equation

$$a +_\star z = 0_\star$$

has at least one solution. Hence,

$$y = y +_\star 0_\star$$

$$
\begin{aligned}
&= y +_\star (a +_\star z) \\
&= (y +_\star a) +_\star z \\
&= (a +_\star y) +_\star z \\
&= b +_\star z \\
&= (a +_\star x) +_\star z \\
&= a +_\star (x +_\star z) \\
&= a +_\star (z +_\star x) \\
&= (a +_\star z) +_\star x \\
&= 0_\star +_\star x \\
&= x.
\end{aligned}
$$

This completes the proof. □

**Theorem 1.21.** *For any $a \in \mathbb{R}_\star$, $a \neq 0_\star$, the equation*

$$a \cdot_\star x = b \tag{1.4}$$

*has a solution.*

*Proof.* Let

$$x = b /_\star a.$$

Then

$$
\begin{aligned}
a \cdot_\star x &= a \cdot_\star (b /_\star a) \\
&= a \cdot_\star e^{\frac{\log b}{\log a}}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{\log a \log e^{\frac{\log b}{\log a}}} \\
&= e^{\log a \cdot \frac{\log b}{\log a}} \\
&= e^{\log b} \\
&= b.
\end{aligned}
$$

This completes the proof. □

**Corollary 1.22.** *If $a \in \mathbb{R}_\star$, $a \neq 0_\star$, then any solution of the equation*

$$a \cdot_\star x = a \tag{1.5}$$

*is a solution to the equation*

$$b \cdot_\star x = b, \quad b \in \mathbb{R}_\star. \tag{1.6}$$

*Proof.* By Theorem 1.21, it follows that the equation (1.5) and the equation

$$a \cdot_\star y = b$$

have at least one solution $x$ and $y$, respectively. Then

$$
\begin{aligned}
b \cdot_\star x &= (a \cdot_\star y) \cdot_\star x \\
&= a \cdot_\star (y \cdot_\star x) \\
&= a \cdot_\star (x \cdot_\star y) \\
&= (a \cdot_\star x) \cdot_\star y \\
&= a \cdot_\star y \\
&= b,
\end{aligned}
$$

i.e., $x$ is a solution to the equation (1.6). This completes the proof. □

**Corollary 1.23.** *Let $a \in \mathbb{R}_\star$, $a \neq 0_\star$. Then the equation (1.5) has a unique solution.*

*Proof.* By Theorem 1.21, it follows that the equation (1.5) has at least one solution. Let $x$ and $y$ be two solutions to the equation (1.5). By Corollary 1.22, it follows that

$$y \cdot_\star x = y$$

and

$$x \cdot_\star y = x.$$

Hence,

$$\begin{aligned} y &= y \cdot_\star x \\ &= x \cdot_\star y \\ &= x. \end{aligned}$$

This completes the proof. □

**Remark 1.24.** *By Corollary 1.23, it follows that the multiplicative unit is unique.*

**Corollary 1.25.** *Let $a \in \mathbb{R}_\star$, $a \neq 0_\star$. Then the equation (1.4) has a unique solution.*

*Proof.* By Theorem 1.21, it follows that the equation (1.4) has at least one solution. Assume that the equation (1.4) has two solutions $x$ and $y$. By Theorem 1.21, we have that the equation

$$a \cdot_\star z = 1_\star$$

has at least one solution. Then

$$\begin{aligned} y &= y \cdot_\star 1_\star \\ &= y \cdot_\star (a \cdot_\star z) \\ &= (y \cdot_\star a) \cdot_\star z \end{aligned}$$

$$\begin{aligned} &= (a \cdot_\star y) \cdot_\star z \\ &= b \cdot_\star z \\ &= (a \cdot_\star x) \cdot_\star z \\ &= (x \cdot_\star a) \cdot_\star z \\ &= x \cdot_\star (a \cdot_\star z) \\ &= x \cdot_\star 1_\star \\ &= x. \end{aligned}$$

This completes the proof. □

**Corollary 1.26.** *Let $a, b \in \mathbb{R}_\star$. Then*

1. $0_\star = -_\star 0_\star$.
2. $a -_\star 0_\star = a$.
3. $a \cdot_\star 0_\star = 0_\star$.
4. $(-_\star 1_\star) \cdot_\star a = -_\star a$.
5. $(-_\star 1_\star) \cdot_\star (-_\star 1_\star) = 1_\star$.
6. $-_\star(a -_\star b) = b -_\star a$.

*Proof.* 1. We have that $0_\star$ is a solution to the equation

$$0_\star +_\star x = 0_\star. \tag{1.7}$$

Since

$$0_\star +_\star (-_\star 0_\star) = 0_\star,$$

we obtain that $-_\star 0_\star$ is also a solution of the equation (1.7). By Corollary 1.18, it follows that the equation (1.7) has a unique solution. Therefore

$$0_\star = -_\star 0_\star.$$

2. We have

$$\begin{aligned} a -_\star 0_\star &= a +_\star (-_\star 0_\star) \\ &= a +_\star 0_\star \\ &= a +_\star 1 \\ &= a \cdot 1 \\ &= a. \end{aligned}$$

3. We have

$$\begin{aligned} a \cdot_\star 0_\star &= a \cdot_\star 1 \\ &= e^{\log a \log 1} \\ &= 1 \\ &= 0_\star. \end{aligned}$$

4. We have

$$\begin{aligned} (-_\star 1_\star) \cdot_\star a &= (-_\star e) \cdot_\star a \\ &= \frac{1}{e} \cdot_\star a \\ &= e^{\log \frac{1}{e} \log a} \\ &= e^{-\log a} \\ &= \frac{1}{a} \\ &= -_\star a. \end{aligned}$$

5. We have

$$
\begin{aligned}
(-_\star 1_\star)\cdot_\star(-_\star 1_\star) &= (-_\star e)\cdot_\star(-_\star e)\\
&= \frac{1}{e}\cdot_\star\frac{1}{e}\\
&= e^{\log\frac{1}{e}\log\frac{1}{e}}\\
&= e\\
&= 1_\star.
\end{aligned}
$$

6. We have

$$
\begin{aligned}
-_\star(a-_\star b) &= -_\star(a+_\star(-_\star b))\\
&= -_\star\left(a+_\star\frac{1}{b}\right)\\
&= -_\star\frac{a}{b}\\
&= \frac{1}{\frac{a}{b}}\\
&= \frac{b}{a}\\
&= b-_\star a.
\end{aligned}
$$

This completes the proof.

□

## 1.2 An Order in $\mathbb{R}_\star$

**Definition 1.27.** *We say that a number $a\in\mathbb{R}_\star$ is multiplicative positive or $\star$ positive if $a>1$. We will write $a>_\star 0_\star$.*

**Example 1.28.** 5 *is $\star$ positive.*

**Definition 1.29.** *A number $a \in \mathbb{R}_\star$ is said to be multiplicative negative or $\star$ negative if it is not equal to $0_\star$ and it is not $\star$ positive. We will write $a <_\star 0_\star$.*

**Example 1.30.** $\frac{1}{2}$ *is a $\star$ negative number.*

**Definition 1.31.** *Let $a, b \in \mathbb{R}_\star$. We say that $a$ is multiplicative greater than $b$ or $\star$ greater than $b$ and we will write $a >_\star b$ if $a -_\star b >_\star 0_\star$. We will denote $a \geq_\star b$ if $a >_\star b$ or $a = b$.*

**Remark 1.32.** *Let $a, b \in \mathbb{R}_\star$. Then*

$$a >_\star b \iff a -_\star b >_\star 0_\star \iff \frac{a}{b} > 1 \iff a > b.$$

**Definition 1.33.** *Let $a, b \in \mathbb{R}_\star$. We say that $a$ is multiplicative less than $b$ or $\star$ less than $b$ and we will write $a <_\star b$ if $a -_\star b <_\star 0_\star$. We will denote $a \leq_\star b$ if $a <_\star b$ or $a = b$.*

**Remark 1.34.** *Let $a, b \in \mathbb{R}_\star$. Then*

$$a <_\star b \iff a -_\star b <_\star 0_\star \iff \frac{a}{b} < 1 \iff a < b.$$

**Theorem 1.35.** *Let $a, b, c, d \in \mathbb{R}_\star$. If*

$$a >_\star b \quad \text{and} \quad c >_\star d, \tag{1.8}$$

*then*

$$a +_\star c >_\star b +_\star d.$$

*Proof.* By (1.8), it follows that

$$a > b > 0 \quad \text{and} \quad c > d > 0.$$

Then

$$\begin{aligned} a +_\star c &= ac \\ &> bd \\ &= b +_\star d. \end{aligned}$$

This completes the proof. □

**Theorem 1.36.** *Let* $a, b, c, d \in \mathbb{R}_\star$. *If*

$$a >_\star b, \quad c >_\star d, \quad c >_\star 0_\star, \quad b >_\star 0_\star, \tag{1.9}$$

*then*

$$a \cdot_\star c >_\star b \cdot_\star d.$$

*Proof.* By (1.9), it follows that

$$a > b, \quad c > d, \quad c > 1, \quad b > 1.$$

Then

$$\begin{aligned} \log a &> \log b, \\ \log c &> \log d, \\ \log c &> 0, \\ \log b &> 0, \end{aligned}$$

whereupon

$$\begin{aligned} \log a \log c &> \log b \log c \\ &> \log b \log d \end{aligned}$$

and

$$e^{\log a \log c} > e^{\log b \log d},$$

i.e.,

$$a \cdot_\star c >_\star b \cdot_\star d.$$

This completes the proof. □

**Theorem 1.37** (Chebyschev Multiplicative Inequality)**.** *Let* $a_1, a_2, b_1, b_2 \in \mathbb{R}_\star$. *Then the inequality*

$$a_1 \cdot_\star b_1 +_\star a_2 \cdot_\star b_2 \leq_\star e^{\frac{1}{2}} \cdot_\star (a_1 +_\star a_2) \cdot_\star (b_1 +_\star b_2)$$

*holds if and only if the inequality*

$$\log \frac{a_2}{a_1} \log \frac{b_2}{b_1} \leq 0$$

*holds.*

*Proof.* Note that

$$\log \frac{a_2}{a_1} \log \frac{b_2}{b_1} \le 0$$

if and only if

$$\log a_1 \log b_1 + \log a_2 \log b_2 \le \frac{1}{2} \log(a_1 a_2) \log(b_1 b_2)$$

if and only if

$$\begin{aligned} e^{\log a_1 \log b_1 + \log a_2 \log b_2} &\le e^{\frac{1}{2} \log(a_1 a_2) \log(b_1 b_2)} \\ &= e^{\log e^{\frac{1}{2}} \log(a_1 a_2) \log(b_1 b_2)} \end{aligned}$$

if and only if

$$a_1 \cdot_\star b_1 +_\star a_2 \cdot_\star b_2 \le_\star e^{\frac{1}{2}} \cdot_\star (a_1 +_\star a_2) \cdot_\star (b_1 +_\star b_2).$$

This completes the proof. □

**Theorem 1.38.** *Let $a, b, c, \lambda \in \mathbb{R}_\star$.*

1. *If $a >_\star b$, then $a +_\star c >_\star b +_\star c$.*
2. *If $a >_\star b$ and $b >_\star c$, then $a >_\star c$.*
3. *If $\lambda >_\star 0_\star$ and $a >_\star b$, then $\lambda \cdot_\star a >_\star \lambda \cdot_\star b$.*
4. *If $\lambda <_\star 0_\star$ and $a >_\star b$, then $\lambda \cdot_\star a <_\star \lambda \cdot_\star b$.*

*Proof.*

1. Let $a >_\star b$. Then $a > b$ and hence,

   $$ac > bc,$$

   or

   $$a +_\star c >_\star b +_\star c.$$

2. Let $a >_\star b$ and $b >_\star c$. Then $a > b$ and $b > c$. Hence, $a > c$ and $a >_\star c$.
3. Let $\lambda >_\star 0_\star$ and $a >_\star b$. Then $\lambda > 1$ and $\log \lambda > 0$, and

   $$\log a > \log b.$$

Hence,

$$\log\lambda \log a > \log\lambda \log b$$

and

$$e^{\log\lambda \log a} > e^{\log\lambda \log b},$$

or

$$\lambda \cdot_\star a >_\star \lambda \cdot_\star b.$$

4. Let $\lambda <_\star 0_\star$ and $a >_\star b$. Then $\lambda \in (0,1)$ and

$$\log a > \log b, \quad \log\lambda < 0.$$

Hence,

$$\log\lambda \log a < \log\lambda \log b$$

and

$$e^{\log\lambda \log a} < e^{\log\lambda \log b},$$

or

$$\lambda \cdot_\star a <_\star \lambda \cdot_\star b.$$

This completes the proof.

□

## 1.3 Multiplicative Absolute Value

In this section, we will define the multiplicative absolute value and we will deduce some of its properties.

**Definition 1.39.** *Let $x \in \mathbb{R}_\star$. The multiplicative absolute value is defined as follows*

$$|x|_\star = \begin{cases} x & \text{if} \quad x \geq_\star 0_\star, \\ \\ \dfrac{1}{x} & \text{if} \quad x \leq_\star 0_\star, \end{cases}$$

*or*

$$|x|_\star = \begin{cases} x & \text{if} \quad x \geq 1, \\ \\ \dfrac{1}{x} & \text{if} \quad x \leq 1. \end{cases}$$

**Example 1.40.** *Let $x = 5$. Then $|x|_\star = 5$.*

**Example 1.41.** *Let* $x = \frac{1}{6}$. *Then* $|x|_\star = 6$.

Below, we will deduce some of the properties of the multiplicative absolute value. Let $a, b \in \mathbb{R}_\star$.

1. $|a|_\star \geq_\star 0_\star$.

   *Proof.* If $a \geq_\star 0_\star$, then $a \geq 1$ and

$$\begin{aligned} |a|_\star &= a \\ &\geq 1 \\ &\geq_\star 0_\star. \end{aligned}$$

   Let $a \leq_\star 0_\star$. Then $a \leq 1$ and

$$\begin{aligned} |a|_\star &= \frac{1}{a} \\ &\geq 1 \\ &\geq_\star 0_\star. \end{aligned}$$

   This completes the proof. □

2. $|a|_\star = |-_\star a|_\star$.

   *Proof.* We have

$$-_\star a = \frac{1}{a}.$$

   Then

$$|a|_\star = \begin{cases} a & \text{if} \quad a \geq 1, \\ \frac{1}{a} & \text{if} \quad a \leq 1 \end{cases}$$

   and

$$|-_\star a|_\star = \begin{cases} \frac{1}{a} & \text{if} \quad a \leq 1, \\ a & \text{if} \quad a \geq 1. \end{cases}$$